A FIXED POINT THEOREM IN REFLEXIVE BANACH SPACES

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In this note we shall prove the following fixed-point theorem.

**Theorem.** Let $B$ be a reflexive Banach space, $K$ a nonempty bounded, closed and convex subset of $B$ and let $T : K \to K$ be a map such that

$$
\text{diam } [T(D)] < \text{diam } (D)
$$

holds for every closed and convex subset $D$ of $K$, containing more than one element and mapped into itself by $T$. Then $T$ has a fixed point in $K$.

**Proof.** Let $\mathcal{F}$ denote a family of all non-empty closed and convex subsets of $K$ which $T$ maps into itself. Then using the result of Smulian [7, p. 327] and Zorn's Lemma it follows that $\mathcal{F}$ has a minimal element, say $C$. Since $T(C) \subseteq C \in \mathcal{F}$ it follows that $\text{Cl} [\text{co } T(C)] \subseteq C$ and hence

$$
T(\text{Cl} [\text{co } T(C)]) \subseteq T(C) \subseteq \text{Cl} [\text{co } T(C)].
$$

This implies $\text{Cl} [\text{co } T(C)] \in \mathcal{F}$ and by the minimality of $C$ we have

$$
\text{Cl} [\text{co } T(C)] = C.
$$

As $\text{diam } (\text{co } S) = \text{diam } (S)$ for every subset $S$ of $K$ [5, p. 17], (2) implies

$$
\text{diam } [T(C)] = \text{diam } (C).
$$

Now, using (1) we conclude that $C$ is a singleton, say $C = u$. Therefore, $u$ is a fixed-point of $T$, and the proof is complete.

We remark that maps considered in [2], [4] and [6] satisfy the condition (1), and therefore our theorem is a certain generalization of corresponding fixed-point theorems. We shall illustrate this on a theorem given in [6].

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Theorem A [6]. Let $K$ and $D$ be as in the previous theorem and $T : K \to K$ mapping satisfying the following conditions:

\[ \|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|, a\|x - Ty\| + b\|y - Tx\|, \}
\]

\[(\|x - y\| + \|x - Tx\| + \|y - Ty\|)/3; \]

\[ x, y \in K, \ a \geq 0, \ b \geq 0, \ a + b < 1, \]

\[ \sup_{z \in D} ||z - Tz|| < r \text{ diam } (D), \ 0 < r = r(D) < 1. \] (5)

Then $T$ has a unique fixed point in $K$.

Proof. If $\text{diam } (D) > 0$, then by (4) and (5) for every $x, y \in D$ we have

\[ ||Tx - Ty|| \leq \max\{r, (a + b), (1 + 2r)/3\} \cdot \text{diam } (D) < \text{diam } (D). \]

Therefore, $T$ satisfies (1), and by the previous theorem, $T$ has a fixed point in $K$. Since condition (4) implies that $T$ may have at most one fixed point, the proof is complete.

REFERENCES


