ON \( \sigma \)-PERMUTABLE \( n \)-GROUPS

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Abstract. In this paper \( \sigma \)-permutable \( n \)-groups are defined and considered. An \( n \)-group \((G, f)\) is called \( \sigma \)-permutable, where \( \sigma \) is a permutation of the set \( \{1, \ldots, n+1\} \), if

\[
    f(x_{\sigma(1)}, \ldots, x_{\sigma(n+1)}) = f(x_1, \ldots, x_n) = x_{n+1}
\]

for all \( x_1, \ldots, x_{n+1} \in G \). Such \( n \)-groups are a special case of \( \sigma \)-permutable \( n \)-groupoids considered in [7] and also they represent a generalization of \( i \)-permutable \( n \)-groups from [6] and some other classes of \( n \)-groups. Examples of \( \sigma \)-permutable \( n \)-groups are given and some of their properties described. Necessary and sufficient conditions for an \( n \)-group to be \( \sigma \)-permutable are determined. Several conditions under which such \( n \)-groups are derived from a binary group are given.

1. Introduction. In [7] \( \sigma \)-permutable \( n \)-groupoids which represent a generalization of several classes of \( n \)-groupoids were considered. Some well known classes of \( n \)-groupoids are special cases of \( \sigma \)-permutable \( n \)-groupoids. Among them are \((i, j)\)-commutative and commutative \( n \)-groupoids, totally symmetric \( n \)-quasigroups, medial \( n \)-groups [2], cyclic \( n \)-quasigroups from [12], [13] and [18], \( i \)-permutable \( n \)-groupoids from [15], alternating symmetric \( n \)-quasigroups ([14], [16]), parastrophy invariant \( n \)-quasigroups ([9], [8]) and some others. Binary groupoids which are \( \sigma \)-permutable for different values of \( \sigma \) are commutative groupoids, semisymmetric groupoids (satisfying the identity \((xy)z = y\)), totally symmetric quasigroups, groupoids satisfying Sade’s left ”key’s” law \((x(xy) = y\) and Sade’s right ”key’s” law \(((xy)y = x\), [1]. In this paper we shall consider \( \sigma \)-permutable \( n \)-groups, but first we give some necessary definitions and notations.

2. Notation and Definitions. We shall use the following abbreviated notation: \( f(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+s}, x_{k+s+1}, \ldots, x_n) = f(x_1^k, x, x_{k+s+1}^n) \), whenever \( x_{k+1} = x_{k+2} = \cdots = x_{k+s} = x \) \((x_0^i \) is the empty symbol for \( i > j \) and for \( i > n \), also \( x^0 \) is the empty symbol).

To avoid repetitions we assume throughout the whole text that \( n \geq 2 \).

AMS Subject Classification (1980): Primary 20N15.
An $n$-groupoid $(G, f)$ is $k$-solvable, where $k \in \{1, \ldots, n\} = N_n$ is fixed, iff the equation $f(a_i^{k-1}, a_j^k) = b$ has a solution $x \in G$ for all $a_i^k, b \in G$. If this equation has a unique solution for every $k \in N_n$ then $(G, f)$ is called an $n$-quasigroup.

An $n$-groupoid $(G, f)$ is $(i, j)$-associative, where $1 \leq i < j \leq n$, iff
\[ f(x_i^{j-1}, f(x_i^{j+i-1}, x_{n+i}^{2n-1})) = f(f(x_i^{j-1}, x_{n+i}^{2n-1}), x_{n+i+j}^{2n-1}) \]
for all $x_i^{2n-1} \in G$. An $n$-groupoid is associative (i.e. it is an $n$-semigroup) iff it is $(i, j)$-associative for every pair $(i, j)$, $i, j \in N_n$.

An associative $n$-quasigroup is an $n$-group. One can prove [10, p. 213] that an $n$-semigroup is an $n$-group iff it is $k$-solvable for $k = 1$ and $k = n$ or for some $k$ other than 1 and $n$. On the other hand Sokolov proved (in [11], but this proof is not complete, cf. [4]) that an $n$-quasigroup is an $n$-group iff it is $(i, i + 1)$-associative for some $i \in N_{n-1}$. A similar characterization of $n$-groups is given in [2] and [4].

An $n$-group is called medial (or abelian) iff it is $(1, n)$-commutative ([2], [3]).

By $S_n$ we denote the symmetric group of degree $n$.

If $\sigma \in S_n$, then $x_{\sigma_i}, x_{\sigma(i+1)}, \ldots, x_{\sigma_j}$ we denote by $x_{\sigma_1}^{\sigma_j}$. If $i > j$, then $x_{\sigma_1}^{\sigma_j}$ is considered empty.

By $w$ we shall always denote the automorphism $x \mapsto x^{-1}$ of a commutative group.

If $G$ is a set, by it we denote the identity mapping of $G$. If $A$ is a set of integers, by $gcd A$ we denote the greatest common divisor of all elements of $A$.

If an $n$-group $(G, f)$ has the form
\[ f(x_i^n) = x_1 x_2 x_3 \ldots x_{n-1} x_n b, \]
where $(G, \cdots)$ is a binary group, $b \in G$, $\theta$ an automorphism of $(G, \cdot)$, such that $\theta b = b$, $\theta^{n-1} x b = bx$ for all $x \in G$, then this $n$-group is called $(\theta, b)$-derived from $(G, \cdot)$ and it is denoted by $\text{der}_{\theta, b}(G, \cdot)$. If $\theta$ is the identity mapping, then this $n$-group is called $b$-derived from $(G, \cdot)$. If $\theta$ is the identity mapping and $b$ is the neutral element of $(G, \cdot)$, then an $n$-group $\text{der}_{\theta, b}(G, \cdot)$ is called derived from $(G, \cdot)$. In this case we say also that an $n$-group operation is a long product of the group operation (see [2] or [5]). By Hosszu theorem every $n$-group is $(\theta, b)$-derived from some binary group.

Let $(G, f)$ be an $n$-group, $a_2^{n-1} \in G$ be fixed and let $x \cdot y = f(x, a_2^{n-1}, y)$. Then the groupoid $(G, \cdot)$ is a group and it is called a (binary) retract of the $n$-group $(G, f)$. As it is well known [19], if an $n$-group $(G, f)$ is $(\theta, b)$-derived from a group $(G, \cdot)$, then $(G, \cdot)$ is isomorphic to some retract of $(G, f)$ and (since all retracts of a given $n$-group are isomorphic, [5]) it is isomorphic to every retract of $(G, f)$.

3. $\sigma$-permutative n-groupoids. $\sigma$-permutative $n$-groupoids were introduced and investigated in [7]. Here we shall give some of the basic definitions and propositions from [7].

Definition 1. Let $\sigma \in S_{n+1}$. An $n$-groupoid $(G, f)$ is called $\sigma$-permutative iff for all $x_i^{n+1} \in G$
\[ f(x_i^n) = x_{n+1} \Leftrightarrow f(x_{\sigma_1}^{\sigma_{n+1}}) = x_{\sigma_{n+1}}. \]
An equivalent form of the preceding definition is the following.

**Definition 2.** Let $\sigma \in S_{n+1}$. If $\sigma i = n+1$ for some $i \in N_n$, then an $n$-groupoid $(G, f)$ is $\sigma$-permutable if and only if for all $x^n_i \in G$

$$f(x^{\sigma(i-1)}_1, f(x^n_i), x^n_{\sigma(i+1)} = x_{\sigma(n+1)}.$$ 

If $\sigma(n+1) = n + 1$, then $(G, f)$ is $\sigma$-permutable if and only if for all $x^{n+1}_1 \in G$

$$f(x^{\sigma(n)}_{\sigma(1)}) = f(x^n_1).$$

Let $\sigma \in S_{n+1}$ and $H$ be the subgroup of $S_{n+1}$, generated by $\sigma$. An $n$-groupoid $(G, f)$ is $\sigma$-permutable if it is $\tau$-permutable for every $\tau \in H$. The set of all permutations $\sigma \in S_{n+1}$ such that

$$f(x^{\sigma(n)}_{\sigma(1)}) = f(x^n_1) \iff f(x^n_1) = x_{n+1}$$

for all $x^{n+1}_1 \in G$ is a subgroup of $S_{n+1}$.

Let $H$ be a subgroup of $S_{n+1}$. If an $n$-groupoid $(G, f)$ is $\sigma$-permutable for every $\sigma \in H$, then $(G, f)$ is called a $H$-permutable $n$-groupoid.

**4. $\sigma$-permutable $n$-groups.** A $\sigma$-permutable $n$-groupoid which is an $n$-group is called a $\sigma$-permutable $n$-group. Now we shall consider such $n$-groups, but first we give some examples.

1. $n$-groups satisfying the cyclic identity $(f(f(x^n_1)x^{n-1}) = x_n)$ from [17] are $\sigma$-permutable $n$-groups where $\sigma = (12 \ldots n + 1)$.

2. Commutative $n$-groups are $H$-permutable where $H \simeq S_n$ and $\sigma(n+1) = n + 1$ for every $\sigma \in H$.

3. Medial $n$-groups ([2], [3]) are $\sigma$-permutable, where $\sigma = (1n)$.

4. Let $(G, +)$ be an abelian group and let $f(x, y, z) = x - y + z$. Then $(G, f)$ is a ternary group which is $H$-permutable, where $H$ is the Klein’s four group generated by the permutations $(1234)$ and $(13)(24)$, and which is not reducible to a binary group.

5. Let $G = Z_2 \times \cdots \times Z_2$ ($k$ times), where $k \geq 3$ and $Z_2$, is a cyclic group of order 2. Then the mapping $\theta(x^k_1) = (x_3, x_2, x_1, x^k_1)$ is an automorphism of the group $G$ such that $\theta^k = \text{id}$, and by Hosszu theorem the set $G$ with the operation

$$f(y_{10}^{10}) = y_0 \theta y_2 \theta^2 y_3 \theta y_4 \theta y_5 \theta^2 y_6 \theta y_7 \theta^2 y_8 \theta y_9 \theta y_{10}$$

is a non-commutative $10$-group. It is non-reducible and it is $H$-permutable, where $H$ is a subgroup of $S_{11}$ generated by the transpositions $(111)$, $(411)$, $(711)$ and $(1011)$.

In the sequel we shall consistently use the following abbreviations: if $\sigma \in S_{n+1}$, then always $i = \sigma^{-1}(n + 1)$ and $j = \sigma(n + 1)$.

**Theorem 1.** If $(G, f) = \text{der}_{ab}(G, \cdot)$ is a $\sigma$-permutable $n$-group, then $(G, \cdot)$ is commutative, $\theta^{(n-1)} = \text{id}$ and $(G, f)$ is medial (abelian). Moreover, if $\sigma(n + 1) \neq n + 1$, then $b = b^{-1}$ and if $x \cdot y = f(x, y_1^{(n-2)}, y)$ for some $a \in G$, then
\( \theta x = f(a, x, e, \frac{(n-3)}{a}) \) and when \( n \) is odd \( b = a \), or when \( n \) is even \( b = e \), where \( e \) is the unit of \((G, \cdot)\).

**Proof.** Let \((G, f)\) be a \(\sigma\)-permutable \(n\)-group. By Hosszú theorem there exist a group \((G, \cdot)\), and an element \( b \| G \| \) such that

\[
f(x^n) = x_1 \theta x_2 \theta^2 x_3 \ldots \theta^{n-1} x_n b,
\]

where \( \theta b = b \) and \( \theta^{n-1} x = b x b^{-1} \) for all \( x \in G \).

We shall consider two cases: 1. \( \sigma(n+1) \neq n + 1 \) and 2. \( \sigma(n + 1) = n + 1 \).

1. Since \((G, f)\) is \(\sigma\)-permutable, from Definition 2 we get the following identity

\[
x_{\sigma_1} \theta x_{\sigma_2} \ldots \theta^{i-2} x_{\sigma(i-1)} \theta^{i-1}(x_1 \theta x_2 \ldots \theta^{n-1} x_n b) \theta x_{\sigma(i+1)} \ldots
\]

\[
\ldots \theta^{n-1} x_{\sigma n} b = x_{\sigma(n+1)}.
\]

If we take \( r \in N_n \setminus \sigma(n+1) \), then fixing by \( e \) all variables in (1) except \( x \), we get \( \theta^k x, \theta^m x, b^2 = e \) or \( \theta^m x, b \theta^k x, b = e \), where \( k = \sigma^{-1} r - 1, m = i + r - 2 \). In either case, by letting \( x_r = e \) we get \( b^2 = e \) and so \( \theta^k x, \theta^m x, = e \) or

\[
\theta^{n}(x, b) \theta^{k}(x, b) = e, \text{ i.e. } \theta^{k-m} x = x^{-1}.
\]

Hence the mapping \( x \mapsto x^{-1} \) is an automorphism of \((G, \cdot)\) which means that \((G, \cdot)\) is commutative.

2. The \(\sigma\)-permutability of \((G, f)\) implies the identity

\[
x_1 \theta x_2 \theta^2 x_3 \ldots \theta^{n-1} x_n = x_{\sigma_1} \theta x_{\sigma_2} \theta^2 x_{\sigma 3} \ldots \theta^{n-1} x_{\sigma n}.
\]

Fixing in (2) all variables by \( e \), except one, we get that the identity \( \theta^{k-1} x = \theta^{r-1} x \) is valid for every \( k \in N_n \). \( \sigma \) is not the identity permutation, hence there exist \( k, m \in N_n \) such that fixing in (2) all variables by \( e \), except \( x_k, x_m \), we get \( \theta^{k-1} x_k \theta^{m-1} x_m = \theta^{r-1} x_k \theta^{r-1} x_m \), i.e. \((G, \cdot)\) is commutative.

So we have proved that in both cases \((G, \cdot)\) is commutative, which means that \( \theta^{n-1} \) is the identity mapping. Thus \( f(x^n) = f(x_n, x_{n-1}, x_1) \), i.e. \((G, f)\) is medial.

From \( xy = f(x, \frac{(n-2)}{a}, y) \), \( x \theta a \theta^2 a \ldots \theta^{n-2} a b \) it follows that

\[
\theta a \theta^2 \ldots \theta^{n-2} a = b.
\]

Applying \( \theta \) to the preceding equality and multiplying by \( \theta a \), we get \( a = \theta a \).

Hence \( b = a^{n-2} \) and

\[
f(a, x, e, \frac{(n-3)}{a}) = \theta x.
\]

Putting in (1) \( x_{\sigma_1} = a \) and \( x_m = e \) for all \( m \in N_n \setminus \{1\} \) is follows \( a^2 = e \).

So, if \( n \) is odd then \( b = a \), if \( n \) is even then \( b = e \).

In the sequel only \(\sigma\)-permutable \(n\)-groups such that \( \sigma(n+1) \neq n + 1 \) will be considered and we shall assume that this condition is always satisfied without stating it explicitly.
Theorem 2. An $n$-group $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ is $\sigma$-permutable iff $(G, \cdot)$ is a commutative group, $b = b^{-1}$ and $\theta$ is such that $\theta^{n-1} = \theta^{j-2} = \omega$ for all $k \in N_n \setminus \{i\}$.

Proof. Let $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ be a $\sigma$-permutable $n$-group. From Theorem 1 it follows that $(G, \cdot)$ is commutative, $b^2 = e$, $\theta^{-1} = \text{id}$ and the $\sigma$-permutability of $(G, f)$ implies (1). If in (1) we fix by $e$ all variables except $x_j$, we get $\theta^{j-2} = \text{id}$. Fixing in (1) all variables by $e$ except $x_k$, $k \in N_n \setminus \{j\}$, we get $\theta^{n-k} x_k \theta^{k-2} x_k = e$, i.e. $\theta^{n-k-k+i+1} = \omega$. But every $\sigma$-permutable $n$-group is also $\sigma^{-1}$-permutable, applying the preceding to $\sigma^{-1}$ we get that $\theta^{k-k-j+1} = \omega$, for all $k \in N_n \setminus \{i\}$.

The converse part of the theorem follows by a straightforward computation.

Corollary 1. If $(G, f) = \text{der}_{\text{id}, b}(G, \cdot)$ is $\sigma$-permutable, then $(G, \cdot)$ is boolean.

Corollary 2. If $n$ is even and $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ is a $\sigma$-permutable $n$-group, then $(G, \cdot)$ is boolean and there exists a boolean group $(G, \ast)$ such that $b$ is the unit of $(G, \ast)$ and $(G, f) = \text{der}_{\theta, b}(G, \ast)$.

Proof. By Theorem 2 $\omega$ is a power of $\theta$ and since $\theta^{-1} = \text{id}$ it follows $\omega^{-1} = \text{id}$. Since $n$ is even we get $\omega = \text{id}$.

If we define a new operation by $x \ast y = xyb$, then $(G, \ast)$ is a boolean group and $b$ is the unit of this group. Since $b^2 = e$ and $n$ is even we have

$$f(x^n) = x_1 \theta x_2 \theta^2 x_3 \ldots \theta^{n-2} x_{n-1} x_n b = x_1 \theta x_2 b \theta^2 x_3 b \ldots \theta^{n-2} x_{n-1} b x_n b = x_1 \ast \theta x_2 \ast \theta^2 x_3 \ast \ldots \ast \theta^{n-2} x_{n-1} \ast x_n.$$  

It is not difficult to see that $\theta$ is an automorphism of the group $(G, \ast)$ and the corollary is proved.

Remark. If $(G, \cdot)$ is a boolean group, $b$ fixed element from $G$, $f(x^n) = x_1 x_2 \ldots x_n b$, then $(G, f)$ is a $\sigma$-permutable $n$-group for every $\sigma$. Since every finite boolean group is of order $2^k$, $k$ nonnegative integer, and for every such $k$ there exists a boolean group of order $2^k$, we get that for every even $n$ there exists a nontrivial $\sigma$-permutable $n$-group of order $g$ iff $g = 2^k$, $k \in N$. (An $n$-group $(G, f)$ is called nontrivial iff $|G| > 1$.)

Definition 3. If $\sigma \in S_{n+1}$, then

$$d(\sigma) = \gcd\{n - 1, i + j - 2, 2(\sigma k - k - j + 1)\}_{k \in N_n \setminus \{i\}},$$

$$d'(\sigma) = \gcd\{n - 1, i + j - 2, \sigma k - k - j + 1\}_{k \in N_n \setminus \{i\}}.$$

Lemma. If $(G, f) = \text{der}_{\theta, b}(G, \cdot)$ is a $\sigma$-permutable $n$-group, then $\theta^{d(\sigma)} = \text{id}$ and $\theta^{d'(\sigma)} = \omega$.

Proof. If we express $d(\sigma)$ and $d'(\sigma)$ as linear combinations of elements of the corresponding sets, by Theorem 2 we get that $\theta^{d(\sigma)} = \text{id}$ and $\theta^{d'(\sigma)} = \omega$.

Theorem 3. Let $(G, f)$ be a $\sigma$-permutable $n$-group.
(i) If \( d(\sigma) = 1 \), then \( (G, f) \) is \( \theta \)-derived from a commutative group.

(ii) If \( d'(\sigma) = 1 \), then \( (G, f) \) is \( \langle \theta, b \rangle \)-derived from a commutative group, where \( \theta = \omega \).

(iii) If \( d(\sigma) = d'(\sigma) \), then \( (G, f) \) is \( \langle \theta, b \rangle \)-derived from a boolean group.

(iv) If \( d(\sigma) = 1 \) and \( n \) is even, then \( (G, f) \) is a long product of a boolean group.

Proof. Follows from Theorem 2, Corollary 2 and the Lemma.

Corollary 3. Let \( (G, f) \) be a \( \sigma \)-prermutable \( n \)-semigroup. If \( i + j - 2 \) and \( n - 1 \) are relatively prime, then \( (G, f) \) is an \( n \)-group which is \( \theta \)-derived from a boolean group.

Proof. Since \( i + j - 2 \) and \( n - 1 \) are relatively prime, then at least one of the integers \( i, j \) belongs to the set \( \{2, \ldots, n-1\} \). Hence by Proposition 5 from [7], this \( n \)-semigroup is an \( n \)-group, which is by Theorem 3 \( \theta \)-derived from a boolean group.

The following theorem gives a more convenient expression for \( d(\sigma) \) and \( d'(\sigma) \).

Theorem 4. If \( \sigma \in S_{n+1} \), then
\[
\begin{align*}
    d'(\sigma) &= \gcd\{n - 1, i - 1, j - 1, \sigma k - k\}_{k \in N_n \setminus \{i\}}, \\
    d(\sigma) &= \gcd\{n - 1, i + j - 2, 2d'(\sigma)\}.
\end{align*}
\]

Proof. To prove the first equality it suffices to prove that \( j \equiv 1 \pmod{d'(\sigma)} \).

Let \( c_m = m d'(\sigma) + 1 \), \( m = 0, 1, \ldots, s \), where \( s = (n - 1)/d'(\sigma) \). Then \( c_m \leq n \) for all \( m \). If for some \( m \) \( c_m = i \), then \( m d'(\sigma) = i - 1 \), i.e. \( i \equiv 1 \pmod{d'(\sigma)} \), which implies that \( j \equiv 1 \pmod{d'(\sigma)} \). If for every \( m = 0, 1, \ldots, s \) \( c_m \neq i \), then since \( \sigma k - k - j - 1 = 0 \pmod{d'(\sigma)} \) for every \( k \neq i, n + 1 \), we get that for all \( m = 0, 1, \ldots, s \), \( \sigma c_m = j \pmod{d'(\sigma)} \). If \( a = \min_m \sigma c_m \), then for all \( m a \leq \sigma c_m \leq n \), hence for all \( m \sigma c_m - a \leq n - a \). We have obtained that each of \( s + 1 \) different nonnegative integers \( \sigma c_m - a \), which are all congruent modulo \( d'(\sigma) \), is not greater than \( n - a \), hence \( n - a \geq s d'(\sigma) = n - 1 \). So \( a = 1 \), i.e. there exists \( m \) such that \( \sigma c_m = 1 \). Hence \( j \equiv 1 \pmod{d'(\sigma)} \).

The second equality follows directly from the definition of \( d'(\sigma) \) and \( d(\sigma) \).

Acknowledgement. The authors are grateful to S. Krstić for helpful suggestions which improved the presentation of this paper.

REFERENCES


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