BASES FROM ORTHOGONAL SUBSPACES OBTAINED
BY EVALUATION OF THE REPRODUCING KERNEL

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Abstract. Every inner operator function \( \theta \) with values in \( B(E, E) \), \( E \) — a fixed (separable) Hilbert space, determines a co-invariant subspace \( H(\theta) \) of the operator of multiplication by \( z \) in the Hardy space \( H^2_E \). "Evaluating" the reproducing kernel of \( H(\theta) \) at "\( U \)-points" of the function \( \theta (U \) is unitary operator\) we obtain operator functions \( \gamma_U(z) \) and subspaces \( \gamma_U E \). The main result of the paper is: Let the operator \( I - \theta (z) U^* \) have a bounded inverse for every \( z, |z| < 1 \). If \( (1 - r)^{-1} \Re \varphi (r) \) for definition of \( \varphi \) see (1) is uniform bounded in \( r, 0 < r < 1 \), for all \( t, |t| = 1 \), except for a countable set, then the family of subspaces \( \gamma_U E \) is orthogonal and complete in \( H(\theta) \). This generalizes an analogous result of Clark \[3\] in the scalar case.

1. Introduction. Throughout this paper we denote by \( D \) the unit disc \( |z| < 1 \) and by \( T \) the unit circle \( |z| = 1 \) of the complex plane \( C \). Given a separable Hilbert space \( E (E \neq \{0\}) \), let \( H^2_E \) be the standard Hardy space of analytic \( E \)-valued functions on \( D \). (See \[1\] or \[2\] for general references.) Writing inner products and norms in \( H^2_E \) we will omit designation of the space in the index. The space \( H^2_E \) possesses a so-called reproducing kernel. This is the function \( k_w(z) = (1 - z \bar{w})^{-1}, w \in D, z \in D \), with the following properties: \( k_w a \in H^2_E \), \( w \in D, a \in E \), \( (k_w a, k_w a) = k_w(\cdot) a \) and \( (f, k_w a) = (f(w), a)_E, f \in H^2_E \), \( w \in D \). If \( \theta \) is an inner operator function \[1\] (defined on \( D \) and with values in \( B(E, E) \)), then let \( H = H(\theta) = H^2_E \otimes \theta H^2_E \). The reproducing kernel for the space \( H \) is the function \( K_w(z) = (1 - z \bar{w})^{-1}(I - \theta (z) \theta (w)^*), w \in D, z \in D \), where by \( I \) is denoted the identity mapping in \( E \).

If \( U \) is a unitary operator in \( E \), then we will also consider the following operator functions:

\[
\varphi(z) = \varphi_U(z) = (I + \theta(z) U^*)(I - \theta(z) U^*)^{-1},
\]

\( z \in D \), (if \( (I - \theta(z) U^*)^{-1} \) exists and

\[
\gamma(z) = \gamma_U(t, z) = (1 - z \bar{t})^{-1}(I - \theta(z) U^*),
\]

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\[ t \in T, \ z \in D. \] In the scalar case \( \dim E = 1 \) \( U \) is a number of modulus 1 and \( U^* \) shall be replaced by \( \overline{U} \).

In [3] Clark considered orthogonal sets in \( H \) obtained by evaluation of the kernel \( K_w(z) \) on \( T \) in the case \( \dim E = 1 \). The purpose of this paper is to generalize the criterion for completeness of such orthogonal sets which is contained in Theorem 7.1 of [3].

2. Bases from subspaces. Let \( T_U \) be the set of all points \( t \in T \) such that \( \gamma a \in H \) for some \( a \in E, \ a \neq 0 \). Given \( t \in T \), we denote by \( \gamma t E \) the closure of the set of all functions of the form \( \gamma a, \ a \in E \), lying in \( H \). All such subspaces form a family which we will denote by \( G_U = \{ \gamma E \mid t \in T_U \} \). The problem we are interested in is: when does the family \( G_U \) form an orthogonal basis from subspaces of \( H \), i.e., when does \( \gamma t E \perp \gamma s E, \ t \neq s \) and \( \text{Cl}(\cup \gamma t E, \ t \in T_U) = H \) hold? (Cl=closure).

We begin with some lemmas.

**Lemma 1.** The mapping \( f \to f(w) \) is a bounded operator from \( H^2_E \) to \( E \) for every \( w \in D \).

**Proof.** The statement follows from the inequality

\[
\|f(w)\|_E = \sup \{ (f,k_w a) : a \in E, \ |a| \leq 1 \} \leq \|f\|(k(w))^{1/2}, \ f \in H^2_E, \ w \in D.
\]

Note that it follows by lemma 1, that if the operator \( I - \theta(z)U^* \) has a bounded inverse for at least one \( z \in D \) then every function in \( \gamma E \) has the form \( \gamma a, \ a \in E \).

**Lemma 2.** Let \( H_1 \) and \( H_2 \) be Hilbert spaces with (scalar) reproducing kernels \[ K^1_w(z) \text{ and } K^2_w(z), \ w \in D, \ z \in D. \] If there exists a function \( h \) (from \( D \) into \( C \)) such that \( h(z) \neq 0, \ z \in D, \) and \( K^2_w(z) = \overline{h(w)}h(z)K^1_w(z), \ w \in D, \ z \in D, \) then multiplication by \( h \) is an isomorphism of spaces \( H_1 \) and \( H_1 \).

**Proof.** We establish the equality

\[
(hf, hg)_2 = (f, g)_1, \ f \in H_1, \ g \in H_1,
\]

first in the case when \( f = \overline{h(w)}K^1_w, \ w \in D, \) and \( g = \overline{h(v)}K^1_v, \ v \in D : (hf, hg)_2 = K^2_w(\nu) = (f, g)_1. \) By linearity it follows that (3) holds also when \( f \) and \( g \) are linear combinations of functions of the form \( \overline{h(w)}K^1_w, \ w \in D. \) The same conclusion follows by continuity of the inner product and by completeness of the set \( \{ \overline{h(w)}K^1_w \mid w \in D \} \) in \( H_1 \) also when \( f \) and \( g \) are arbitrary functions in \( H_1 \). Thus multiplication by \( h \) preserves the inner product. Since the set \( \{ K^2_w \mid w \in D \} \) is complete in \( H_2, \ hH_1 = H_2, \) i.e., multiplication by \( h \) is an isomorphism of spaces \( H_1 \) and \( H_2. \)

**Lemma 3.** Let \( \theta \) be a scalar inner function and \( t \in T \). Then the following are equivalent

(a) \( \gamma t \in H \) for some complex number \( U \) of modulus 1,
(b) the limit $\lim_{r \to 1-} K_{rt}$ exists in the $H$-norm,

(c) $\|K_{rt}\|$ is bounded for \( r < 1 \).

Proof. (a) $\Rightarrow$ (b). If $\gamma_t \in H$ for some $U$, $|U| = 1$, then every function $f \in H$ has a nontangential limit $f(t)$ at $t$ and the functional $f \to f(t)$ is bounded [3]. By the existence of the limit $\lim_{r \to 1-} \gamma_t(rt) = \lim_{r \to 1-}(\theta(rt)U)(1-r)^{-1}$ it follows that $\lim_{r \to 1-} \theta(rt) = U$ and that $\lim_{r \to 1-} K_{w}(rt) = \gamma_t(w) = (K_{w}, \gamma_t)$, \( w \in D \).

This means that $K_{w}(t) = (K_{w}, \gamma_t)$, \( w \in D \), so $f(t) = (f, \gamma_t)$ for every $f \in H$. In particular

$$\lim_{r \to 1-} (1 - \theta(rt)U)(1-r)^{-1} = \lim_{r \to 1-} \gamma_t(rt) = \|\gamma_t\|^2.$$  

This implies that

$$K_{rt}(rt) = (1 - \theta(rt)U)(1-r^2)^{-1} + \theta(rt)U(1-\theta(rt)U)(1-r^2)^{-1}$$

$$\text{tends to } \|\gamma_t\|^2 \text{ as } r \to 1-.$$  

Thus $\|K_{rt} - \gamma_t\|^2 = K_{rt}(rt) - \gamma_t(rt) - \gamma_t(rt) + \|\gamma_t\|^2 \to 0$, as $r \to 1-$, i.e. (b) holds.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). Let $\theta$ have the representation $\theta(z) = \nu B(z)S(z)$, \( z \in D \), where $|\nu| = 1$,

$$B(z) = \prod_{k=1}^{l} b_k(z) = \prod_{k=1}^{l} \left| z_k \right|^l \left| z_k(z_k - z)(1 - z_k) \right|^{-1},$$

\( z \in D \), with $z_k \in D$ for $k = 1, 2, \ldots$, \( l (1 \leq l \leq \infty) \mid \left| z_k \right| / z_k = 1, \text{if } z_k = 0 \) (if $\theta$ has no zeros then $B(z) \equiv 1$), and

$$S(z) = \exp \left( - \int_{0}^{2\pi} (s + z)(s - z)^{-1} d\mu(x) \right), \text{ if } z \in D, (s = e^{ix}),$$

where $\mu$ is a finite, non-negative singular measure on $T$. From boundedness of $\|K_{rt}\|^2 = K_{rt}(rt)$ and from $|B(rt)| \geq |\theta(rt)|$ and $|\theta(rt) - \theta(rt) |$ it follows that $$(1 - \left| B(rt) \right|^2)(1 - r^2)^{-1} \text{ and } (1 - \left| S(rt) \right|^2)(1 - r^2)^{-1} \text{ are bounded}.$$  

Since

$$(1 - \left| B(rt) \right|^2)(1 - r^2)^{-1} = (1 - \left| z_l \right|^2) \left| 1 - rz_l \right|^2 +$$

$$+ \sum_{k=1}^{l} \prod_{j=1}^{k-1} \left| b_j(rt) \right|^2 \left( 1 - \left| z_k \right|^2 \right) \left| 1 - rz_k \right|^2 \to$$

$$\to \sum_{k=1}^{l} \left( 1 - \left| z_k \right|^2 \right) \left| 1 - rz_k \right|^2 \text{ as } r \to 1-,$$

it follows that

$$(4) \sum_{k=1}^{l} \left( 1 - \left| z_k \right|^2 \right) \left| 1 - rz_k \right|^2 < \infty.$$  

Since \( |S(rt)|^2 = \exp\left(-2(1-r^2) \int_0^{2\pi} |s-rt|^{-2} d\mu(x)\right)\), it follows from boundedness of \((1-|S(rt)|^2 (1-r^2)^{-1}\) that \(\int_0^{2\pi} |s-rt|^{-2} d\mu(x)\) is bounded for \(r\) sufficiently near to 1, which gives

\[
\int_0^{2\pi} |s-t|^{-2} d\mu(x) < \infty.
\]

Now, (4) and (5) imply that \(\gamma_t \in H\) for some \(U\), \(|U|=1\), [3]. This completes the proof.

**Remark 1.** Let \(\Re \varphi(rt)(1-r^{-1})\) be bounded, \(t \in T(\varphi = \varphi_t)\). Then \(|K_{rt}|\) is bounded also. This is evident from the relation

\[
K_{rt}(rt) = \Re \varphi(rt)(1-r^{-1}) |1 - \theta(rt)|^2.
\]

**Lemma 4.** Let the operator \(I - \theta(z)\) have a bounded inverse for every \(z \in D\) and let \(\Re \varphi(rt) \to 0\), \(r \to 1-\), \((\varphi = \varphi_t)\) (at least in the weak operator convergence) for a.e. \(t \in T\). Fix a.a. \(t \in T\), and put \(\varphi_a(z) = (\varphi(z)a, a)_E, z \in D\). Then the function \(\theta_a(\varphi_a - 1)(\varphi_a + 1)^{-1}\) is a (scalar) inner function and the corresponding space \(H_a = H(\theta_a)\) is isometrically isomorphic to the subspace \(K_a\) of \(H\) generated by functions of the form \(K_{w}(z)(I - \theta(w)^*)^{-1}a, w \in D\). An isomorphism \(\Phi\) from \(K_a\) to \(H_a\) is given by \(\Phi f(z) = (1 - \theta(z))(I - \theta(z))^* f(z), a)_E, z \in D\), \(f \in K_a\).

**Proof.** Since \(|\theta(z)|| \leq 1\), \(z \in D\), and

\[
\Re \varphi(z) = (I - \theta(z))^{-1}(I - \theta(z))\theta(z)(I - \theta(z))^{-1},
\]

it follows that \(\Re \varphi(z) \geq 0\) and \(\Re \varphi_a(z) \geq 0\), which implies \(|\theta_a| \leq 1\), \(z \in D\). Since \(\Re \varphi(rt) \to 0\), \(r \to 1-\), for a.e. \(t \in T\), it follows that the same holds for \(\Re \varphi_a\) and so radial limits of \(\theta_a\) have modulus 1 for a.e. \(t \in T\). Thus \(\theta_a\) is an inner function.

Now consider the mapping \(\Phi_1\) defined by \(\Phi_1 f(z) = ((I - \theta(z))^{-1} f(z), a)_E\), \(z \in D\), \(f \in K_a\). Because of \(\Phi_1 f(z) = (f, K_{z}(I - \theta(z)^*)^{-1}a)\) \(\Phi_1\) is a regular mapping, i.e. \(\Phi_1 f = 0\) iff \(f = 0\). So \(\Phi_1\) maps \(K_a\) one-to-one onto a set \(L = L_a\) of scalar analytic (in \(D\)) functions. If we define in \(L\) the inner product by \((h_1, h_2)_L = (\Phi_1^{-1} h_1, \Phi_1^{-1} h_2), h_1, h_2 \in L\), then \(L\) becomes a Hilbert space isometrically isomorphic to \(K_a\). The space \(L\) possesses also reproducing kernel. This is the function

\[
J_w(z) = \Phi_1 K_{w}(z)(I - \theta(W)^*)^{-1} a = (\varphi_a(z) + \overline{\varphi_a(w)})^{-1}(1 - z\overline{w})^{-1}, z \in D, w \in D.
\]

Finally, multiplication by the function \(1 - \theta_a(z)\) is an isometrical isomorphism from \(L\) onto \(H_a\) (Lemma 2). Thus \(\Phi\) is really an isometrical isomorphism from \(K_a\) onto \(H_a\).

**Lemma 5.** Let the assumptions of Lemma 4 be satisfied and let \(t \in T_t\). Then there exists an operator \(\alpha(t) \in B(E, E)\) such that

\[
\lim_{r \to 1-} (1 - r)(I - \theta(rt)^*)^{-1} = \alpha(t)
\]
in the strong operator convergence. If \( a \in A \backslash \{0\} \), then the function \( \gamma_t(\alpha(t)a) = \gamma_t(z) \), see (2) belongs to the subspace \( Ka \) (defined in Lemma 4) and it holds
\[
\lim_{r \to 1} (1 - r)K_{rt}(I - \theta(rt)^*)^{-1}a = \gamma_t(\alpha(t)a)
\]
in the \( H \)-norm. If \( \theta, H_a \) and \( \Phi \) are as in Lemma 4 and if \( \gamma^0_t \) denotes the function \( (1 - \theta(z))(1 - zI)^{-1} \), then \( \gamma^0_t \in N_a \) iff \( (\alpha(t)a, a)_E \neq 0 \) and it is also
\[
\Phi \gamma_t(\alpha(t)a) = (\alpha(t)a, a)_E \gamma^0_t.
\]

Proof. Since \( t \in T_I \), it follows that \( \gamma_t b \in H \) for some \( b \in E \backslash \{0\} \). Let \( a \in E \) and \( b, a)_E \neq 0 \). Denote by \( P \gamma_t b \) the projection of \( \gamma_t b \) to the subspace \( Ka \). Because of
\[
((I - \theta(z))^{-1} P \gamma_t(z)b, a)_E = (\gamma_t b, K_z(I - \theta(z)^*)^{-1}a) = (b, a)_E (1 - zI)^{-1}, \ z \in D,
\]
it must be
\[
\Phi P \gamma_t b = (b, a)_E \gamma^0_t.
\]
Since \( (b, a)_E \neq 0 \), the function \( \gamma_t^0 \) lies in \( H_a \). If \( K^0_a(z) \) denotes the reproducing kernel in \( H_a \), then by Lemma 3 \( \gamma^0_t = \lim_{r \to 1} \frac{K^0_a}{K^a_{rt}} \) in the \( H_a \)-norm. Since \( \Phi \) is an isomorphism (Lemma 4) and \( \Phi^{-1} K^a_{rt} = (1 - \theta_a(rt))K_{rt}(I - \theta(rt)^*)^{-1}a \) we have also
\[
\Phi^{-1} \gamma_t^0 = \lim_{r \to 1} (1 - \theta_a(rt))K_{rt}(I - \theta(rt)^*)^{-1}a
\]
in the \( H \)-norm. Regarding the fact that
\[
\lim_{r \to 1} (1 - \theta_a(rt))(1 - r)^{-1} = \lim_{r \to 1} (\gamma_t^0, K^a_{rt})_{H_a} = ||\gamma_t^0||^2,
\]
we obtain
\[
\Phi^{-1} \gamma_t^0 = ||\gamma_t^0||^2 \lim_{r \to 1} (1 - r)K_{rt}(I - \theta(rt)^*)^{-1}a.
\]
If we consider pointwise convergence (Lemma 1) in the last relation, we can conclude that there exists
\[
\lim_{r \to 1} (1 - r)(I - \theta(rt)^*)^{-1}a \overset{\text{def}}{=} \alpha(t)a
\]
in the \( E \)-norm and that (7) must hold, which gives also \( \gamma_t \alpha(t)a \in Ka \). In fact, the limit (11) exists and the relation (7) holds for every \( a \in E \), for if \( (b, a)_E = 0 \) we can write \( a = (a + b) - b \). Since \( a \) in (11) may be arbitrary, \( \alpha(t) \) is a (bounded) operator and (6) follows. Putting \( b = \alpha(t)a \) in (9) we obtain (8). Now (10) and (7) imply
\[
\Phi^{-1} \gamma_t^0 = ||\gamma_t^0||^2 \alpha(t)a.
\]
Comparing this with (8) we see that \( (\alpha(t)a, a)_E = ||\gamma_t^0||^{-2} \). Hence it is evident that \( \gamma_t^0 \in H_a \) implies \( (\alpha(t)a, a)_E \neq 0 \) and (8) shows that the converse is also true.

Lemma 6. In Lemma 5 all functions of the form \( \gamma_t(\alpha(t)a), a \in E \), form a complete set in \( \gamma_E \).
Proof. If $\gamma b \in \gamma E$ and $\gamma b \perp \gamma \alpha(t)a$, $a \in E$, then by (7) $0 = (\gamma b, \gamma \alpha(t)a) = \lim_{r \to 1^-} (1 - r)(\gamma b, K_{rt}(I - \theta(rt)*^{-1})a) = (b, a)_E$, i.e. $b = 0$ and $\gamma b = 0$.

Lemma 7. Let the assumptions of Lemma 4 be satisfied. Then the set $G = G_I$ is orthogonal.

Proof. Let $t \in T_I$, $s \in T_I$, $t \neq s$, and let $\gamma \alpha(t)a \in \gamma_1 E$ and $\gamma s b \in \gamma_2 E$. Then it follows by (7) that

$$(\gamma \alpha(t)a, \gamma s b) = \lim_{r \to 1^-} (1 - r)(1 - rt)^{-1}(a, b)_E = 0.$$ 

By completeness of the set $\{\gamma \alpha(t)a \mid a \in E\}$ in $\gamma_1 E$ it follows that $\gamma_1 E \perp \gamma_2 E$. Thus the family $G$ is orthogonal.

Theorem. Let $\theta$ be an inner operator function, $U$ a unitary operator in $E$ and let the operator $I - \theta(z)U^*$ have a bounded inverse for every $z \in D$. If $(1 - r)^{-1} \Re \varphi(rt)$ is bounded in $r$ for all $t \in T$ except for a countable set, then the family $G_U$ is orthogonal and complete in $H$.

Proof. Since $H(\theta U^*) = H(\theta)$ for each unitary operator $U$ (in $E$), it is enough to give the proof only in the case $U = I$. Thus let $U = I$. The assumption on boundedness of $(1 - r)^{-1} \Re \varphi(rt)$ implies that $\lim_{r \to 1^-} \Re \varphi(rt) = 0$ in the strong operator convergence for all $t \in T$ except for a countable set. So the assumptions of Lemmas 4, 5, 6, 7 are satisfied.

Orthogonality of the family $G$ is proved in Lemma 7. Let us prove the completeness of $G$. It is clear that whenever $(1 - r)^{-1} \Re \varphi(rt)$ is bounded then $(1 - r)^{-1} \Re \varphi(rt)$ is too, for $a \in E$ ($\varphi_a$ as in Lemma 4). By Remark 1 and by Lemma 3 it follows that the condition (a) in Lemma 3 is satisfied for all $t \in T$ except for a countable set. By Theorem 7.1 and Lemma 3.1 in [3] it follows that the set of functions of the form $\gamma_2^a(t)$, $t \in T$, which belong to $H_a$ is complete in $H_a$. By Lemma 5 (relation (8)), $\Phi$ maps the set of all functions of the form $\nu \gamma_2(t)a$, $t \in T_I$, $\nu \in C$ (a fixed), onto the set of all functions of the form $\nu \gamma_2^a(t)a$, $t \in T$, $\nu \in C$, which belong to $H_a$. This implies that the set of functions of the form $\gamma_2(t)a$, $t \in T_I$, is complete $K_a$. If a function $f$ in $H$ is orthogonal to all subspaces of the type $\gamma_2 E$, $t \in T_I$, it is orthogonal also to all functions of the form $\gamma_2(t)a$, $t \in T_I$, $a \in E$. Since the above set of functions for fixed $a$ is complete in $K_a$, that implies $f \perp K_a$ for every $a \in E$. However, this implies that $(I - \theta(w))^{-1} f(w), a) = (f, K_w(I - \theta(w)^{-1})a) = 0$ for every $a \in E$ and every $w \in D$, so that $f = 0$. Thus, the set $G$ is complete in $H$. This completes the proof.

Remark 2. If the function $\theta$ admits analytic continuation across some point $t \in T$ and if $\theta(t) = U$, then $\gamma a \in H$ for every $a \in E$ and $\gamma a$ is obtained by evaluation of the (analytically continued) reproducing kernel $K_w(z)$ for $w = t$. In the general case the situation is, in a sense, similar. Namely, it follows easily by (7) that, for $t \in T_I$, $a \in E$ and $z \in D$, $\lim_{r \to 1^-} K_{rt}(z)a(t)a = \gamma(z)\alpha(t)a$ in the $E$-norm. With the help of the last relation $K_w(z)$ can be extended for every $t \in T_I$ along the radius $\{rt \mid 0 \leq r \leq q\}$ at least as an operator function with values in
the set of bounded operators from $\alpha(t)E$ into $\alpha(t)E$, so that we can consider $\gamma_t(z)$ also in the general case as an evaluation of $K_w(z)$ for $w = t$.

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