UNIONS AND INTERSECTIONS OF ISOMORPHIC IMAGES
OF NONSTANDARD MODELS OF ARITHMETIC

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Abstract. We consider those initial segments of a nonstandard model \( M \) of Peano arithmetic (abbreviated by \( P \)) which can be obtained as unions or intersections of initial segments of \( M \) isomorphic to \( M \). For any consistent theory \( T \supseteq P \) we find models of \( T \) having collections of initial segments densely ordered by inclusion so that for any segment \( I \) from such collection and any \( k \in \omega \) the family \( A^M_k = \{ M \mid M \subseteq M, M \prec_\Sigma_k M, M \equiv M \} \) can be partitioned into two disjoint parts \( A_1 \), and \( A_2 \) satisfying \( I = \bigcup A_1 = \bigcap A_2 \) i.e. \( I \) is a "point of accumulation" for all families \( A^M_k \). We investigate the order type of such collections of segments in the case of recursively saturated models of \( P \).

We denote nonstandard models of \( P \) by \( M, N \) and \( \mathfrak{A} \) and their domains by \( |M|, |N| \) and \( |\mathfrak{A}| \), respectively; \( L_p \) denotes the language of \( P, N \) denotes the structure of natural numbers and \( \omega \) stands for its domain. If \( M \) is a model of \( P \) and \( \mathfrak{A} \) a structure for \( L_p \) such that \( \mathfrak{A} \subseteq M \), then by \( M \) we denote the smallest initial segment of \( M \) containing \( \mathfrak{A} \). \( M \subseteq \mathfrak{A} \) means that \( M \) is an end extension (elementary end extension) of \( \mathfrak{A} \), while \( M \prec_\Sigma_k \mathfrak{A} \) means that for all \( \Sigma_k \) formulas \( \varphi \) and all \( a_1, \ldots, a_n \in |M| \), \( M \models \varphi[a_1, \ldots, a_n] \) holds iff \( \mathfrak{A} \models \varphi[a_1, \ldots, a_n] \) holds. For any \( M \models P \) and \( a \in M \), let \( I_a = \{ b \in |M| \mid b < a \} \). We use the consequence of Matijasevič’s theorem asserting that for any models \( M, N \) of \( P, M \subseteq N \) implies \( M \prec_\Sigma_k N \). Thus, \( A^0_0 = \{ M \mid M \subseteq N, M \equiv M \} \). If \( \Gamma \) is a set of sentences of \( L_p \) then \( \text{Th}_P(\mathfrak{A}) \) denotes the set of all sentences from \( \Gamma \), which are true in \( M \). We use the fact that for any models \( M, N \) of \( P, M \subseteq N \) implies \( \text{SSy}(M) = \text{SSy}(N) \). The following hierarchical refinement of Gaifman’s Splitting Theorem is Theorem 1.2 from [3].

Proposition 0.1. Let \( M \) and \( \mathfrak{A} \) be models of \( P \) and \( M \prec_\Sigma_k \mathfrak{A} \). Then \( \mathfrak{A} \prec_\Sigma_k \mathfrak{A} \mathfrak{A} \mathfrak{A} \mathfrak{A} \) and \( \mathfrak{A} \prec_\Sigma_k M \).

This paper is a revised part of author’s master thesis. I would like to express my gratitude to Žarko Mijajlović, my advisor, for many helpful discussions on this subject.

AMS Subject Classification (1980): Primary 03H15
The following proposition is a hierarchical generalization of Theorem 2.4 (ii) from [5], and can be proved in the same way.

**Proposition 0.2.** The following are equivalent: (i) for arbitrary \( a \in \mathcal{M} \), \( \mathcal{M} \) is isomorphic to an initial segment of \( \mathcal{M} \) \( \Sigma_k \)-elementarily embedded in \( \mathcal{M} \) which contains \( a \). (ii) \( \text{Th}_{\Pi_{k+2}} \mathcal{M} \subseteq \text{Th}_{\Pi_{k+1}}(\mathcal{M}) \) and \( \text{SSy}(\mathcal{M}) = \text{SSy}(\mathcal{M}) \).

Let us consider those initial segments of a model \( \mathcal{M} \) of \( P \) which can be obtained as unions or intersections of initial segments of \( \mathcal{M} \) isomorphic to \( \mathcal{M} \). As it was shown in [3], (see also [1]) \( \bigcap A^\mathcal{M}_k \) is the smallest initial segment of \( \mathcal{M} \) containing all \( \Sigma_{k+1} \)-definable points of \( \mathcal{M} \) on the other hand \( \bigcup A^\mathcal{M}_k = \mathcal{M} \).

**Lemma 1.1.** Let \( I_1 \) and \( I_2 \) be initial segments of a nonstandard model \( \mathcal{M} \) of \( P \) such that \( \omega \subset I_2 \subset I_1 \). Then \( I_1 \) contains a model \( \mathcal{N} \) of \( P \) such that \( \mathcal{N} \subset \mathcal{M} \), \( \mathcal{N} \prec \mathcal{M} \), \( \mathcal{M} \models \mathcal{N} \models I_2 \) iff it contains a model \( \mathcal{R} \models \mathcal{N} \) such that \( \mathcal{R} \prec \mathcal{M} \), \( \text{Th}_{\Pi_{k+2}}(\mathcal{R}) \supseteq \text{Th}_{\Pi_{k+2}}(\mathcal{M}) \) and \( \mathcal{R} \not\subset I_2 \).

**Proof.** Suppose that there is a model \( \mathcal{R} \) satisfying the conditions from Lemma 1.1, and let \( a \in [\mathcal{R}] \setminus I_2 \). Proposition 0.1 implies \( \mathcal{R} \prec \mathcal{M} \), \( \mathcal{R} \subseteq \mathcal{M} \); thus \( \text{Th}_{\Pi_{k+2}}(\mathcal{R}) \supseteq \text{Th}_{\Pi_{k+2}}(\mathcal{M}) \) and \( \text{SSy}(\mathcal{R}) = \text{SSy}(\mathcal{M}) \) holds. According to Proposition 0.2, model \( \mathcal{M} \) is isomorphic to a submodel \( \mathcal{R} \) of \( \mathcal{M} \) such that \( \mathcal{R} \subseteq \mathcal{M} \), \( \mathcal{R} \prec \mathcal{M} \) and \( a \in [\mathcal{R}] \). Since \( \mathcal{R} \prec \mathcal{M} \), \( \mathcal{R} \subseteq \mathcal{M} \), and \( a \in [\mathcal{M}] \setminus I_2 \), we conclude that \( \mathcal{R} \) satisfies the conditions from Lemma 1.1. The converse is obvious.

**Corollary 1.2.** Let \( I \) be an initial segment of a nonstandard model \( \mathcal{M} \) of \( P \) and \( I \neq \omega \). Then:

(i) There is a subfamily \( \mathcal{A} \subseteq A^\mathcal{M}_k \) such that \( I = \bigcup \mathcal{A} \) iff for all \( a \in I \) there is a model \( \mathcal{R}_a \models \mathcal{M} \) such that \( \mathcal{R}_a \prec \mathcal{M} \) and \( \text{Th}_{\Pi_{k+2}}(\mathcal{R}_a) \supseteq \text{Th}_{\Pi_{k+2}}(\mathcal{M}) \).

(ii) There is a subfamily \( \mathcal{A} \subseteq A^\mathcal{M}_k \) such that \( I = \bigcap \mathcal{A} \) iff \( I \models \mathcal{M} \) or for all \( a \in [\mathcal{M}] \setminus I \) there is a model \( \mathcal{R}_a \subseteq \mathcal{M} \) such that \( \mathcal{R}_a \subseteq I \), \( \mathcal{R}_a \models P \), \( \mathcal{R}_a \prec \mathcal{M} \) and \( \text{Th}_{\Pi_{k+2}}(\mathcal{R}_a) \supseteq \text{Th}_{\Pi_{k+2}}(\mathcal{M}) \).

**Proof.** (i) We apply Proposition 2.1. to all pairs \( I, I_a, a \in I \).

(ii) \( I \models \mathcal{M} \), let \( \mathcal{A} = \{I\} \); otherwise, we apply Proposition 2.1. to all pairs \( I, I_a, a \in [\mathcal{M}] \setminus I \).

The following lemma, which is useful for applications of Corollary 1.2, can be proved easily.

**Lemma 1.3.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be arbitrary models for the same language. Then \( \mathcal{M} \prec \mathcal{N} \) implies \( \text{Th}_{\Pi_{k+2}}(\mathcal{M}) \subseteq \text{Th}_{\Pi_{k+2}}(\mathcal{N}) \). □

**Corollary 1.4.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be nonstandard models of \( P \) and \( \mathcal{M} \prec \mathcal{M} \), then there is a subfamily \( \mathcal{A} \subseteq A^\mathcal{M}_k \) such that \( \mathcal{M} = \bigcup \mathcal{A} \).

**Proof.** Immediately from Proposition 0.1, Corollary 1.2 (i) and Lemma 1.3.

As a consequence, we get the following proposition.

**Proposition 1.5.** Let \( I \) be an initial segment of a nonstandard model \( \mathcal{M} \) of \( P \) and \( B \) a family of initial segments of \( \mathcal{M} \) such that for any \( \mathcal{N} \) from \( B \), \( \mathcal{N} \models P \) and \( \mathcal{N} \prec \mathcal{M} \) holds. Then:
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(i) If $I = \bigcup B$, then there is a subfamily $A \subseteq \mathcal{A}_k^\mathcal{M}$ such that $I = \bigcup A$;
(ii) If $I = \bigcap B$ and $I \notin B$, then there is a family $A \subseteq \mathcal{A}_k$ such that $I = \bigcap A_k^\mathcal{M}$.

**Proposition 1.6.** Let $\mathcal{M}$ and $\mathcal{N}$ be nonstandard models of $P$, such that $\mathcal{N} \subseteq \mathcal{M}$ and let $\mathcal{N}_1, \mathcal{N}_2, \ldots$, be a strictly decreasing $\Sigma_{k+1}$-elementary chain of initial segments, i.e. $\mathcal{N}_i \subseteq \mathcal{M}$, $\mathcal{N}_1 \supset \mathcal{N}_2 \supset \mathcal{N}_3 \supset \mathcal{N}_{k+1} \ldots$, such that $\bigcap_{i \in \omega} \mathcal{N}_i = \mathcal{M}$. Then, the family $A_{\mathcal{M}}^{\mathcal{M}}$ can be divided into two disjoint subfamilies $A_{\mathcal{M}}^1$ and $A_{\mathcal{M}}^2$ such that $\mathcal{M} = \bigcup A_{\mathcal{M}}^1 = \bigcap A_{\mathcal{M}}^2$.

**Proof.** Since $P$ has definable Skolem functions, the hierarchal refinement of the Tarski-Vaught Theorem implies $\bigcap_{i \in \omega} \mathcal{N}_i = \mathcal{M} \supset \mathcal{N}_{k+1}$. Thus, letting $A_{\mathcal{M}}^1 = \{R | R \in A_{\mathcal{M}}^1, R \subseteq \mathcal{M}\}$ and $A_{\mathcal{M}}^2 = \{R | R \in A_{\mathcal{M}}^2, R \supset \mathcal{M}\}$, we get from Corollary 1.4 and Proposition 1.5 $\mathcal{M} = \bigcup A_{\mathcal{M}}^1 = \bigcap A_{\mathcal{M}}^2$.

**Corollary 1.7.** Let $\mathcal{M}$ be a nonstandard model of $P$ and $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \mathcal{N}_3 \ldots$, a strictly decreasing elementary chain of initial segments of $\mathcal{M}$, such that $\bigcap_{i \in \omega} \mathcal{N}_i = \mathcal{M}$. Then for all $n \in \omega$ the family $A_{\mathcal{M}}^{\mathcal{M}}$ can be divided into two disjoint subfamilies $A_{\mathcal{M}}^1$, $A_{\mathcal{M}}^2$ such that $\mathcal{M} = \bigcup A_{\mathcal{M}}^1 = \bigcap A_{\mathcal{M}}^2$.

**Proof.** Since $P$ has definable Skolem functions, $\bigcap_{i \in \omega} \mathcal{N}_i \subset \mathcal{M}$ holds, and consequently, $\bigcap_{i \in \omega} \mathcal{N}_i \models P$. Since $\bigcap_{i \in \omega} \mathcal{N}_i \models P$, we can apply Proposition 1.6.

We now look for models having such chains.

**Lemma 1.8.** For any consistent extension $T$ of $P$ there is a countable model $\mathcal{M}$ of $T$ having a family of initial segments densely ordered by inclusion such that any member of this family is an intersection of a strictly decreasing elementary chain of initial segments of $\mathcal{M}$.

**Proof.** Let $\ll$ be any recursive dense ordering on $\omega$, and $U(x,y,z), V(x,y)$ two new predicate symbols. We consider the theory $T' = T \cup A_1 \cup A_2 \cup A_3 \cup A_4$ of the language $L = L_\ll \cup \{U, V\}$, where $A_1 \cdots A_4$ are defined as follows:

- $A_1 = \{\forall x \exists y (V(x,n) \land y < x \rightarrow V(y,n)) ; n \in \omega, \text{ and the same for } U(x,m,n)\};$
- $A_2 = \{\forall x_1 \cdots x_k V(x_1,n) \land \cdots \land V(x_k,n) \land \exists \varphi(x_1,\ldots,x_k) \rightarrow \exists \varphi(V(x,n) \land \varphi(x_1,\ldots,x_k)) \} \text{ for all } k,n \in \omega, \text{ all formulas } \varphi \text{ of } L_\ll, \text{ and the same for } U(x,m,n), m,n \in \omega\};$
- $A_3 = \{\forall z((V(x,n) \rightarrow U(x,n,m) \land U(x,n,m+1) \rightarrow U(x,n,m))) \land \exists x U(x,n,m) \land
- U(x,n,m+1), n,m \in \omega\};$
- $A_4 = \{\forall x U(x,n,1) \rightarrow V(x,m)) ; \text{ for all } m,n \in \omega, \text{ such that } n \gg m\}.$

Theory $T'$ is consistent because any finite subtheory of $T'$ is realized in a model $\mathcal{M}$ obtained as a finite chain of elementary end extensions of any model $\mathcal{M}$ of $T$. Any countable model of $T'$, with the family of initial segments which are interpretations in this model of $U(x,n,m)$ and $\bigcap_{i \in \omega} U(x,n,m), n,m \in \omega$, obviously satisfies the conditions from Lemma 1.8.
Since $T'$ is a recursive theory, the same argument shows that any resplendent countable model of $T$ can be expanded to a model of $T'$ because $T'$ is consistent with $\text{Th}(\mathfrak{M})$ for any $\mathfrak{M}, \mathfrak{M} \models T$.

From Lemma 1.8 and Corollary 1.7 the following proposition immediately follows.

**Proposition 1.9.** For any consistent extension $T$ of $P$ there is a countable model $\mathfrak{M}$ of $T$ having a collection of initial segments densely ordered by inclusion, so that, for any segment $I$ from the collection and any $k \in \omega$ the family $A^\mathfrak{M}_k$ can be divided into two disjoint parts $A^\mathfrak{M}_1$ and $A^\mathfrak{M}_2$ so that $I = \bigcup A^\mathfrak{M}_1 = \bigcap A^\mathfrak{M}_2$.

Using a Kotlarski's result [2] we can prove that in the case of recursively saturated countable models of $P$, we can find such a collection of initial segments of the power $2^\omega$. Namely, in that case, the set $Y = \{ \mathfrak{M} | \mathfrak{M} \prec \mathfrak{M} \}$ is of the order type of Cantor set $2^\omega$ with its lexicographical ordering, and any $\mathfrak{M}$ from $Y$ is isomorphic to $\mathfrak{M}$. We call a pair $(Y_1, Y_2)$ a cut in $Y$ iff $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 = Y$ and for all $I_1, I_2$ from $Y$, $I_1 \subseteq Y_1$ and $I_2 \subseteq Y_2$ implies $I_1 \subseteq I_2$. Since for two different cuts $(I_1, I_2)$ and $(I'_1, I'_2)$ the sets $\bigcap I_2 \setminus \bigcap I_1$ and $\bigcap I'_2 \setminus I'_1$ are disjoint and since $\mathfrak{M}$ is countable, there are only countably many cuts $(Y_1, Y_2)$ such that $\mathfrak{M} \setminus (\bigcap Y_2 \setminus \bigcup Y_1) \neq \emptyset$. It is easy to see that for any cut $(Y_1, Y_2)$ such that $\mathfrak{M} \setminus (\bigcap Y_2 \setminus \bigcup Y_1) = \emptyset$, the segment $I = \bigcap Y_2 = \bigcup Y_1$ satisfies the conditions from Proposition 1.9, and that the family of such segments is of power $2^\omega$ and is densely ordered by inclusion.

**References**


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(Received 23 09 1985)