HYPERSURFACES OF C2-LIKE FINSLER SPACES

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Abstract. The notion of C2-like Finsler spaces has been introduced by Matsumoto and Numata [1]. The purpose of the present paper is to study the properties of hypersurfaces immersed in C2-like Finsler spaces. We prove that each non-Riemannian hypersurface of a C2-like Finsler space is C2-like. The condition under which a hypersurface of a C2-like Landsberg space is Landsberg is obtained. Finally, after using the so-called T-conditions [6] we explore the situation in which a hypersurface of a C2-like Finsler space $F_n$ satisfying the T-conditions also satisfies the T-condition.

1. Introduction Let $F_n$ be a Finsler space of dimension $n$ with the fundamental function $F(x, y), (y^i = \dot{x}^i)$. The following are the two well-known properties of Finsler spaces:

$(P1)$ The Berwald connection parameter $G^i_{jk}$ [3] is not, in general, independent on the direction element $y^i$.

$(P2)$ $g_{ij(k)} = -2C^i_{jk0} \neq 0$ in general, where (k) stands for Berwald's process of covariant derivation, $C^i_{jk} = 1/2 \cdot dg_{ij}(x, y)$ and suffix 0 stands for transvection with respect to $y^i$.

A Finsler space in which $G^i_{jk}$ is independent on $y$ is called a Berwald space. This space is characterized by the condition $C^i_{jk0} = 0$.

A Finsler space in which $g_{ij(k)} = 0$ is called a Landsberg space. This space is characterized by $C^i_{jk0} = 0$.

It is obvious that each Berwald space is a Landsberg space. Further, the relation $\Gamma^i_{jk} = G^i_{jk} - C^i_{jk0}$ [3] involving Cartan's connection parameter $\Gamma^i_{jk}$ proves,

Lemma 1. In a Landsberg space, Cartan's and Berwald's connection parameters are identical and in Berwald's space the Cartan's connection parameter is independent on $y$.

Definition 1. Finsler space $F_n$ ($n \geq 2$) with $C^2 = C^iC_i \neq 0$ is called C2-like [1], if the (h) hv-torsion tensor $C_{ijk}$ can be written in the form

\begin{equation}
C_{ijk} = C_iC_jC_k/C^2 \quad \text{where} \quad C_i = g^{jk}C_{ijk}.
\end{equation}

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The following lemma can be easily deduced with the help of the equation (1.1) and definition of Berwald and Landsberg spaces.

**Lemma 2.** The necessary and sufficient condition that a C2-like Finsler space be a Berwald space (or Landsberg space) is that $C_{ijkl} = 0$ (or $C_{ij0} = 0$).

2. **Hypersurfaces of a C2-like Finsler space.** Consider a non-Riemannian hypersurface $F_{n-1}$ of $F_n$ ($n \geq 3$), characterized by the equation $x^i = x^I(u^\alpha)$, where we assume that all the Latin indices $i, j, \ldots$ take values $1, 2, \ldots n$, while all the Greek indices $\alpha, \beta, \ldots$ take values $1, 2, \ldots n-1$. The fundamental tensor of $F_{n-1}$ is given by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, y)B^i_{\alpha}B^j_{\beta}, \quad \text{where} \quad B^i_{\alpha} = \partial x^i / \partial u^\alpha.$$  

We shall use the notation, $B^{ij...k}_{\alpha\beta...\gamma} = B^i_{\alpha}B^j_{\beta} \ldots B^k_{\gamma}$.

Now in a hypersurface $F_{n-1}$ of $F_n$ we have

$$C_{\alpha\beta\gamma} = C_{ijkl}B^{ij...k}_{\alpha\beta...\gamma}.$$  

If $F_n$ is C2-like, then by means of the equation (1.1), the equation (2.2) reduces to

$$C_{\alpha\beta\gamma} = \frac{\overline{C}_\alpha}{\overline{C}_\beta} \overline{C}_\gamma / C^2,$$

where

$$\overline{C}_\alpha = C_iB^i_{\alpha} = C^2 / C^2 \cdot C_\alpha,$$

where we have put $\overline{C}^2 = \overline{C}_\alpha \overline{C}^\alpha \neq 0$ and $C_\alpha = g^{\beta\gamma}C_{\alpha\beta\gamma}$.

The equations (2.3) and (2.4) give the following

$$C_{\alpha\beta\gamma} = C^4 / \overline{C}^6 \cdot C_\alpha C_\beta C_\gamma$$

A direct calculation will give

$$C^4 / \overline{C}^6 = 1 / \overline{C}^2$$

where $\overline{C}$ stands for $C_\alpha C^\alpha$ and this must be non-zero, for if it is zero then $C_\alpha = 0$. Therefore by Diecke’s theorem the hypersurface is Riemannian, which is a contradiction to our assumption. Thus (2.5) reduces to $C_{\alpha\beta\gamma} = C_\alpha C_\beta C_\gamma / \overline{C}^2$ which proves the following

**Theorem 2.1.** The hypersurface $F_{n-1}$ of a C2-like Finsler space $F_n$ is C2-like.

Throughout the paper it will be assumed that $\overline{C}^2 \neq 0$.

The differences between the intrinsic and induced connection parameters $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ and $\Gamma^{\alpha}_{\beta\gamma}$ of a hypersurface has been obtained by Rund [2]. If the space $F_n$ is C2-like then this difference tensor $\Lambda^{\alpha}_{\beta\gamma} = \tilde{\Gamma}^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\beta\gamma}$ reduces to the form

$$\Lambda^{\alpha}_{\beta\gamma} = \rho C^2 / \overline{C}^4 \cdot [C_\beta C_\gamma \Omega_{\alpha0} + C_\alpha C_\beta \Omega_{\gamma0} - C_\gamma C_\beta \Omega_{\alpha0} - C_\gamma C_\alpha \Omega_{\beta0} - C_\alpha C_\beta C_\gamma \Omega_{00}]$$
where \( \rho = N^i C_i \), \( \Omega_{\alpha \beta} \) are the components of the second fundamental tensor of \( F_{n-1} \), and \( N^i \) are the components of the unit vector normal to \( F_{n-1} \). If we suppose that intrinsic and induced connection parameters of \( F_{n-1} \) are identical, then (2.7) gives either \( \rho = 0 \), or \( \Omega_{\alpha 0} = 0 \), or \( C_{\alpha} = 0 \). But \( C_{\alpha} = 0 \) gives that the hypersurface is Riemannian, which is a contradiction to our assumption. This proves the following.

**Theorem 2.2.** The necessary and sufficient condition that intrinsic and induced connection parameters of a hypersurface of a \( C^2 \)-like Finsler space be equal is that either \( \Omega_{\alpha 0} = 0 \) or the vector \( C_i \) is tangential to the hypersurface.

In order to derive a condition under which a hypersurface of a \( C^2 \)-like Landsberg space is a Landsberg space we note that the induced covariant differentiation of the relation \( C_{\alpha} = \overrightarrow{C}^2 / C^2 \cdot C_i B^{i \alpha} \) yields

\[
C_{\alpha i} = \overrightarrow{C}^2 / C^2 (C_{i0} B^{i \alpha} + \partial C_i / \partial \hat{u}^\alpha \cdot \Omega_{\beta 0} N^1 + \rho \Omega_{\alpha \beta}) + \overrightarrow{C} (\overrightarrow{C}^2 / C^2)_{|| \beta} \]

where we have used the fact that \( \partial C_i / \partial y^i \) is symmetric in the indices \( i, j \). (The double vertical bar stands for induced covariant derivative). The transvection of the relation (2.8) with respect to \( \hat{u}^\alpha \) gives

\[
C_{\alpha i} = \overrightarrow{C}^2 / C^2 (C_{i0} B^{i \alpha} + \partial C_i / \partial \hat{u}^\alpha \cdot \Omega_{\beta 0} N^1 \rho \Omega_{\alpha \beta}) + \overrightarrow{C} (\overrightarrow{C}^2 / C^2)_{|| \alpha} \]

If we take \( \rho = 0 \) then equation (1.1) shows that the tensor defined by, \( M_{\alpha \beta} = C_{ijk} B^{ij}_{\alpha \beta} N^k \) vanishes. The properties of the hypersurfaces in this case have been discussed by Brown [4]. He has shown that in this case

\[
\partial N^1 / \partial \hat{u}^\alpha = -M_{\alpha} N^1, \quad \text{where} \quad M_{\alpha} = C_{ijk} B^{ij}_{\alpha} N^j N^k. 
\]

This relation and the condition \( \rho = C_1 N^1 = 0 \) give

\[
\frac{\partial C_1}{\partial \hat{u}^\alpha} N^1 = -C_1 \frac{\partial N_1}{\partial \hat{u}^\alpha} = C_1 N^1 M_{\alpha} = 0.
\]

A direct calculation will give \( \overrightarrow{C} = C^2 - \rho^2 \).

This shows that the condition \( \rho = 0 \) will reduce the equation (2.9) to \( C_{\alpha i} = C_{i0} B^{i \alpha}_{\alpha} \). Again, Brown [4] has shown that for \( M_{\alpha \beta} = 0 \), the intrinsic and induced connection parameters are identical. Hence and from Lemma 2 we obtain the following

**Theorem 2.3.** A hypersurface of a \( C^2 \)-like Landsberg space will be a Landsberg space if the vector \( C_i \) is tangential to the hypersurface.

Now we want to find the condition under which the induced connection parameter \( \Gamma^\beta_{\gamma \delta} \) of a hypersurface of a \( C^2 \)-like Berwald space is independent of \( \hat{u}^\alpha \). Rund [3] has given the following relation for induced connection parameter of \( F_{n-1} \),

\[
\Gamma^\beta_{\gamma \delta} = B^j_i \left( \frac{\partial^2 x^j}{\partial u^\delta \partial u^\gamma} + \Gamma^k_{ji} B^{jk}_{\beta} \right), \quad \text{where} \quad B^j_i = g^{\alpha \beta} g_{ij} B^{ij}_{\alpha \beta},
\]
If the Finsler space $F_n$ is Berwald, then equation (2.10) by means of Lemma 1 gives that $\Gamma^\alpha_{\beta\gamma}$ is independent of $\hat{u}^\alpha$ if and only if $B^{\alpha}_{\rho}$ is independent of $\hat{u}^\alpha$. Rund [3] has given the following relation
\[
\frac{\partial B^\alpha_{\rho}}{\partial \hat{u}^\lambda} = 2g^{\rho\delta}B^{\delta}_{\lambda k}C^{kl}_{\lambda i}N_i
\]
which in view of (1.1) and (2.4) reduces to
\[
(2.11) \quad \frac{\partial B^\alpha_{\rho}}{\partial \hat{u}^\lambda} = 2\rho C^2/\sqrt{\xi} \cdot C^\alpha C_{\lambda} N_i.
\]
Hence we have the following:

**Theorem 2.4.** The necessary and sufficient condition that the induced connection parameter of a hypersurface of a C2-like Berwald space be independent on the direction element is that the vector $C_i$ is tangential to the hypersurface.

Theorems 2.2. and 2.4 give the following:

**Theorem 2.5.** If the included connection parameter of a hypersurface of a C2-like Berwald space is independent on the direction element then the induced and intrinsic connection parameters are equal.

The two normal curvature vectors denoted by $I^\alpha_\beta$ and $\check{H}^\alpha_\beta$ are given by Rund [3] and Davies [5]. These vectors are related by [3] as follows.

\[
(2.12) \quad \check{H}^\alpha_\beta = I^\alpha_\beta + N^i C^j_{hk} B - \beta^h \check{H}^\alpha_\lambda \hat{u}^\lambda
\]
The relation (2.12) after transvection with respect to $\hat{u}^\beta$ gives
\[
(2.13) \quad \check{H}^\alpha_\beta \hat{u}^\beta = I^\alpha_\beta = \Omega_{\alpha\beta} N^i.
\]
The equations (1.1), (2.4), (2.12) and (2.13) give
\[
\check{H}^\alpha_\beta = I^\alpha_\beta + (\rho^2/\sqrt{\xi}) \Omega_{\alpha\beta} N^i
\]
which proves the following:

**Theorem 2.6.** The necessary and sufficient condition that Rund's and Davies's normal curvature vectors of the hypersurface of a C2-like Finsler space are identical is that either $\Omega_{\alpha\beta} = 0$, or the vector $C_i$ is tangential to the hypersurface.

The following theorems is a consequence of theorems 2.2 and 2.6.

**Theorem 2.7.** A necessary and sufficient condition that Rund's and Davies's normal curvature vectors of the hypersurface of a C2-like Finsler space are identical is that their induced and intrinsic connection parameters, are identical.

Theorems 2.4 and 2.6 yield the following:

**Theorem 2.8.** If the induced connection parameter of a hypersurface of a C2-like Berwald space is independent on the direction element then Rund's and Davies's normal curvature vectors of hypersurface are identical.
3. T-Conditions. We now consider the T-tensor (Matsumoto [6] and Kawaguchi [7]) given by

\[ T_{\alpha ijk} = FC_{\alpha ijk|k} + C_{\alpha ijk}l_k + C_{\alpha ijk}l_j + C_{\alpha ijk}l_i + C_{\alpha ijk}l_h \]

where \( C_{\alpha ijk} \) stands for the \( v \)-covariant derivative of \( C_n \) with respect to \( y^h \). The corresponding expression for the T-tensor \( T_{\alpha \beta \gamma \delta} \) in \( F_{n-1} \) can be written as

\[ T_{\alpha \beta \gamma \delta} = FC_{\alpha \beta \gamma \delta} + l_{\alpha}C_{\beta \gamma \delta} + l_{\beta}C_{\alpha \gamma \delta} + l_{\gamma}C_{\alpha \beta \delta} + l_{\delta}C_{\alpha \beta \gamma} \]

The relation (2.2) yields

\[ C_{\alpha \beta \gamma \delta} = C_{\alpha ijk|l}B_{\alpha \beta \gamma l}^{ijh} + C_{\alpha ijk}B_{\alpha \beta \gamma l}^{jkh} - C_{\alpha ijk}B_{\alpha \beta \gamma l}^{ihj} + C_{\alpha ijk}B_{\alpha \beta \gamma l}^{kij} - C_{\alpha ijk}B_{\alpha \beta \gamma l}^{ijh} \]

where \( Z_{\alpha l}^i = B_{\alpha l}^i = N^jM_{\alpha l} \). A direct calculation will give

\[ \text{By virtue of the equations (1.1), (2.4), (3.4) and (3.5) the relation (3.3) reduces to the form} \]

\[ C_{\alpha ijk|l} = C_{\alpha ijk}B_{\alpha \beta \gamma l}^{jkh} + 3\rho^2C^4/C \cdot C_{\alpha}C_{\beta}C_{\gamma}C_{\delta} \]

The equations (2.2), (3.1), (3.2), (3.6) and the well known relation \( l_\alpha = i_iB_{\alpha}^i \) give

\[ T_{\alpha \beta \gamma \delta} = C_{\alpha ijk}B_{\alpha \beta \gamma l}^{jkh} + 3\rho^2C^4/C \cdot C_{\alpha}C_{\beta}C_{\gamma}C_{\delta} \]

The space \( F_n \) is said to satisfy the T-condition if and only if \( T_{\alpha ijk} = 0 \). Therefore we have the following theorem.

THEOREM 3.2. If a \( C^2 \)-like Finsler space \( F_n \) satisfies the T-condition then the necessary and sufficient condition for its hypersurface \( F_{n-1} \) to satisfy the T-condition is that the vector field \( C_i \) is tangential to the space \( F_{n-1} \).

The theorems 2.2, 2.3, 2.4, 2.6, and 3.1 yield the following:

THEOREM 3.7. If a \( C^2 \)-like Bernald space \( F_n \) and its hypersurface \( F_{n-1} \) satisfy the T-condition then the induced connection parameter of \( F_{n-1} \), is independent on the direction element, its intrinsic and induced connection parameters are identical, its Rund's and Davies's normal curvature vectors are identical and the hypersurface is a Landsberg space.

It can be easily shown that in a hypersurface of a \( C^2 \)-like space the \( v \)-curvature tensor \( S_{\alpha \beta \gamma \delta} \) = 0, which proves the following

THEOREM 3.3. The hypersurface of a \( C^2 \)-like Finsler space is a flat space.

REFERENCES


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