ISOTROPIC SECTIONS AND CURVATURE PROPERTIES
OF HYPERBOLIC KAHLERIAN MANIFOLDS

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Abstract. In [4,2] curvature properties of pseudo-Riemannian manifolds were investigated with respect to isotropic vectors and isotropic sections. Further, analogous properties have been treated in [1] for Kaehlerian manifolds with an indefinite metric. In this paper we consider hyperbolic Kaehlerian manifolds, and study how the curvature properties of one- and two-dimensional isotropic tangential spaces determine the curvature properties of the manifold.

1. Preliminaries

Let $M$ be a $2n$-dimensional hyperbolic Kaehlerian manifold, i.e. $M$ is a Riemannian manifold with an indefinite metric $g$ and an almost product structure satisfying the conditions:

$$p^2 = id, \quad g(PX, PY) = -g(X, Y)$$

for arbitrary vector fields $X, Y$ and $\nabla P = 0$. The metric $g$ is of signature $(n, n)$ and $P$ trace $= 0$.

$R, \rho$ and $T$ will stand for the curvature tensor, the Ricci tensor and the scalar curvature respectively. The curvature tensor $R$ satisfies the condition

$$R(X, Y, Z, U) = -R(X, Y, PU, PZ)$$

for arbitrary vectors in the tangential space $T, M, p$ in $M$. The Ricci tensor $\rho$ has the property

$$\rho(X, Y) = -\rho(PX, PY); \quad X, Y \in T_p M.$$

Further, we consider the tensors:

$$\varphi(Y, Z, U) = g(Y, Z)\rho(X, U) - g(X, Z)\rho(Y, U) + g(X, U)\rho(Y, Z) - g(Y, U)\rho(X, Z);$$

$$\psi(X, Y, Z, U) = -g(Y, PZ)\rho(X, PU) + g(X, PZ)\rho(Y, PU) - g(X, PU)\rho(Y, PZ) + g(Y, PU)\rho(X, PZ) + 2g(X, PY)\rho(Z, PU) + 2g(Z, PU)\rho(X, PY);$$

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\[ \pi_1(X, Y, Z, U) = g(Y, z)g(X, U) - g(X, Z)g(Y, U); \]
\[ \pi_2(X, Y, Z, U) = -g(Y, PZ)g(X, PU) + g(X, PZ)g(Y, PU) \\
+ 2g(X, PY)g(Z, PU). \]

Let \( \alpha \) be a section (2-plane) in \( T_pM \). The section \( \alpha \) is said to be nondegenerate, weakly isotropic, strongly isotropic, if the rank of the restriction of the metric \( g \) on \( \alpha \) is 2, 1, 0 respectively. With respect to the structure \( P \) a section \( \alpha \) is said to be holomorphic (totally real) if \( P\alpha = \alpha(P\alpha \neq \alpha, P\alpha \perp \alpha) \).

We shall use two kinds of special bases of \( T_pM \):

1) An adapted basis \( \{a_1, \ldots, a_n; x_1, \ldots, x_n\} \) is characterized with the property that the matrices \( g \) and \( P \) with respect to such a basis are

\[ g = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad P = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \]

where \( I_n \) is the unit matrix.

2) A separate basis \( \{\eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_n\} \) consists of eigen vectors of \( P \), so that \( \{\xi_1, \ldots, \xi_n\} \) form a basis of the eigen space \( V^+ \), corresponding to the eigen value +1 of \( P \). The vectors \( \{\eta_1, \ldots, \eta_n\} \) form a basis of the eigen space \( V^- \). With respect to a separate basis the matrices \( g \) and \( P \) are

\[ P = \begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix}, \quad g = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}. \]

The following equation is fulfilled \( T_pM = V^+ \oplus V^- \) (nonorthogonal). The second condition of (1) implies that every eigen vector \( \xi \) of \( P \) is isotropic, i.e. \( g(\xi, \xi) = 0 \). Given an adapted basis, one obtains a separate basis by the formulae:

\[ \xi_i = (a_i + x_i)/\sqrt{2}, \quad \eta_i = (a_i - x_i)/\sqrt{2}; \quad i = 1, \ldots, n. \]

These formulae give also an inverse transition.

In what follows, \( x, y, z \) will denote unit space-like vectors, i.e. \( g(x, x) = 1 \); \( a, b, c \) will denote unit time-like vectors, i.e. \( g(a, a) = -1 \); \( u, r, v \) will denote isotropic vectors which are not eigen vectors, i.e. \( g(u, u) = 0 \), \( Pu \neq \pm u \); \( \xi, \eta, \zeta \) will denote eigenvectors of \( P \), i.e. \( P\xi = \pm\xi \).

Taking into account both structures, we find the following types of holomorphic and totally real sections in \( T_pM \):

A. **Holomorphic sections.**

A1. **Nondegenerate holomorphic sections.** These sections have an orthonormal basis of type \( \{x, Px\} \) or \( \{a, Pa\} \) and a basis of type \( \{\xi, \eta\} \), \( P\xi = \xi, P\eta = -\eta, g(\xi, \eta) \neq 0 \).
A2. **Strongly isotropic holomorphic sections of hybrid type.** These sections exist by \( n \geq 2 \), and have a basis of type \( \{ u, Pu \} \). Another kind of useful bases for such sections are \( \{ \xi, \eta \}, P \xi = \xi, P \eta = -\eta, g(\xi, \eta) \neq 0 \).

A3. **Strongly isotropic holomorphic sections of pure type.** By \( n \geq 2 \) these sections are the sections in \( V^+ \) and in \( V^- \).

B. **Totally real sections.**

B1. **Nondegenerate totally real sections of pure type.** These sections exist by \( n \geq 2 \) and have an orthonormal basis of type \( \{ x, y \} \), \( g(x, Py) = 0 \) or \( \{ a, b \} \), \( g(a, Pb) = 0 \).

B2. **Nondegenerate totally real sections of hybrid type.** These sections exist by \( n \geq 2 \) and have an orthonormal basis of type \( \{ x, a \} \), \( g(x, Pa) = 0 \).

B3. **Weakly isotropic totally real sections of the I type.** These sections exist by \( n \geq 2 \) and have a basis of type \( \{ x, \xi \} \), \( g(x, \xi) = 0 \); \( \{ a, \xi \} ; g(a, \xi) = 0 \).

B4. **Weakly isotropic totally real sections of the II type.** These sections exist by \( n \geq 3 \) and have a basis of type \( \{ x, u \} \), \( g(x, u) = g(x, Pu) = 0 \); \( \{ a, u \} \), \( g(a, u) = g(a, Pu) = 0 \).

B5. **Strongly isotropic totally real sections of the I type.** These sections exist by \( n \geq 3 \) and have a basis of type \( \{ \xi, u \} \), \( g(\xi, u) = 0 \).

B6. **Strongly isotropic totally real sections of the II type.** These sections exist by \( n \geq 4 \) and have a basis of type \( \{ u, v \} \), \( g(u, v) = g(u, Pv) = 0 \).

2. **Holomorphic curvatures**

If \( \alpha \) is a nondegenerate section in \( T_p M \) with a basis \( \{ X, Y \} \), its curvature is given by

\[
K(\alpha, p) = K(X, Y) = R(X, Y, Y, X) / \pi_1(X, Y, Y, X).
\]

For an isotropic section such a curvature cannot be defined. If \( \{ X, Y \} \) forms a basis of an isotropic section \( \alpha \) and

\[
R(X, Y, Y, X) = 0,
\]

this is a geometric property of the section \( \alpha \).

Now, let \( \alpha \) be a nondegenerate holomorphic section. Curvatures of such sections will be called holomorphic sectional curvatures. As for Kaehlerian manifolds, we have.

**Lemma 1.** Let \( T \) be a tensor of type \( (0, 4) \) over \( T_p M \) with the properties:

1) \( T(X, Y, Z, U) = -T(Y, X, Z, U) \);
2) \( T(X, Y, Z, U) = -T(X, Y, U, Z) \);
3) \( T(X, Y, Z, U) + T(Y, Z, X, U) + T(Z, X, Y, U) = 0 \);
4) \( T(X, Y, Z, U) = -T(X, Y, ZU, PU) \).

(5)
If $T$ has zero holomorphic sectional curvatures, then $T = 0$.

Proof. From the condition of the lemma it follows that

$$
T(X, PX, PX, X) = 0
$$

for an arbitrary nonisotropic vector $X$ in $T_p M$. Let $Y$ be an arbitrary isotropic vector. Then $Y = \lambda(x + a)$, $\lambda$ - real number, $g(x, x) = -g(a, a) = 1$, $g(x, a) = 0$. Substituting the vector $x + ta$, $|t| < 1$ in (6), we obtain a polynomial identity

$$
f(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 = 0.
$$

for $|t| < 1$. This implies $c_0 = \cdots = c_4 = 0$ and in particular $f(1) = 0$, i.e. $T(Y, PY, PY, Y) = 0$. Thus, (6) is fulfilled for an arbitrary vector. Now, as in the case of a Kaehlerian manifold [5], it follows that $T = 0$.

A hyperbolic Kaehlerian manifold is said to be of constant holomorphic sectional curvature $\mu$ if $K(\alpha, \nu) = \mu$, does not depend on the choice of the nondegenerate holomorphic section $\alpha$ in $T_p M$, $\nu$ in $M$. The curvature identity characterizing these manifolds has been found in [7] with respect to local coordinates. We shall derive this identity from Lemma 1.

**Proposition.** [7] A hyperbolic Kaehlerian manifold is of constant holomorphic sectional curvature $\mu$ if and only if

$$
R = \mu(\pi_1 + \pi_2)/4, \quad \mu = \tau/n(n + 1).
$$

Proof. The proposition follows by applying Lemma 1 to the tensor $T = R - (\mu/4)(\pi_1 + \pi_2)$.

The equality (7) implies $\rho = \mu((n + 1)/2)g$, i.e. $M$ is Einsteinian. Hence, if $M$ is connected, $\mu$ is a constant on $M$.

**Remark.** In [7], hyperbolic Kaehlerian manifolds of constant holomorphic sectional curvature have been called manifolds of almost constant curvature.

Let $\mathcal{K}$ be the vector space of the tensors over $T_p M$ having the properties (5). For $T$ in $\mathcal{K}$, $\rho(T)$ and $\tau(T)$ will stand for the Ricci tensor and the scalar curvature with respect to $T$. The metric $g$ induces in a natural way an inner product in $\mathcal{K}$. Using the same method as in [6, 8], we obtain the following decomposition theorem for $\mathcal{K}$.

**Theorem 1.** The following decomposition of $\mathcal{K}$ is orthogonal:

$$
\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_w,
$$

where

1) $\mathcal{K}_1 = \{T \in \mathcal{K}| T = \mu(\pi_1 + \pi_2)/4\}$;
2) $\mathcal{K}_w = \{T \in \mathcal{K}| \rho(T) = 0\}$;
3) $\mathcal{K}_2$ is the orthogonal complement $\mathcal{K}_w$ in $\mathcal{K}_1^\perp$;
4) $\mathcal{K}_1 \oplus \{T \in \mathcal{K}| \rho(T) = \tau(T)g/2n\}$;
5) $\mathcal{K}_2 \oplus \{T \in \mathcal{K}| \tau(T) = 0\}$.
The curvature tensor $R$ of a hyperbolic Kaehlerian manifold has the properties (5). The component $B(R)$ of $R$ in $K_w$ (Weyl component) is said to be the Bochner curvature tensor. It is easy to check that this component is

$$B(R) = R - \frac{1}{2n(n + 2)}(\varphi + \psi) + \frac{\tau}{4(n + 1)(n + 2)}(\pi_1 + \pi_2).$$

**Corollary 1.** A hyperbolic Kaehlerian manifold $M(2n \geq 4)$ is of constant holomorphic sectional curvature if and only if $M$ is Einsteinian and $B(R) = 0$.

The Ricci curvature of a direction determined by a nonisotropic vector $X$ is given by $\rho(X) = \rho(X, X)/g(X, X)$. Applying Lemma 1 we obtain

**Corollary 2.** Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. $M$ has a vanishing Bochner curvature tensor if and only if

$$K(X, PX) - \frac{4}{n + 2}\rho(X) + \frac{\tau}{(n + 1)(n + 2)} = 0$$

for an arbitrary nonisotropic vector $X$ in $T_p M$, $p$ in $M$.

**Theorem 2.** Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent.

1) $R(u, Pu, Pu, u) = 0$ for arbitrary $u$ in $T_p M$, i.e. the strongly isotropic holomorphic sections of hybrid type have the property (4);

2) $B(R) = 0$.

**Proof.** Let $\{a_1, \ldots, a_n; x_1, \ldots, x_n\}$ be an adapted basis for $T_p M$. From the condition 1) of the theorem we have $R(a_i + x_j, a_j + x_i, a_j + x_i, a_i + x_j) = 0, i \neq j$. These equalities imply

$$6K(a_i, x_j) + 2K(a_i, a_j) = K(a_i, x_i) + K(a_j, x_j); \quad i \neq j.$$  

Using $u = a_i + x_i + a_j - x_j, i \neq j$ and the condition 1) we obtain

$$K(a_i, a_j) = K(a_i, x_j), \quad i \neq j.$$  

The equalities (10) and (11) give

$$K(x_i, Px_i) - \frac{4}{n + 2}\rho(x_i) + \frac{\tau}{(n + 1)(n + 2)} = 0,$$

which is equivalent to (9) and hence $B(R) = 0$. The inverse is a simple verification.

3. **Totally real sections**

The curvatures of nondegenerate totally real sections are said to be totally real sectional curvatures.
Lemma 2. Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1) $R(x,a,a,x) = 0$ whenever $a \perp x$, $Px$, i.e. the totally real sectional curvatures of hybrid type are zero;

2) $R(x,y,y,x) = 0$ whenever $x \perp y$, $Py$, i.e. the totally real sectional curvatures of pure type are zero;

3) $R = 0$.

Proof. Let $\{x, y, a\}$ be an orthogonal triple spanning a 3-dimensional totally real space. For the pair $\{x, a' = (a + ty)/\sqrt{1 - t^2}\}$, $|t| < 1$ we have $a' \perp x$, $Px$. Substituting this pair into the condition 1) of the lemma, we get $R(x, a + ty, a + ty, x) = 0$. The corresponding polynomial identity gives $R(x, y, y, x) = 0$, i.e. 1) implies 2). The inverse follows in a similar way.

Now, let $\{x, y, z\}$ be orthogonal and span a 3-dimensional totally real space. Applying 1) to the vectors $(x - y)/\sqrt{2}$, $(Px + Py)/\sqrt{2}$ and using 2) we find $K(x, Pz) + K(y, Py) = 0$. Analogously, $K(y, Py) + K(z, Pz) = K(z, Pz) = 0$. Therefore $K(x, Pz) = 0$ and Lemma 1) implies $R = 0$.

The following theorem has an easy proof using Lemma 2.

Theorem 3. Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1) $M$ is of constant totally real sectional curvature of hybrid type, i.e. $K(a, x) = v$, whenever $a \perp x$, $Px$;

2) $M$ is of constant totally real sectional curvature of pure type, i.e. $K(x, y) = v(K(a, b) = v)$, whenever $x \perp y$, $Py$ (a $\perp b$, $Pb$);

3) $M$ is of constant holomorphic sectional curvature $\mu = 4v$.

Theorem 4. Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1) $R(x, \xi, \xi, x) = 0$ whenever $\{x, \xi\}$ spans a weakly isotropic totally real section of $I$ type;

2) $B(R) = 0$.

Proof. Let the pair $\{x, y\}$ be orthogonal and span a totally real section. Applying the condition 1) of the theorem to the pair $\{x, \xi = y + Py\}$ we obtain

$$R(x, y, y, x) + R(x, Py, Py, x) = 0. \quad (12)$$

Now, we substitute the pair $\{x, y\}$ in (12) by $\{(x + y)/\sqrt{2}, (x - y)/\sqrt{2}\}$ and linearizing we find

$$8K(x, y) = K(x, Pz) + K(y, Py). \quad (13)$$

Further, as in the proof of Theorem 2, (12) and (13) give $B(R) = 0$. 

The inverse follows immediately by taking into account that \( \rho(\xi, \xi) = 0 \).

**Theorem 5.** Let \( M(2n \geq 6) \) be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1. \( R(x, u, u, x) = 0 \), whenever \( \{x, u\} \) spans a weakly isotropic totally real section of the II type;
2. \( M \) is of constant holomorphic sectional curvature.

**Proof.** Let \( \{a_1, \ldots, a_n; x_1, \ldots, x_n\} \) be an adapted basis for \( T_pM \). Applying the condition 1 of the theorem to the pairs \( \{x_i, x_j + a_k\} \) \((i, j, k \text{ - different})\), we find \( K(x_i, x_j) = \text{const}; \ i \neq j \). This is equivalent to the condition 1 of Theorem 3. Hence, \( M \) is of constant holomorphic sectional curvature.

The inverse is easy to check.

**Theorem 6.** Let \( M(2n \geq 6) \) be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1. \( R(\xi, u, u, \xi) = 0 \), whenever \( \{\xi, u\} \) spans a strongly isotropic totally real section of the I type;
2. \( B(R) = 0 \).

**Proof.** Let \( \{\eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_n\} \) be a separate basis for \( T_pM \). Applying the condition 1 to the pair \( \{\xi_i, \eta_j + \lambda \xi_\ell\}, \lambda \neq 0 \) \((i, j, k \text{ - different})\) we obtain

\[
0 = R(\xi_i, \eta_j, \eta_j, \xi_i); \quad i \neq j.
\]

The pairs \( \{\xi_i, \eta_j\}, \ i \neq j \) span strongly isotropic holomorphic sections of hybrid type and (14) is equivalent to the condition 1 of Theorem 2. Hence, \( B(R) = 0 \).

**Theorem 7.** Let \( M(2n \geq 8) \) be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1. \( R(u, v, v, u) = 0 \), whenever \( \{u, v\} \) spans a strongly isotropic totally real section of the II type;
2. \( B(R) = 0 \).

**Proof.** Let \( \{\eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_n\} \) be a separate basis for \( T_pM \). Substituting \( \{u = \xi_i + \lambda \eta_j, v = \lambda \xi_k + \eta_j\}, \lambda \neq 0 \) \((i, j, k, l \text{ - different})\) in the condition 1, we get \( R(\xi_i, \eta_j, \eta_j, \xi_i) = 0, \ i \neq 1 \), which is (14) and therefore \( B(R) = 0 \).

**Theorem 8.** Let \( M(2n \geq 8) \) be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1. \( R(x_i, x_j, x_k, x_l) = 0 \), \((i, j, k, l \text{ - different})\), whenever \( \{a_1, \ldots, a_n; x_i, \ldots, x_n\} \) is an adapted basis;
2. \( K(x_i, x_j) + K(x_k, x_l) = K(x_j, x_k) + K(x_j, x_l) \), \((i, j, k, l \text{ - different})\) whenever \( \{a_1, \ldots, a_n; x_i, \ldots, x_n\} \) is an adapted basis;
3. \( B(R) = 0 \).
This theorem is analogous to a theorem in [9] for Kaehlerian manifolds and it can be checked in a similar way taking into account the properties of the structure $P$.

4. Pinching problems

A Ricci curvature cannot be defined for an isotropic direction. If $X$ is an isotropic vector and $\rho(X,X) = 0$, this is a geometric property of the isotropic direction, defined by $X$.

The following statement is a slight modification of a result in [3].

**Lemma 3.** Let $M$ be a hyperbolic Kaehlerian manifold. The following conditions are equivalent:

1) $\rho(u,u) = 0$, for arbitrary $u$;
2) $\rho = (\tau/2n)g$, i.e. $M$ is Einsteinian.

**Theorem 9.** Let $M(2n \geq 4)$ be a hyperbolic Kaehlerian manifold. If the holomorphic sectional curvatures in every point are bounded, i.e. for an arbitrary nondegenerate holomorphic section $\alpha$ in $T_pM$

$$|K(\alpha,p)| \leq c(p),$$

then $M$ is of constant holomorphic sectional curvature.

**Proof.** Let $x = u + a$, $a \perp u, Pu$ and $\alpha$ be the holomorphic section spanned by $(x + ta/\sqrt{1 - t^2}, (Px + tPa)/\sqrt{1 - t^2}, |t| < 1$. From condition (15) we get

$$|R(x + ta, Px + tPa, Px + tPa, x + ta) \leq (1 - t^2)^2 c(p).$$

Hence, $R(u, Pu, Pu, u) = 0$ and Theorem 2 implies $B(R) = 0$, i.e.

$$\frac{4}{n+2} \rho(x) = K(x, Px) + \frac{\tau}{(n+1)(n+2)}.$$

This equality gives that the Ricci curvatures in every point are bounded

$$|\rho(x)| \leq c'(p).$$

Substituting $x$ by $(x + ta)/\sqrt{1 - t^2}$, $|t| < 1$ in (16), we find $\rho(u) = 0$ and Lemma 3 implies that $M$ is Einsteinian. Now, the statement follows from Corollary 1.

**Theorem 10.** Let $M(2n \geq 6)$ be a hyperbolic Kaehlerian manifold. If the totally real sectional curvatures of hybrid type are bounded in every point, i.e. if

$$|K(x, a)| \leq c(p); \quad a \perp x, Px,$$

then $M$ is of constant holomorphic sectional curvature.

**Proof.** Let $u = x + a$ and $\{x, a, b\}$ span a totally real 3-dimensional space. Substituting the pair $\{x, a\}$ in (17) by $(x + ta)/\sqrt{1 - t^2}$, $|t| \leq 1$, we obtain

$$|R(x + ta, b, b, x + ta) \leq (1 - t^2)^2 c(p).$$
Therefore, \( R(u, b, b, u) = 0 \), and Theorem 5 implies that \( M \) is of constant holomorphic sectional curvature.

**Remark.** The totally real curvatures of hybrid type in Theorem 10 can be replaced by totally real curvatures of pure type.

**Theorem 11.** Let \( M(2n \geq 6) \) be a hyperbolic Kaehlerian manifold. If the totally real sectional curvatures are bounded from above, i.e. if

\[
K(x, a) \leq c(p); \quad a \perp x, Px,
\]

\[
K(x, y) \leq c(p); \quad x \perp y, Py,
\]

then \( M \) is of constant holomorphic sectional curvature.

**Proof.** Let \( u = y + a \) and \( \{x, y, a\} \) span a 3-dimensional totally real space. The first condition of (18) implies \( R(x, a, a, x) \geq -c(p) \). Substituting here the vector \( a \) by \( (a + ty)/\sqrt{1 - t^2} \), \(|t| < 1\), we get \( R(x, u, u, x) \geq 0 \). Using, the inequality \( R(x, y, y, x) \leq c(p) \) and substituting the vector \( y \) by \( (y + ta)/\sqrt{1 - t^2} \), \(|t| < 1\), we obtain \( R(x, u, u, x) \leq 0 \). Therefore \( R(x, u, u, x) = 0 \) and the theorem follows now from Theorem 5.

5. Plane axioms

Let \( M \) (\( \dim M = m \geq 3 \)) be a differentiable manifold with a linear connection of zero torsion. \( M \) is said to satisfy the axiom of \( r \)-planes (\( 2 \leq r < m \)), if, for each point \( p \) and for any \( r \)-dimensional subspace \( E \) of \( T_pM \), there exists an \( r \)-dimensional totally geodesic submanifold \( N \) containing \( p \) such that \( T_pN = E \).

**Theorem 12.** (Axiom of nondegenerate totally real 2-planes of hybrid type) Let \( M(2n \geq 6) \) be a hyperbolic Kaehlerian manifold. If for any nondegenerate totally real section \( \alpha \) in \( T_pM \) of hybrid type there exists a 2-dimensional totally geodesic submanifold \( N \) containing \( p \) such that \( T_pN = \alpha \), then \( M \) is of constant holomorphic sectional curvature.

**Proof.** Let \( \{x, y, b\} \) be orthogonal and let it span a 3-dimensional totally real space in \( T_pM \). The pair \( \{x, y = (b + ty)/\sqrt{1 - t^2}\}, \(|t| < 1\) \) spans a 2-plane \( \alpha \), which is nondegenerate totally real of hybrid type. By the condition of the theorem, it follows that \( R(y', x)x \) is in \( \alpha \) and \( R(y', x)x \perp y' \), where \( y' = y + tb \). From this, it follows that \( R(x, u, u, x) = 0 \), where \( u = y + b \). Now, the proposition follows from Theorem 5.

**Remark.** The nondegenerate totally real 2-planes of hybrid type in Theorem 12 can be replaced with nondegenerate totally real 2-planes of pure type.

**Theorem 13.** (Axiom of weakly isotropic totally real 2-planes of the I type) Let \( M(2n \geq 6) \) be a hyperbolic Kaehlerian manifold. If for any weakly isotropic totally real 2 plane \( \alpha \) in \( T_pM \) of the I type there exists a 2-dimensional totally
geodesic submanifold $N$, containing $p$ such that $T_p N = \alpha$, then $M$ has a vanishing Bochner curvature tensor.

Proof. Let $a$ be an arbitrary weakly isotropic totally real 2-plane of the I type with a basis $\{\xi, x\}$ $\xi \perp x, \xi$ - eigen. By the condition of the theorem, it follows that $R(\xi, x)x$ is in a and therefore, $R(\xi, x, x, \xi) = 0$. Now, the proposition follows from Theorem 4.

**Theorem 14.** (Axiom of weakly isotropic totally real 2-planes of the II type) Let $M(2n \geq 6)$ be a hyperbolic Kählerian manifold. If for every weakly isotropic totally real 2-plane in $T_p M$ of the II type there exists a 2-dimensional totally geodesic submanifold $N$ containing $p$ such that $T_p M = \alpha$, then $M$ is of constant holomorphic sectional curvature.

The proof is similar to the proof of Theorem 13 and we omit it.

**Theorem 15.** (Axiom of strongly isotropic totally real 2-planes of the I type (II type)) Let $M(2n \geq 8)$ be a hyperbolic Kählerian manifold. If for every strongly isotropic totally real 2-plane $a$ in $T_p M$ of the I type (II type) there exists a 2-dimensional totally geodesic submanifold $N$ containing $p$ such that $T_p N = \alpha$, then $M$ has a vanishing Bochner curvature tensor.

The proof is similar to the proof of Theorem 13 and it is based on Theorem 6 (Theorem 7).

**Theorem 16.** (Axiom of nondegenerate holomorphic 2-planes) Let $M(2n \geq 4)$ be a hyperbolic Kählerian manifold. If for every nondegenerate holomorphic 2-plane $\alpha$ in $T_p M$ there exists a 2-dimensional totally geodesic submanifold $N$ containing $p$ such that $T_p N = \alpha$, then $M$ is of constant holomorphic sectional curvature.

Proof. Let $x$ be arbitrary and $a \perp x, Px$. If $\alpha$ is the holomorphic section spanned by $\{x, Px\}$, from the condition of the theorem it follows that $R(x, Px)Px$ is in $a$. Hence,

$$R(x, Px, Px, a) = 0. \quad (19)$$

Substituting the pair $\{x, a\}$ in (19) by $\{(x + ta)/\sqrt{1 - t^2}, (a + tx)/\sqrt{1 - t^2}\}$, $|t| < 1$, we obtain $R(u, Pu, Pu, u) = 0$, where $u = a + x$. Theorem 2 implies $B(R) = 0$. By using (19) and formula (8) we find

$$\rho(x, a) = 0, \quad (20)$$

Substituting the pair $\{x, a\}$ as above, we get $\rho(u, u) = 0$. Now, from Lemma 3 it follows that (20) implies $\rho = (\gamma/2n)g$. This condition and $B(R) = 0$ give the proposition.

**Theorem 17.** (Axiom of strongly isotropic holomorphic 2-planes) Let $M(2n \geq 4)$ be a hyperbolic Kählerian manifold. If for every strongly isotropic...
holomorphic 2-plane $a$ in $T_p M$ of hybrid type there exists a 2-dimensional totally geodesic submanifold $N$ containing $p$ such that $T_p N = \alpha$, then $M$ a vanishing Bochner curvature tensor.

REFERENCES


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