ON THE ZEROS OF A POLYNOMIAL

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Abstract. Using an extension of Hölder's inequality, we obtained an upper bound for the moduli of zeros of a polynomial with complex coefficients.

According to a classical result of Kuniyada, Montel and Toya [2, p. 124], on the location of zeros of a polynomial, all the zeros of the polynomial \( P(z) = \sum_{k=0}^{n} a_k z^k \) (\( a_n \neq 0 \)) of degree \( n \), lie in the circle

\[
|z| < \left( 1 + \left( \frac{1}{n} \sum_{j=0}^{n-1} \frac{a_j}{a_n} \right)^{p/q} \right)^{1/q}
\]

where \( p > 1 \), \( q > 1 \), \( 1/p + 1/q = 1 \).

Hölder's inequality was used to obtain the bound (1). In this note, we have used an extension of Hölder's inequality to obtain the following:

Theorem. All the zeros of the polynomial \( p(z) = \sum_{k=0}^{n} a_k z^k \), of degree \( n \), lie in the circle

\[
|z| < \kappa^{1/q},
\]

where \( \kappa \) is the unique root of the equation

\[
x^3 - (1 + DN)x^2 + DNx - D = 0,
\]

in \((1, \infty)\),

\[
D = \left( \sum_{j=0}^{n-1} \frac{|a_j|^p}{|a_n|^p} \right)^{q/p}
\]

\[
N = (|a_{n-1}| + |a_{n-2}|)q(|a_{n-1}^p + |a_{n-2}|^p)^{-(q-1)},
\]

\( p > 1 \), \( q > 1 \), \( 1/p + 1/q = 1 \).

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Remark 1. The bound (2) is better than the bound (1), for the polynomials, whose coefficients satisfy the inequality
\[ D < (2 - N)/(N - 1), \]
(4)
as can be easily seen from the following:
\[ f(x) = x^3 - (1 + DN)x^2 + DNx - D, \]
\[ f(x) > 0, \text{ for } x > K; \ f(x) < 0, \text{ for } 1 < x < K \]
\[ f(1 + D) = (1 + D)^3 - (1 + DN)(1 + D)^2 + DN(1 + D) - D \]
\[ = D^2[(2 - N) - D(N - 1)] > 0 \quad \text{(by (3))}, \]
which implies \((1 + D) > K\).

The polynomial \(P(z) = a_0 + a_1 z + \cdots + a_5 z^5\) with \(|a_0| = 1, |a_1| = 2, |a_2| = 1, |a_3| = 1, |a_4| = 4, |a_5| = 6\), satisfies (4), for \(q = p = 2\).

Remark 2. It is quite easy to show that the equation (3) has only one real root in \((1, \infty)\).

Let
\[ F(y) = f(1 + y) = (1 + y)^3 - (1 + DN)(1 + y)^2 + DN(1 + y) - D \]
\[ = y^3 + (2 - DN)y^2 + (1 - DN)y - D. \]

Now, let us consider three different possibilities:

1. \(0 < DN \leq 1;\)
2. \(1 < DN \leq 2;\)
3. \(2 < DN.\)

If we consider (i), then \(F(y)\) has only one change of sign. Hence, according to Descartes’ rule of signs, the equation \(F(y) = 0\) will have at most one positive root. The same is also true for (ii) and (iii). Hence, whatever be the value of \(DN\), the equation \(F(y) = 0\) will have at most one positive root. Furthermore \(F(0) = -D; F(y) \to +\infty\) as \(y \to \infty\). Hence, the equation \(F(y) = 0\) has exactly one positive root. This obviously implies that the equation \(f(x) = 0\) i.e., the equation (3) has only one root in \((1, \infty)\).

For the proof of the theorem we require the following extension of Hölder’s inequality:

Lemma. (Beckenbach [1]) Let \(\alpha_j > 0, \beta_j > 0\) for \(j = 1, 2, \ldots, n\) and \(p > 1, q > 1\) with \(1/p + 1/q = 1\). Then
\[ \sum_{j=1}^{n} \alpha_j \beta_j \leq \left( \left( \sum_{j=1}^{n} \beta_j^p \right)^{1/p} \left( \sum_{j=1}^{m} \beta_j^q \right)^{-1/p} \right) \cdot \left[ \left( \sum_{j=1}^{m} \alpha_j \beta_j \right)^{-q} + \left( \left( \sum_{j=1}^{m} \beta_j^p \right)^{-q} \right) \left( \sum_{j=m+1}^{n} \alpha_j^q \right)^{-1/q} \right] \quad (4) \]
Proof of the theorem. For \(|z| > 1\), we have

\[
|P(z)| \geq |a_n| |z|^n - \sum_{j=0}^{n-1} |z|^j |a_j|
\]

\[
\geq |a_n| |z|^n - \left( \sum_{j=0}^{n-1} |a_j|^p \right)^{1/p} \left( |a_{n-1}|^p + |a_{n-2}|^p \right)^{-1/p} 
\]

\[
\cdot \left[ (|a_{n-1}| |z|^{n-1} + |a_{n-2}| |z|^{n-2})^q + (|a_{n-1}|^p + |a_{n-2}|^p)^{q-1} \left( \sum_{j=0}^{n-1} |z|^{jq} \right)^{1/q} \right] , \quad \text{(by (5))}
\]

\[
\geq |a_n| |z|^n - \left( \sum_{j=0}^{n-1} |a_j|^p \right)^{1/p} \cdot \left\{ \left( |z|^{(n-1)q} (|a_{n-1}| + |a_{n-2}|)^q (|a_{n-1}|^p + |a_{n-2}|^p)^{-q+1} + \left( \sum_{j=0}^{n-3} |z|^{jq} \right) \right\}^{1/q}
\]

\[
= |a_n| |z|^n \left[ 1 - \frac{1}{D^{1/q} \left\{ (N|z|^{-q} + \left( \sum_{j=3}^{n} |z|^{-jq} \right)^{1/q} \right\}} \right] 
\]

\[
> |a_n| |z|^n \left[ 1 - \frac{1}{D^{1/q} \left\{ (N|z|^{-q} + \left( \sum_{j=3}^{\infty} |z|^{-jq} \right)^{1/q} \right\}} \right] 
\]

\[
= |a_n| |z|^n \left[ 1 - \frac{1}{D^{1/q} \left\{ (N|z|^{-q} + 1 + 1) / (|z|^{2q}(|z|^{-q} - 1)) \right\}^{1/q} \right] 
\]

\[
= |a_n| |z|^n \left[ 1 - \frac{(DN|z|^{2q} - DN|z|^{-q} + D)/(|Z|^{2q} - |z|^{2q}))^{1/q} \right] 
\]

\[
\geq 0, \quad \text{if} \quad 1 \geq (DN|z|^{2q} - DN|z|^{-q} + D)/(|Z|^{2q} - |z|^{2q}) \).
\]

So, we conclude that \(P(z) \neq 0\), for \(|z| > 1\), if

\[
(|z|^q)^3 - (|z|^q)^2 - DN(|z|^q)^2 + DN(|z|^q) - D > 0.
\]

Replacing \(|z|^q\) by \(x\), we get the result.

REFERENCES