SPECTRAL TYPE OF SOME TRANSFORMATIONS
OF CERTAIN STOCHASTIC PROCESSES

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Abstract. We introduce a stochastic process with multiplicity equal to one which satisfies certain conditions and consider spectral type of the derivative process and of the non-anticipative integral transformations for the given process.

0. The technique used in the paper is the same as in [1] or [3]. Let the process \( x(t) \) be given by Cramer representation:

\[
x(t) = \int_{0}^{t} g(t, u)dz(u),
\]

\( u \leq t, t \in T = (a, b) \) where \( z(u) \) is a process of orthogonal increments such that

\( Ez(u) = 0 \) and \( Ez^2(u) = F(u) \) and \( g(t, u) \) is a nonrandom function for \( u \leq t \) from \( L^2(dF) \) space. We suppose that the second order process \( x(t) \) is continuous to the left and purely nondeterministic.

Let us introduce the following conditions for \( g(t, u) \) and \( z(u) \):

\( (R_1) \) The functions \( g(t, u) \) and \( g'(t, u) \) are continuous and bounded for \( u \leq t, u, t \in T \).

\( (R_2) \) \( g(t, t) = 0 \) for all \( t \in T \).

\( (R_3) \) The function \( F(u) = Ez^2(u) \) is absolutely continuous and not identically constant and \( f(u) = F'(u) \) has at most a finite number of discontinuity points in any finite subinterval of \( T \).

In [4] we proved the following theorem: the process \( x(t), t \in T \) given by (1) and satisfying \( (R_1), (R_2), (R_3) \) has multiplicity \( N = 1 \). Furthermore we suppose that \( x(t) \) satisfies conditions above.

1. As it is well known a form of correlation function for \( x(t) \) is:

\[
r(s, t) = \int_{a}^{\min(s, t)} g(s, u) \cdot g(t, u)f(u)du,
\]

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and \( r(s,t) \) is continuous everywhere in the interval \( T \times T \). By the condition \( R_1 \),
\( r(s,t) \) has partial derivatives \( r^t_s \) and \( r^t_t \) which are continuous everywhere except perhaps on the diagonal \( s = t \). But we have:
\[
\lim_{s \to t} \frac{r(s,t) - r(t,t)}{s-t} = \int_a^t g(t,u)g'_t(t,u)f(u)du,
\]
\[
\lim_{s \to t} \frac{r(s,t) - r(t,t)}{s-t} = \int_a^t g(t,u)g'_t(t,u)f(u)du + g^2(t,t) \cdot f(t).
\]
By the condition \( R_2 \) this two limiting values will be equal. Hence \( r^t_s \) and \( r^t_t \) are continuous at every point. Similarly the partial derivative \( r''_{s,t} \) is a continuous function for \( s, t \) and its form is:
\[
r''_{s,t} = \int_s^{\min(s,t)} g'_s(s,u)g''_t(t,u)f(u)du,
\]
\( s, t \in T \). The expression above is the correlation function of the derivative process:
\[
x'(t) = \int_a^t g'_t(t,u)du, \quad u \leq t, \ u, t \in T.
\]

**Theorem 1.** The derivative process \( x'(t) \) given by (2) is continuous and has the same spectral type as the process \( x(t) \).

**Proof.** Continuity of \( x'(t) \) follows from the fact that its correlation function is continuous. By the theorem from [2] and from the form of \( r(s,t) \), for \( x'(t) \) it is sufficient to show that \( g'_t(s,u) \), \( u \leq t, \ u, t \in T \) is complete in \( L^2(dF) \).
If \( \int_a^s g'_s(s,u)\psi(u)f(u)du = 0 \) for all \( s \in (a,t] \), where \( \psi(u) \) is any function from \( L^2(dF) \) space, and \( t \) is any point from \( T \), then for all \( s \in (a,t] \) the following holds: \( (\int_a^s g(s,u)\psi(u)f(u)du)'_a = 0 \). That means \( \int_a^s g(s,u)\psi(u)f(u)du = 0 \) for all \( s \in (a,t] \). Since \( g(t,u) \) is complete in \( L^2(dF) \), \( u, t \in T \) then \( \psi(u) = 0 \) almost everywhere related to the measure \( dF \) and that is what we have to show. The spectral measure for \( x'(t) \) is \( dF \), multiplicity equal to one and the expression (2) is Cramer representation of \( x'(t) \).

**Example 1.** Let \( x(t) = \int_a^t (P(t) - P(u)) \cdot du, \ u \leq t, \ u, t \in (a,b) \) be a process with absolutely continuous \( F(u) \), where \( P(t) \) is a polynomial of any degree \( n \geq 1 \). If \( g(t,u) = P(t) - P(u) \), \( u \leq t, \ u, t \in (a,b) \) is complete in \( L^2(dF(u)) \), then the process \( x'(t) \) exists, has multiplicity one and its spectral measure is \( dF \).

**Example 2.** The same fact holds for the process \( x(t) = \int_a^t Q(t-u) \cdot du, \ u \leq t, \ u, t \in T, \) where \( F(u) \) is absolutely continuous, \( Q(t) \) is a polynomial of degree \( n \geq 1 \), and \( Q(0) = 0 \).

**2.** Let us introduce now \( y(t), \ t \in T \) as a nonanticipative integral transformation of \( x(t) \) given by (1):
\[
y(t) = \int_a^t \varphi(t,u)x(u)du,
\]
\[ y(t) = \int_a^t \varphi(t, u) \left( \int_a^u g(u, \nu) d\nu \right) du = \int_a^t \left( \int_\nu^t \varphi(t, u) g(u, \nu) du \right) d\nu, \quad t \in T. \]

Let us denote \( \int_\nu^t \varphi(t, u) g(u, \nu) du \) by \( G(t, \nu) \) where \( a < \nu \leq u \leq t < b \).

**Lemma.** The functions \( G(t, \nu) \) and \( G'_t(t, \nu) \) are continuous if \( \varphi(t, u), \varphi'_t(t, u), \)
\( g(t, u) \) are continuous on \( t \) and \( u, \quad u \leq t, u, t \in T. \)

**Proof.** The continuity of the function \( G(t, \nu) \) on \( t \) for all \( \nu \) follows from:

\[ |G(t_2, \nu) - G(t_1, \nu)| \leq \int_{\nu_1}^{t_1} |\varphi(t_2, u) - \varphi(t_1, u)| \cdot |g(u, \nu)| du \\
+ \int_{t_1}^{t_2} |\varphi(t_2, u)| \cdot |g(u, \nu)| du, \]

when \( \nu_1 \to \nu_2 \) and \( \nu_1 \leq \nu_2 \). In a similar way we can show that by conditions of lemma the function \( G'_t(t, \nu) \) is continuous for \( t \) and \( \nu \). Here is:

\[ G'_t(t, \nu) = \int_\nu^t \varphi'_t(t, u) g(u, \nu) du + \varphi(t, t) g(t, \nu), \quad \nu \leq u \leq t, \quad \nu, t \in T. \]

**Theorem 2.** The nonanticipative integral transformation \( y(t) \) defined by (3) has the same spectral type as \( x(t) \) from (1) if the functions \( \varphi(t, u) \) and \( \varphi'_t(t, u) \) are continuous and bounded for \( u, t \in T, \quad u \leq t \).

**Proof.** From the continuity and the limitation of \( \varphi(t, u) \) and \( \varphi'_t(t, u) \) on \( t \) and \( u \), and from the fact that \( x(t) \) satisfies the condition \( R_1 \) it follows by lemma that \( R_1 \) holds for \( G(t, u), \quad u \leq t, u, t \in T \). The condition: \( G(t, t) = 0 \) for all \( t \in T \) is valid too. Finally since all of the three conditions \( R_1, R_2, R_3 \) hold for the process \( y(t) \) then \( y(t) \) has multiplicity equal to one. The spectral measure \( dF \) is the same as for \( x(t) \). That implies the same spectral type. (See theorem 5.2 in [1], and the remark in [4]).

**Example 3.** The process \( y(t) = \int_a^t x(u) du, \quad u, t \in T \) has the same spectral type as \( x(t) \). Here is \( \varphi(t, u) \equiv 1 \) for \( u \leq t \), \( G(t, \nu) = \int_a^t g(u, \nu) du \) and \( G'_t(t, \nu) = g(t, \nu) \) where \( a < \nu \leq u \leq t < b \).

**Example 4.** The process \( x(t) = z(t), \quad t \in [0, \tau] = T \) with the absolutely continuous function \( F(u), \quad u \in T \) has multiplicity \( N = 1 \). The process \( y(t) \) from (3) has the same spectral type as \( x(t) \) if \( \varphi(t, u) \) and \( \varphi'_t(t, u) \) are continuous on \( t \) and \( u, t, u \in T \). They are bounded because \( T \) is compact. Here \( G(t, \nu) = \int_\nu^t \varphi(t, u) du \), and \( G'_t(t, \nu) = \int_\nu^t \varphi'_t(t, u) du + \varphi(t, t), \quad 0 \leq \nu \leq u \leq t \leq \tau \).
Remark. If we want to prove that the process $y(t)$ has multiplicity equal to one when multiplicity of $x(t)$ is unknown, then we may omit the assumptions that $g_1(t, u)$ is continuous and bounded for $u, t \in T$ and $g(t, t) = 0$ for all $t \in T$. Namely the next theorem is valid.

Theorem 3. Let the process $x(t)$ be given by expression (1), let $g(t, u)$ be a continuous and bounded function on $t$ and $u, u, t \in T$ and let the condition $R_3$ be satisfied. Then the process $y(t)$ given by (3) has multiplicity equal to one if the functions $\varphi(t, u)$ and $\varphi'_1(t, u)$ are continuous and bounded for $u, t \in T$.

Proof. Since the conditions $R_1$, $R_2$, $R_3$ hold for $y(t), t \in T$ then the statement is valid [1, 5.2].

Example 5. The process which has multiplicity equal to two, while its nonanticipative integral transformation has multiplicity equal to one. Let $x(t)$ be represented by $x(t) = w_1(t) + h(t) \cdot w_2(t)$, where $w_1$ and $w_2$ are two independent Wiener processes for $t \geq 0$. A function $h(t)$ is absolutely continuous with $h'(t) > 0$, so that $h'(t)$ does not belong to $L^2((l, m))$ for any open interval $(l, m) \subset [0, \infty)$ but does belong to $L^1([0, t])$ for any $t > 0$. By [5] multiplicity of this process is two and the spectral type is $dt < dt$. Let us define the nonanticipative integral transformation of $x(t)$ as above in which $\varphi(t, u) = 1, u \leq t, u, t \in [0, \infty)$. That means:

$$
y(t) = \int_0^t x(u)du = \int_0^t (w_1(u) + h(u) \cdot w_2(u))du
$$

$$
= \int_0^t w_1(u)du + \int_0^t h(u)w_2(u)du
$$

$$
= \int_0^t \int_0^u dw_1(\nu)du + \int_0^t h(u) \int_0^u dw_2(\nu)du
$$

$$
= \int_0^t \left( \int_\nu^t du ight) dw_1(\nu) + \int_0^t \left( \int_\nu^t h(u)du \right) dw_2(\nu)
$$

where $0 \leq \nu \leq u \leq t < \infty$. The functions

$$
G(t, \nu) = (G_1(t, \nu), G_2(t, \nu)) = \left( t - \nu, \int_\nu^t h(u)du \right) \quad \text{and} \quad G'_1(t, \nu) = (1, h(t))
$$

are continuous and bounded for $t, \nu, 0 \leq \nu \leq t < \infty$. It is easy to see that

$$
\int_0^t G^2(t, \nu)du < \infty \quad \text{holds for} \quad t \in [0, \infty) \quad \text{and} \quad \text{the nonanticipative transformation}
$$

exists in the quadratic mean. Hence by the last theorem, multiplicity of $y(t)$ is one and its spectral measure is $dt$.
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