ON A PROOF OF THE ERDŐS-MONK THEOREM

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Abstract. We prove an elementary proposition of combinatorial analysis, which with some use of model theory of Boolean algebras gives immediately the Erdős-Monk theorem. We shall prove also a generalization of this theorem.

Assuming the Continuum Hypothesis (CH) holds, Erdős and Monk proved [2] that \( P(\omega)/I_0 \cong P(\omega)/I \), where \( P(\omega) \) is the Boolean algebra of all subsets of \( \omega \)-the set natural numbers, and \( I_0, I \) are the following ideals of \( P(\omega) \):

\[
I_0 = \{ a \subseteq \omega : a \text{ is finite} \}, \quad I = \{ a \subseteq \omega : \sum_{n \in a} 1/n < \infty \},
\]

1. An elementary statement of combinatorial analysis.

If \( f, g : \omega \to 2 \), \( 2 = \{0, 1\} \), then \( f \leq g \) denotes \( \forall n \in \omega f(n) \leq g(n) \). The following proposition may have an independent interest, so this is the reason why we extracted it.

**Theorem 1.1.** 1° Let \( f_n \in 2^\omega \), \( n \in \omega \), be a sequence of functions such that

\[
(1) \ldots f_2 \leq f_1 \leq f_0, \quad (2) \sum_{f_i(n)=1} 1/n = \infty, \quad i \in \omega.
\]

Then there is an \( h \in 2^\omega \) such that

\[
(1') \sum_{h(n)=1} 1/n = \infty, \quad (2') \sum_{f_i(n)<h(n)} 1/n < \infty, \quad i \in \omega.
\]

2° Let \( f_n, g_n \in 2^\omega \), \( n \in \omega \), be two sequences of functions such that

\[
(3) g_0 \leq g_1 \leq g_2 \leq \cdots \text{ and } \sum_{f_i(n)<g_i(n)} 1/n < \infty, \quad i \in \omega.
\]

Then there is an \( h \in 2^\omega \) such that

\[
(3') \sum_{h(n)<g_i(n)} 1/n < \infty, \quad i \in \omega, \quad (4') \sum_{f_i(n)<h(n)} 1/n < \infty, \quad i \in \omega.
\]

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Proof. 1° Let \( a_i = \{n \in \omega : f_i(n) = 1\}, n \in \omega \). Then by the assumption on the functions \( f_n \), we have

\[ a_0 \supseteq a_1 \supseteq a_2 \supseteq \cdots \]  

Define a sequence \( b_n \subseteq \omega \) by induction in the following way. Let \( b_0 \subseteq a_0 \) be the (finite) subset of first elements in \( a_0 \) such that \( \sum_{n \in b_0} 1/n \geq 1 \). Let \( b_{i+1} \) be the subset of first elements in \( a_{i+1} \) \( - (b_0 \cup \cdots \cup b_i) \) so that \( \sum_{n \in b_{i+1}} 1/n \geq 1 \), \( i \in \omega \). The sets \( b_i \) exist by (2) and (5). Let \( b = \cup_i b_i \), and define \( h \) to be characteristic function of \( b \). Then

\[ \sum_{n \in b} 1/n = \sum_{n \in b_i} 1/n = \sum_{i \in \omega} \sum_{n \in b_i} 1/n = \infty \]

i. e. (1') holds. Further, \( \{n \in \omega : f_i(n) < h(n)\} \subseteq b_0 \cup \cdots \cup b_i \}; \) so, the sum

\[ \sum_{n \in b_i} 1/n \]

is finite, i. e. (2') holds.

2° By (4) there exists a strictly increasing sequence \( 0 < s_0 < s_1 < s_2 \cdots \) of natural numbers such that

\[ \sum_{f_k(n) < g_k(n)} 1/n \leq 1/(k+1)^2, \quad k \in \omega. \]

Let \( h \in 2^\omega \) defined by

\[ h(n) = \begin{cases} 0 & \text{if } n < s_0 \\ g_k(n) & \text{iff } s_k \leq n < s_{k+1}. \end{cases} \]

Then

\[ \sum_{n \in b_0} 1/n = \sum_{h(n) < g_k(n)} 1/n + \sum_{h(n) < g_k(n)} 1/n = A + B, \]

Then \( A \) is a finite sum and \( B = 0 \); so (3') holds. Furthermore, let

\[ \sum_{f_k(n) < h(n)} 1/n = \sum_{f_k(n) < h(n)} 1/n + \sum_{k \leq i f_k(n) < h(n)} 1/n = A + B, \]

Then \( A \) is a finite sum, and \( s_k < \infty \). Furthermore,

\[ B = \sum_{k \leq i f_k(n) < g_k(n)} 1/n \leq \sum_{k \leq i f_k(n) < g_k(n)} 1/n \leq \sum_{k \leq i} 1/(i+1)^2 < \infty \]

i. e. (4') holds.
2. $\omega_1$-saturated Boolean algebras

In [3; Prop. 2.27] it is proved that an atomless Boolean algebra $B$ is $\omega_1$-saturated iff $B$ satisfies the following condition:

$H_{\omega_1}$ (1) If $0 < \cdots < a_2 < a_1 < a_0$ is a sequence of elements of $B$, then there exists a $c \in B$ such that $0 < c < a_n$, $n \in \omega$.

(2) If $0 < a_0 < a_1 < \cdots \cdots < b_1 < b_0$ are two sequences of elements in $B$, then there is a $c \in B$ such that $a_n < c < b_n$, $n \in \omega$.

Using $H_{\omega_1}$ we proved in [3; Example 2.28] that

(1) $P(\omega)/I_0$ is an $\omega_1$-saturated Boolean algebra.

Let $D$ be the dual filter of i. e. $D = \{a^c : a \in I\}$. We first observe that

(2) If $f_l, g_l \in 2^\omega/I$ are such that $f_l \leq g_l$, then there is a $h \in 2^\omega$ such that $f_l = h_l$ and $h \leq g$.

To see that, let $a = \{i \in \omega : f(i) \leq g(i)\}$. Then $a \in D$, and the function $h$ defined by $h(i) = f(i)$ if $i \in a$, and $h(i) = g(i)$ if $i \notin a$, satisfies the required condition.

Let $f_l, g_l \in P(\omega)/I$ be such that $f_l < g_l$. By (2) we may assume that $f \leq g$. Since $f_l < g_l$ we have $f_i \neq g_l$, i. e. $\{i \in \omega : f(i) = g(i)\} \notin D$, so $\{i \in \omega : f(i) \neq g(i)\} \notin I$. As $f \leq g$, then $f(i) \neq g(i)$ implies $f(i) < g(i)$, so

$$\sum_{f(i) < g(l)} 1/n = \infty.$$ 

(3) If $f \leq g$, then $f_l < g_l$ is equivalent to

$$\sum_{f(n) < g(n)} 1/n = \infty.$$ 

Finally, for $f, g \in 2^\omega$ we have $f_l \leq g_l$ iff $\{n : g(n) \leq f(n)\} \in D$ iff $\{n : g(n) \leq f(n)\}^c \in I$ iff $\{n : f(n) < g(n)\} \in I$ iff

$$\sum_{f(n) < g(n)} 1/n < \infty,$$ 

(4) $g_l \leq f_l$ is equivalent to

$$\sum_{f(n) < g(n)} 1/n < \infty.$$ 

Using (2), (3), (4) and Theorem 1.1 it follows immediately that $P(\omega)/I$ satisfies the condition $H_{\omega_1}$, therefore we have

**Theorem 2.1.** $P(\omega)/I$ is an atomless $\omega_1$-Boolean algebra.

If $CH$ is assumed, then $|P(\omega)/I| =|P(\omega)/I_0| = \omega_1$; so $P(\omega)/I_0$ and $P(\omega)/I$ are saturated Boolean algebras of the complete theory of atomless Boolean algebras; therefore by uniqueness of elementary equivalent saturated models of the given cardinality [1; Theorem 5.1.13] we have at once

**Corollary 2.2.** If $CH$ is assumed, then $P(\omega)/I_0 \cong P(\omega)/I$. 
Let us now give a generalization of the Erdős-Monk theorem. In [4] the notion of saturative filters is introduced. A filter $F$ over a set $J$ is $k$-saturative iff for every family of models $A_i$, $i \in J$, the reduced product $\prod_{i \in J} A_i / F$ is $k$-saturated. In [4] it is proved

**Theorem 2.3.** Assume $F$ is a filter over a set $J$. Then $F$ is $k$-saturative ($k > \omega$) iff $F$ satisfies the following conditions: 1° $F$ is $k$-good, 2° The reduced product $2^J / F$ is $\omega_1$-saturated, 3° $F$ is incomplete.

As the proof of Lemma 4.2.2. in [3] shows, every filter over $\omega$ is $\omega_1$-good. Since $I_0 \subseteq I$ by Theorem 2.1 and Theorem 2.2 we have

**Proposition 2.4.** The dual filter $D$ of $I$ is $\omega_1$-saturative.

**Corollary 2.5.** Let $B_i \in \omega$, be the Boolean algebras. Then

1° $\prod_i B_i / D$ is an $\omega_1$-saturated Boolean algebra.

2° If CH is assumed and if for all $i \in \omega$ \mid $B_i \mid \leq \omega_1$, then $\prod_i B_i / D$ is an atomless saturated Boolean algebra of cardinality $\omega_1$, and therefore $\prod_i B_i / D \cong 2^\omega / D(= (P\omega) / I)$ if subsets of $\omega$ are identified by their characteristic functions.

**References**


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