NONEXISTENCE OF NONMOLECULAR GENERIC SETS

Donald D. Steiner and Alexander Abian

Abstract. Generic subsets of partially ordered sets play an important role in giving significant examples of Zermelo-Fraenkel set-theoretical models. The significance of these models lies in the fact that a generic subset \( G \) of a partially ordered set \( P \), in general, does not exist in a model \( M \) in which \( P \) exists. Thus, by adjoining \( G \) to \( M \) an interesting extended model may ensue which has properties not shared by \( M \). Thus, in considering generic extensions of set-theoretical models it is quite relevant to know whether or not a generic subset of a partially ordered set \( P \) exists in the same model in which \( P \) exists. In this paper, we give a necessary and sufficient condition for \( P \) to have a generic subset in the same model.

Let \( (P, \leq) \) be a partially ordered set. As usual, when no confusion is likely to arise, we represent \( (P, \leq) \) simply by \( P \). If \( P \) has a minimum (i.e., the smallest) element, we represent it by 0 and we call it the zero element of \( P \).

DEFINITION 1. A subset \( D \) of a partially ordered set \( (P, \leq) \) is called a dense (or, a coinitial) subset of \( P \) if and only if for every nonzero element \( x \) of \( P \) there exists a nonzero element \( y \) of \( D \) such that \( y \leq x \).

It is an interesting fact that, as shown in [1], a partially ordered set \( P \) has either finitely many or else continuum many dense subsets. Clearly, every nonzero minimal element of \( P \) is an element of every dense subset of \( P \). Clearly, every nonzero minimal element of \( P \) is an element of every dense subset of \( P \). On the other hand, if 0 of \( P \) exists, 0 need not be an element of every dense subset of \( P \). However, if \( D \) is a dense subset of \( P \) then \( D - \{0\} \) as well as \( D \cup \{0\} \) is a dense subset of \( P \). Indeed, every superset of a dense subset of \( P \) is a dense subset of \( P \).

Based on the notion of a dense subset of a partially ordered set, we introduce the notion of a generic subset of a partially ordered set as follows:

DEFINITION 2. Let \( (P, \leq) \) be a partially ordered set. A subset \( G \) of \( P \) is called a generic subset of \( P \) (or, simply generic) if and only if: (1) \( 0 \notin G \) and \( G \) is nonempty; (2) For every element \( x \) and \( y \) of \( P \) if \( x \in G \) and \( x \leq y \) then \( y \in G \); (3) Every two elements of \( G \) have a lower bound in \( G \); (4) \( G \) has a nonempty intersection with every dense subset of \( P \).

AMS Subject Classification (1980): Primary 06A10.
We observe that by (1) and (3) every two elements of \( G \) have a nonzero lower bound in \( G \). Also, we observe that (1), (2) (3) imply that \( G \) is a filter. Therefore, a generic subset \( G \) of a partially ordered set \( P \) is a filter of \( P \) such that \( G \) has a nonempty intersection with every dense subset of \( P \).

As shown below, not every partially ordered set has a generic subset. In fact, according to Theorem 1 below, a partially ordered set has a generic subset if and only if it has a molecule. In the literature, a modified notion of a generic subset is usually considered where (4) is replaced by:

\((4a)\) \( G \) has a nonempty intersection with every dense subset of \( P \) belonging to a preassigned collection \( M \) of dense subsets of \( P \).

If in Definition 1, condition (4) is replaced by (4a), then \( G \) is called \emph{generic over} \( M \) and is denoted by \( G/M \). It can be readily verified \[3\], that if \( M \) is a denumerable collection \( \{D_0,D_1,D_2,D_3,\ldots\} \) of dense subsets \( D_i \) of \( (P,\leq) \), then \( P \) always has a \emph{generic over} \( M \) subset \( G \) with \( p \in G \) for any nonzero element \( p \) of \( P \). Indeed, since \( D_i \)'s are dense in \( P \) there exists a nonincreasing sequence:

\[
\cdots \leq d_3 \leq d_2 \leq d_1 \leq d_0 \leq p
\]

where \( d_i \in D_i \). But then clearly \( G = \{x \mid x \in P \text{ and } d_i \leq x \text{ for some } i\} \) is a generic over \( M \) subset of \( P \).

**Definition 3.** A nonzero element \( m \) of a poset \( P \) is called a molecule of \( P \) if and only if every two nonzero elements of \( P \) which are less than or equal to \( m \) have a nonzero lower bound.

Clearly, every nonzero element of a simply ordered set \( S \) is a molecule of \( S \).

A typical partially ordered set without a molecule is provided with the following example. Let \( H \) be the set of all finite sequences of the natural numbers \( 0,1,2,3,\ldots \). Let us consider the partial order \( (H,\leq) \) where \( p \leq q \) if and only if \( p \) is an extension of \( q \). Thus, \((0,3,5,2,1) \leq (0,3,5) \) and \((0,3,5,7) \leq (0,3,5). \) It can be readily verified that \( (H,\leq) \) has no molecule.

**Definition 4.** Let \( m \) be a molecule of a partially ordered set \( (P,\leq) \). Then a subset \( G(m) \) of \( P \) is called molecular, or generated by the molecule \( m \) if and only if:

\[
G(m) = \{x \mid (x \in P) \land (\exists y)((y \neq 0) \land (y \in P) \land (y \leq m) \land (y \leq x))\}
\]

Accordingly, \( G(m) \) consists of those elements of \( P \) each of which is greater than or equal to some element of \( P \) which is less than or equal to \( m \).

From (5) it obviously follows that

\[
m \in G(m)
\]

Next, we prove \[cf. 4, p. 26\].

**Theorem 1.** A subset \( G \) of a partially ordered set \( (P,\leq) \) is generic if and only if \( G \) is generated by a molecule of \( P \).
Proof. Let $G$ be a generic subset of $P$. First, we show that there exists $m \in G$ such that

\[(7) \quad \{y \mid (y \in P) \land (0 \neq y) \land (y \leq m)\} \subseteq G \text{ and } m \text{ is a molecule of } P.\]

Assume on the contrary that for every $m \in G$ there exists a nonempty subset $N(m)$ of $P$ such that $z \in N(m)$ if and only if $z \leq m$ and $z \in (P - G)$. But then clearly, $P - G$ is a dense subset of $P$ which has an empty intersection with $G$, contradicting (4). Thus, the first part of (7) is established. Now, from this and (3) and the fact that $G$ is generic, it follows that $m$ is nonzero and that every two nonzero elements of $P$ which are less than or equal to $m$ have a nonzero lower bound in $G$ (and a fortiori in $P$). Thus, $m$ is a molecule of $P$, according to Definition 3 and the proof of (7) is complete. Next, we show that $G = G(m)$. Let $x \in G$. By (3), we see that $x$ and $m$ must have a lower bound, say, $y$ in $G$. Thus, $y \leq x$ and $y \leq m$. But then, from (5) it follows that $x \in G(m)$. But then, from (5) and (7), it follows that $x \geq y$ for some $y \in G$ which by (2) implies that $x \in G$. Hence, indeed $G = G(m)$.

To complete the proof of the theorem, it remains to show that $G(m)$ as given by (5) is a generic subset of $P$.

We observe that (1) follows directly from (5). To establish (2), it is enough to observe that if $z \in G(m)$ then by (5) we see that $z \geq y$ for some nonzero $y \leq m$. Therefore, if $z \leq x$ then $z \leq y$ with $y \leq m$, which implies $x \in G(m)$. To establish (3), it is enough to observe that if $x_1 \in G(m)$ and $x_2 \in G(m)$, then by (5) we see that $y_1 \leq x_1$ and $y_2 \leq x_2$ for some nonzero $y_1 \leq m$ and nonzero $y_2 \leq m$. But since $m$ is a molecule, $y_1$ and $y_2$ have a nonzero lower bound, which by (5) is an element of $G(m)$. Hence, $x_1$ and $x_2$ have a lower bound in $G(m)$. To establish (4), let $D$ be a dense subset of $P$. But then $D$ has a nonzero element $d$ such that $d \leq m$. From (5) it follows that $d \in G(m)$ and therefore $G(m)$ has a nonempty intersection with every dense subset of $P$.

Corollary 1. If $k$ is a nonzero minimal element of a partially ordered set $(P, \leq)$ then $k$ is a molecule of $P$ and \( \{x \mid x \in P \text{ and } k \leq x\} \) is a generic subset of $P$.

Proof. Since $k$ has no nonzero predecessors, $k$ is trivially a molecule of $P$. But then the conclusion of the corollary follows from (5).

Corollary 2. A partially ordered set has a generic subset if and only if it has a molecule.

This is an immediate consequence of Theorem 1 and (6). Accordingly, the partial order $(H, \leq)$ mentioned above, has no generic subset.

Let us recall the following:

Definition 5. A Boolean algebra $(A, \leq)$ is a complemented distributive lattice with a minimum 0 and a maximum 1. Moreover, a nonzero element $a$ of $A$ is called an atom of $A$ if and only if for every $x \in A$ it is the case that $x < a$ implies $x = 0$. 

Nonexistence of nonmolecular generic sets
Furthermore, a subset $U$ of $A$ is called an ultrafilter of $A$ if and only if:

(8) $0 \notin U$; (9) For every element $x$ and $y$ of $A$ if $x \in U$ and $x \leq y$ then $y \in U$;
(10) Every two elements of $U$ have a lower bound in $U$; (11) For every element $x$ of $A$ either $x \in U$ or else $x' \in U$ where $x'$ is the complement of $x$.

The reader is advised to compare (8), (9), (10), (11) with (1), (2), (3), (4).

Clearly, $1 \in U$ by (11) so that $U$ is nonempty.

**Lemma 1.** Let $(A, \leq)$ be a Boolean algebra. An element $a$ of $A$ is an atom of $A$ if and only if $a$ is a molecule of $A$.

**Proof.** Let $a$ be an atom of $A$. Clearly, $a$ is a nonzero minimal element of $A$ and therefore by Corollary 1, we see that $a$ is a molecule of $A$. Conversely, let $a$ be a molecule of $A$. To prove that $a$ is an atom of $A$ it is enough to show that $x < a$ for no nonzero element $x$ of $A$. Assume on the contrary that $x < a$ for some nonzero element $x$ of $A$. But then, since $A$ is a Boolean algebra $a - x$ exists, is nonzero and $(a - x) < a$. However, from Definition 3 it follows that $x$ and $a - x$ must have a nonzero lower bound. But this leads to a contradiction since the only lower bound of $x$ and $a - x$ in $A$ is $0$. Thus, our assumption is false and $a$ is an atom of $A$.

Definitions 2 and 5 indicate that an ultrafilter of a Boolean algebra somewhat resembles a generic subset of it. However, in view of the discrepancies between (4) and (11), we must not expect that every ultrafilter of a Boolean algebra is also a generic subset of it. Indeed, in view of Theorem 1 and Lemma 1, we have:

**Theorem 2.** A subset $G$ of a Boolean algebra $A$ is generic if and only if $G$ is generated by an atom of $A$ (i.e., if and only if $G$ is a principal ultrafilter of $A$).

**Proof.** By Theorem 1 and Lemma 1, we see that $G$ must be generated by an atom $a$ of $A$ which in view of (5) implies:

$$G = G(a) = \{x \mid x \in A \text{ and } a \leq x\}$$

But then it is a routine matter to verify that the above equality implies that $G(a)$ is an ultrafilter of $A$ (as defined by (8), (9), (10), (11)) generated by the atom $a$ of $A$.

The following lemmas show some significant properties of dense subsets of a Boolean algebra.

**Lemma 2.** Let $(A, \leq)$ be a Boolean algebra and $D$ a dense subset of $A$. Then $\text{hub } D = 1$.

**Proof.** Clearly, $1$ is an upper bound of $D$. To prove the lemma we show that every upper bound $u$ of $D$ is equal to $1$. Assume on the contrary that $u$ is an upper bound of $D$ and $u < 1$. But then $1 - u$ (i.e., the complement $u'$ of $u$) is a nonzero element of $A$. However, no element of $D$ is less than or equal to $1 - u$, contradicting the denseness of $D$. Hence $u = 1$, as desired.

**Lemma 3.** Let $(A, \leq)$ be a Boolean algebra and $H$ a subset of $A$ such that $\text{hub } H = 1$. Then the subset $D$ of $A$ given by

$$D = \{x \mid x \in A \text{ and } x \leq h \text{ for some } h \in H\}$$

(12)
is a dense subset of $A$.

Proof. We must show that for every nonzero element $p$ of $A$ there exists a nonzero element $d$ of $D$ such that $d \leq p$. Since $\text{lub } H = 1$ and since $(A, \leq)$ is a Boolean algebra, we see that

\begin{equation}
(13) \quad p = p \wedge (\text{lub } H) = \text{lub} (p \wedge h)
\end{equation}

Since $p \neq 0$, from (13) it follows that $(p \wedge h) \neq 0$ for some $h \in H$. Let $d = p \wedge h$. Thus, $d$ is nonzero and since $d \leq h$, we see by (12) that $d \in D$. Clearly, $d \leq p$ as desired.

As mentioned earlier, we called a nonempty subset $G$ of a partially ordered set a filter if and only if $G$ satisfies (1), (2), (3). Very often, in the literature [2], condition (3) is replaced by “the greatest lower bound of every two elements of $G$ exists and is an element of $G$”. However, this point is immaterial for our purposes.

Theorem. Let $A$ be a Boolean algebra. Then a filter $G$ of $A$ is a generic subset of $G$ if and only if for every family $(a_i)_{i \in E}$ of $A$ it is the case that

\begin{equation}
(14) \quad \text{lub}_{i \in E} a_i = 1 \text{ implies } a_i \in G \text{ for some } i \in E
\end{equation}

Proof. Let $G$ be a generic subset of $A$. But then by Theorem 2 we see that $A$ has an atom $a$ and $a \in G$. Now, let $\text{lub}_{i \in E} a_i = 1$. Since $A$ is a Boolean algebra, we have:

\begin{equation}
(15) \quad a = a \wedge (\text{lub}_{i \in E} a_i) = \text{lub}(a \wedge a_i) \text{ with } i \in E
\end{equation}

However, since $a$ is an atom, $a \wedge a_i = 0$ or $a \wedge a_i = a$. But then from (15) it follows that $a \wedge a_i = a$ for some $i \in E$. Hence, $a \leq a_i$ and since $a \in G$ and $G$ is a filter, $a_i \in G$ by (2). Thus, (14) is established.

Conversely, let $G$ be a filter of $A$ satisfying (14). We show that $G$ is a generic subset of $A$. To this end, in view of (4) it is enough to prove that $G$ has a nonempty intersection with every dense subset of $A$. Now, let $D$ be a dense subset of $A$. By Lemma 2 we see that $\text{lub } D = 1$ and by (14) we derive that $d \in G$ for some $d \in D$. Thus, indeed $G$ has a nonempty intersection with every dense subset of $G$, as desired. In view of Theorem 2, clearly $G$ in also a principal ultrafilter of $A$.

REFERENCES


MCC, 9430 Research Blvd., Austin, Texas 78759, U.S.A. (Received 14 11 1983)

Department of Mathematics, Iowa State University, Ames, Iowa 50011, U.S.A. (Revised 19 06 1984)