SOME SPECIAL PRODUCT SEMISYMMETRIC
AND SOME SPECIAL
HOLOMORPHICALLY SEMISYMMETRIC F-CONNECTIONS

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Abstract. In the present paper we investigate two special product semisymmetric F-connections: PS-concircular and PS-coharmonic ones and we find the conditions for product semisymmetric connection to be PS-concircular or PS-coharmonic. In the same manner we investigate two special holomorphically semisymmetric F-connections.

Introduction. The connection of an n-dimensional differentiable manifold is termed semisymmetric if its torsion tensor $S$ satisfies

\[ S^i_{jk} = \delta^i_j S_k - \delta^i_k S_j. \]

A semisymmetric connection is generalized on a locally decomposable Riemannian space in [4] and [5]. In [4] the product semisymmetric metric F-connection is defined and studied. In [5] something similar is done for the holomorphically semisymmetric F-connection.

In the present paper we investigate two special holomorphically semisymmetric and two special product semisymmetric F-connections. In 1 we recall what is a product semisymmetric F-connection. In 2 we define PS-concircular connections and prove that each of the relations (2.7), (2.9) and (2.17) is a necessary and sufficient condition for a product semisymmetric connection to be PS-concircular. In 3 we define PS-coharmonic connections and prove that a product semisymmetric connection is PS-coharmonic iff (3.3) holds. In 4 we recall what is a holomorphically semisymmetric connection. In 5 we define HS-concircular connection and prove that each of the relations (5.6) and (5.8) is a necessary and sufficient condition for a holomorphically semisymmetric connection to be HS-concircular. In 6 we investigate another HS-connection.

These results generalize, for the locally decomposable Riemannian space, the results obtained by P. Strave in [6].

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1. **Product semisymmetric metric F-connection.** An n-dimensional differentiable manifold $M_n$ of class $C^\infty$ is called a locally decomposable Riemannian space [8] if in $M^n$ a tensor field $F^i_j \neq \delta^i_j$ and a positive definite Riemannian metric $ds^2 = g_{ij}(x^k)dx^i dx^j$ are given, satisfying the conditions

(1.1) \[ F^i_k F^k_j = \delta^i_j, \quad g_{ab} F^a_i F^b_j = g_{ij}, \quad \nabla_k F^i_j = 0, \]

where $\nabla_k$ is the operator of the covariant derivative with respect to the Riemannian metric. If we put $F^i_k g_{ab} = F_{abij}$, then $F_{bij} = F_{jib}$ and the condition $\nabla_k F^i_j = 0$ is equivalent to the condition $\nabla_k F_{ij} = 0$.

Locally decomposable space can be covered by a separating coordinate system, that is, by such a system of coordinate neighborhoods $(x^i)$ that in any intersection of two coordinate neighborhoods $(x^i)$ and $(x^\tilde{i})$ we get

\[ x^a = x^a(x^i), \quad x^\tilde{b} = x^\tilde{b}(x^\tilde{i}), \]

where the indices $a, b, c$ run over the range $1, 2, \ldots, p$ and the indices $x, y, z$ run over the range $p + 1, \ldots, p + q = n$.

With respect to a separating coordinate system, the metric of the space has the form $ds^2 = g_{ab}(x^c)dx^a dx^b + g_{xy}(x^z)dx^x dx^y$, while the tensors $F_{abij}$ and $F_{aij}$ have the forms:

\[ (F_{abij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\delta^0_0 \quad 0 \quad -\delta^0_y) \]

Therefore

(1.2) \[ \varphi \equiv p - q. \]

In the following we suppose $p > 2, q > 2$.

The product semisymmetric metric F-connection (PS-connection) has the form [4]:

(1.3) \[ \Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_j S_k - g_{jk} S^i + F^a_{ij} S^a_{jk} - S^a F^i_{jk}, \]

where $S_i$ is a decomposable vector field in $M_n$, i.e. the field satisfying the condition

(1.4) \[ F^a_{ij} \nabla_j S_i + i = F^a_{ij} \nabla_j S_a, \]

and $S^i = g^{ab} S_a$. This condition can be expressed in the form

(1.5) \[ F^a_{ij} F^i_{jk} \nabla_k S_a = \nabla_j S_i. \]

Also, we suppose in the following that $S_i$ is locally a gradient vector field.

With respect to a separating coordinate system, all the $\Gamma^i_{jk}$ are zero except

\[ \Gamma^a_{bc} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + 2S_i \delta^a_i - 2g_{bc} S^a, \quad \Gamma^x_{yz} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2S_i \delta^x_i - 2g_{xy} S^x, \]
that is, both connections $\Gamma_{bc}^a$ and $\Gamma_{b}^z$ are semisymmetric metric connections: the first with respect to the metric $g_{ab}(x^c)dx^a dx^b$, the second with respect to the metric $g_{xy}(x^z)dx^x dx^y$.

The curvature tensor $R^i_{rjk}$ of the connection (1.3) can be expressed in the form [4]

$$
R^i_{rjk} = K^i_{rjk} + \delta^i_j \rho_{rk} - g_{jr} \rho_k^i + F^i_{jr} F^a_r \rho_{ak} - F^i_{jr} g^b_k F^a_b \rho_{ak} - \\
- \delta^i_k \rho_{rj} + g_{kr} \rho_j^i - F^i_{kr} F^a_r \rho_{aj} + F^i_{kr} g^b_r F^a_b \rho_{aj},
$$

where

$$
\rho_{rk} = \nabla_r S_k - S_r S_k + S^a S_a g_{r k}/2 - S_r S_k F^a_r F^b_k + S^a S_a F^b_k F_{r k}/2,
$$

and $K^i_{rjk}$ is the curvature tensor of the Riemannian space having $g_{ij}$ as a metric tensor. Since $S_i$ is a gradient, tensor $\rho_{rk}$ is symmetric. Also, it satisfies the condition

$$
\rho_{ab} F^a_i F^b_j = \rho_{ij},
$$

because of (1.5). If we put

$$
R_{rk} = R^a_{rka}, \quad R^*_r = R^a_{ra} F^a_r, \quad K_{rk} = K^a_{rka}, \quad K^*_r = K^a_{ra} F^a_r,
$$

we obtain from (1.6), using (1.8)

$$
R_{rk} = K_{rk} + (n - 4) \rho_{rk} + \varphi^a_r \rho_{ak} + g_{rk} \rho^a_a + F^a_r \rho^b_k + \\
F_{rk} F^b_k \rho^a_r,
$$

$$
R^*_r = K^*_r + (n - 4) \rho_{ra} F^a_r + \varphi^a_r \rho_{rk} + F^a_r \rho^b_a + g_{rk} F^b_k \rho^a_r.
$$

The Ricci tensor $K_{ij}$ of the locally decomposable Riemannian space is a pure tensor, i.e. $K_{ab} F^a_i F^b_j = K_{ij}$, from which $K_{aj} F^a_i = K_{bj} F^b_j$, i.e. $K^*_i$ is a symmetric tensor. The tensor $\rho_{ak} F^b_r$ is a symmetric tensor too, because of (1.8). Therefore, both $R_{rk}$ and $R^*_r$ are symmetric tensors.

We obtain from (1.6)

$$
R_{rjk} = K_{rjk} + g_{i j} \rho_{rk} - g_{jr} \rho_{ik} + F^a_{i j} F^a_r \rho_{ak} - F^a_{jr} F^a_r \rho_{ak} - \\
+ g_{ik} \rho_{rj} - F^a_{kr} \rho_{ij} + F^a_{kh} \rho_{aj} + F^a_{kr} \rho_{aj},
$$

and see that

$$
R_{rikj} = - R_{rjk}.
$$

Eliminating $\rho_{ij}$ from (1.6), we obtain [4]

$$
R^{i}_{rjk} + b [s^{i}_{rjk} - 2(b R - a R^*) r^{i}_{rjk} + 2(a R - b R^*) r^{i}_{akj} F^a_r] - \\
- a [s^{i}_{akj} F^a_r - 2(b R - a R^*) r^{i}_{akj} F^a_r + 2(a R - b R^*) r^{i}_{rjk}] = \\
= K^{i}_{rjk} + b [s^{i}_{rjk} - 2(b K - a K^*) r^{i}_{rjk} + 2(a K - b K^*) r^{i}_{akj} F^a_r] - \\
- a [s^{i}_{akj} F^a_r - 2(b K - a K^*) r^{i}_{akj} F^a_r + 2(a K - b K^*) r^{i}_{rjk}],
$$
where we have put

\[ R = R_a^a, \quad K = K_a^a, \quad R^* = R_a^a, \quad K^* = L_a^a, \]

\[ s_{rkj} = \delta^i_j R_{rk} + g_{rk} R_{kj}^i + F_j^i R_{rk}^* - F_{rk} R_{kj}^i - \]

\[ -\delta^i_j R_{rk} + g_{rk} R_{kj}^i - F_j^i R_{rk}^* + F_{rk} R_{kj}^i, \]

\[ s_{rkj} = \delta^i_j K_{rk} - g_{rk} K_{kj}^i - F_j^i K_{rk}^* - F_{rk} K_{kj}^i - \]

\[ -\delta^i_j K_{rk} + g_{rk} K_{kj}^i - F_j^i K_{rk}^* + F_{rk} K_{kj}^i, \]

\[ r_{rk} = \delta^i_j g_{rk} - \delta^i_j g_{rk} + F_j^i F_{rk} - F_j^i F_{rk} \]

\[ a = \frac{\varphi}{2[\varphi^2 - (n-2)^2]}, \quad b = \frac{n-2}{2[\varphi^2 - (n-2)^2]}, \]

\[ a_i = \frac{\varphi}{\varphi^2 -(n-4)^2}, \quad b_i = \frac{n-2}{\varphi^2 -(n-4)^2}. \]

2. Product semisymmetric concircular connection. In this section we consider a special product semisymmetric metric F-connection, namely the connection (1.3) satisfying the condition

\[ (2.1) \quad \rho_{ik} = f g_{ik} + h F_{ik}, \]

where \( f \) and \( h \) are some scalar functions. From (1.6) and (2.1) it results that

\[ R_{rkj}^i = K_{rkj}^i + 2f[\delta^j_i g_{kr} - \delta^j_i g_{rk} + F_j^i F_{kr} - F_j^i F_{kr}] + \]

\[ + 2h(\delta^j_i F_{kr} - \delta^j_i F_{rk} + F_j^i g_{kr} - F_j^i g_{kr}). \]

Contracting (2.2) with respect to \( i \) and \( j \) we obtain

\[ (2.3) \quad R_{rk} = K_{rk} + 2f[(n-2)g_{kr} + \varphi g_{kr}] + 2h((n-2)F_{kr} + \varphi F_{kr}). \]

It follows from (2.3) that

\[ (2.4) \quad R^*_{rk} = K_{rk} + 2f[(n-2)F_{kr} + \varphi g_{kr}] + 2h((n-2)g_{kr} + \varphi F_{kr}). \]

Transvecting (2.3) and (2.4) with \( g^{rk} \), we find that

\[ R - K = 2f[n(n-2) + \varphi^2] + 4h\varphi(n-1), \]

\[ R^* - K^* = 4f\varphi(n-1) + 2h[n(n-2) + \varphi^2]. \]

Since \( p > 2 \) and \( q > 2 \), \( n(n-2) + \varphi^2 \neq 0 \), and the above relations give

\[ (2.5) \quad 2f = -\alpha(R - K) - \beta(R^* - K^*), \quad 2h = -\beta(R - K) - \alpha(R^* - K^*), \]

where

\[ \alpha = \frac{n(n-2) + \varphi^2}{[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2}, \quad \beta = \frac{2\varphi(n-1)}{[n(n-2) + \varphi^2]^2 - 4\varphi^2(n-1)^2}. \]
Substituting (2.5) into (2.2) and taking into account the notation (1.14), we get
\[
R_{r k} - (\alpha R + \beta R^*) r_{r k} - (\beta R + \alpha R^*) R_{r k} F^a_r = K_{r k} - (\alpha K + \beta K^*) r_{r k} - (\beta K + K^*) r_{r k} F^a_r. \tag{2.7}
\]

Conversely, we suppose that for (1.3) we have (2.7). Then, substituting \( R_{r k} - K_{r k} \) from (2.7) into (1.6), we obtain

\[
[\alpha(R - K) + \beta(R^* - K^*)] r_{r k} + \beta[(R - K) + \alpha(R^* - K^*)] r_{r k} F^a_r = \delta^a_f \rho_{r k} - \delta^a_k \rho_{r j} - g_{r j} \rho^a_k + g_{k r} \rho^a_j + F^a_k F^b_r \rho_{a k} - F^a_k F^b_r \rho_{a j} - F_{r j} g^b F^a_k \rho_{a k} + F_{k r} g^b F^a_l \rho_{a j}.
\]

Contracting this with respect to \( i \) and \( k \) and taking into account (1.8), we get

\[
(4 - n)\rho_{r j} - \varphi F^l_{r k} \rho_{a j} =
\]

\[
= \{ (n - 2)[\alpha(R - K) + \beta(R^* - K^*)] + \beta[(R - K) + \alpha(R^* - K^*)] + \rho^a_k \} g_{r j} +
\]

\[
+ \{ \varphi[\alpha(R - K) + \beta(R^* - K^*)] + (n - 2)[\beta(R - K) + \alpha(R^* - K^*)] + F^a_k F^b_r \} F_{r j}.
\]

or
\[
(4 - n)\rho_{r j} - \varphi F^l_{r k} \rho_{a j} = f_1 g_{r j} + h_1 F_{r j}, \tag{2.8}
\]

where we have put

\[
f_1 = (n - 2)[\alpha(R - K) + \beta(R^* - K^*)] + \beta[(R - K) + \alpha(R^* - K^*)] + \rho^a_k,
\]

\[
h_1 = \varphi[\alpha(R - K) + \beta(R^* - K^*)] + (n - 2)[\beta(R - K) + \alpha(R^* - K^*)] + F^a_k F^b_r \rho^a_k.
\]

From (2.8) we obtain

\[-\varphi \rho_{r j} + (4 - n) F^a_{r k} \rho_{a j} = h_1 g_{r j} + \varphi f_1 F_{r j}. \]

From (2.8) and this last equation we easily find that

\[
[(4 - n)^2 - \varphi^2] \rho_{r j} = [(4 - n) f_1 + \varphi h_1] g_{r j} + [(4 - n) h_1 + \varphi f_1] F_{r j}.
\]

Since \( p > 2 \) and \( g > 2 \), \( (4 - n) f_1 - \varphi^2 \neq 0 \), and we can express the preceding relation in the form

\[
\rho_{r j} = \frac{(4 - n) f_1 + \varphi h_1}{(4 - n)^2 - \varphi^2} g_{r j} + \frac{(4 - n) h_1 + \varphi f_1}{(4 - n)^2 - \varphi^2} F_{r j}.
\]

This shows that \( \rho_{r j} \) has the form (2.1). Therefore, we have

**Theorem 1.** The connection (1.3) satisfies (2.1) if and only if (2.7) holds.

The tensor on the right hand side of (2.7) is the product concircular curvature tensor [3]. Because of that we introduce the following

**Definition.** The connection (1.3) satisfying (2.1) is called PS-concircular (product semisymmetric concircular) connection.
Contracting (2.7) with respect to $i$ and $j$, we find
\[
R_{rk} + \varphi(\beta R + \alpha R^*) + (n - 2)(\alpha R + \beta R^*)g_{kr} + \\
+ [(n - 2)(\beta R + \alpha R^*) + \varphi(\alpha R + \beta R^*)]F_{kr} = \\
= K_{rk} + [\varphi(\beta K + \alpha K^*) + (n - 2)(\alpha K + \beta K^*)]g_{kr} + \\
+ [(n - 2)(\beta K + \alpha K^*) + \varphi(\alpha K + \beta K^*)]F_{kr}.
\]
(2.9)

Conversely, we suppose that for (1.3), we have (2.9). Then, substituting $R_{rk} - K_{rk}$ from (2.9) into (1.9), we obtain (2.8). Therefore, we have

**Theorem 2.** (1.3) is a PS-concircular connection iff (2.9) holds.

Using the abbreviation
\[
(2.10) \quad A = \alpha R + \beta R^*, \quad B = \beta R + \alpha R^*, \quad P = \alpha K + \beta K^*, \quad Q = \beta K + \alpha K^*.
\]
we express (2.9) in the form
\[
R_{rk} + [\varphi B + (n - 2)A]g_{kr} + [(n - 2)B + \varphi A]F_{kr} = \\
= K_{rk} + [\varphi Q + (n - 2)P]g_{kr} + [(n - 2)Q + \varphi P]F_{kr}
\]
From this, we easily obtain
\[
R_{rk}^* = [\varphi B + (n - 2)A]F_{kr} + [(n - 2)B + \varphi A]g_{kr} = \\
= K_{rk}^* + [\varphi + (n - 2)P]F_{kr} + [(n - 2)Q + \varphi P]g_{kr}.
\]
From these two relations it follows that
\[
\delta_j^i R_{rk} + [\varphi B + (n - 2)A]\delta_j^i g_{kr} + [\varphi A + (n - 2)B]\delta_j^i F_{kr} = \\
= \delta_j^i K_{rk} + [\varphi Q + (n - 2)P]\delta_j^i g_{kr} + [\varphi P + (n - 2)Q]\delta_j^i F_{kr},
\]
(2.11)
\[
F^j_i R_{rk} + [\varphi B + (n - 2)A]F^j_ig_{kr} + [\varphi A + (n - 2)B]F^j_ig_{kr} = \\
= F^j_i K_{rk} + [\varphi Q + (n - 2)P]F^j_ig_{kr} + [\varphi P + (n - 2)Q]F^j_ig_{kr},
\]
(2.12)
\[
\delta_j^i R_{rk}^* + [\varphi B + (n - 2)A]\delta_j^i g_{kr} + (n - 2)B + \varphi A]\delta_j^i g_{kr} = \\
= \delta_j^i K_{rk}^* + [\varphi Q + (n - 2)P]\delta_j^i F_{kr} + [(n - 2)Q + \varphi P]\delta_j^i g_{kr},
\]
(2.13)
\[
F^j_i R_{rk}^* + [\varphi B + (n - 2)A]F^j_ig_{kr} + (n - 2)B + \varphi A]F^j_ig_{kr} = \\
= F^j_i K_{rk}^* + [\varphi Q + (n - 2)P]F^j_ig_{kr} + [(n - 2)Q + \varphi P]F^j_ig_{kr}.
\]
(2.14)
We multiply (2.11) with $n - 2$ and (2.13) with $\varphi$ and subtract the second from the first. Then, taking into account the notations (1.15), we have
\[
A\delta_j^i g_{kr} + B\delta_j^i F_{kr} - P\delta_j^i g_{kr} - Q\delta_j^i F_{kr} = \\
2b(\delta_j^i R_{rk} - \delta_j^i K_{rk} - 2a(R_{rk} \delta_j^i - K_{rk} \delta_j^i).
\]
(2.15)
Now, we multiply (2.12) with $\varphi$ and (2.14) with $n - 2$ and subtract the first from the second. We find
\[
B F^j_ig_{kr} + A F^j_ig_{kr} - Q F^j_ig_{kr} - P F^j_ig_{kr} = \\
- 2a(F^j_i R_{rk} - F^j_i K_{rk}) + 2b(F^j_i R_{rk} - F^j_i K_{rk}).
\]
(2.16)
On the other hand, we can express (2.7), using the notation (2.10), in the form:

\[
R^i_{rke} = K^i_{rke} + (A\delta^i_k g_{jr} + B\delta^i_k F_{jr}) - (P\delta^i_k g_{kr} + Q\delta^i_k F_{kr}) - \\
- (A\delta^i_k g_{kr} + B\delta^i_k F_{kr}) + (P\delta^i_k g_{kr} + Q\delta^i_k F_{kr}) + \\
+ (AF^i_k F_{jr} + BF^i_k g_{jr}) - (PF^i_k F_{kr} + QF^i_k g_{kr}) - \\
- (AF^i_k F_{kr} + BF^i_k g_{kr}) + (PF^i_k F_{kr} + QF^i_k g_{kr}).
\]

Substituting from (2.15) and (2.16), we obtain

\[
R^i_{rke} = -2b(\delta^i_k R_{rj} - \delta^i_j R_{rk} + F^j_k R^*_{rj} - F^j_i R^*_{rk}) + \\
+ 2a(\delta^i_k R_{rj} - \delta^i_j R_{rk} + F^j_k R^*_{rj} - F^j_i R^*_{rk}) = \\
K^i_{rke} = -2b(\delta^i_k K_{rj} - \delta^i_j K_{rk} + F^j_k K^*_{rj} - F^j_i K^*_{rk}) + \\
+ 2a(\delta^i_k K^*_{rj} - \delta^i_j K^*_{rk} + F^j_k K^*_{rj} - F^j_i K^*_{rk}).
\]

But the tensor on the right-hand side is the product projective curvature tensor [7]. Thus we have

**Theorem 3.** The product projective curvature tensor is an invariant of the PS-concircular connection.

Lowering the index \(i\) in the preceding equation and taking into account (1.11), and then raising the index \(r\), we obtain

\[
R^r_{ikj} = -2b(g_{ik} R^r_{j} - g_{ij} R^r_{k} + F^r_{ik} R^*_{j} - F^r_{ij} R^*_{k}) + \\
+ 2a(g_{ik} R^r_{j} - g_{ij} R^r_{k} + F^r_{ik} R^*_{j} - F^r_{ij} R^*_{k}) = \\
K^r_{ikj} = -2b(g_{ik} K^r_{j} - g_{ij} K^r_{k} + F^r_{ik} K^*_{j} - F^r_{ij} K^*_{k}) + \\
+ 2a(g_{ik} K^*_{j} - g_{ij} K^*_{k} + F^r_{ik} K^*_{j} - F^r_{ij} K^*_{k}),
\]

where \(R^r_{ik} = R^r_{ik}, K^r_{ik} = K^r_{ik}\).

Conversely, we suppose that for (1.3) we have (2.17). Contracting (2.17) with respect to \(r\) and \(j\), we get

\[
- [n(n-2) - \varphi^2] R_{ik} + 2 \varphi R_{ik} + [n(n-2) - \varphi^2] K_{ik} - 2 \varphi K^*_{ik} = \\
- [n(n-2) - \varphi^2] R_{ik} + 2 \varphi R_{ik} + [n(n-2) - \varphi^2] K_{ik} - 2 \varphi K^*_{ik} = \\
- [n(n-2) - \varphi^2] R_{ik} + 2 \varphi R_{ik} + [n(n-2) - \varphi^2] K_{ik} - 2 \varphi K^*_{ik} =
\]

\[
2 \varphi R_{ik} - [n(n-2) - \varphi^2] R_{ik} + 2 \varphi K_{ik} + [n(n-2) - \varphi^2] K^*_{ik} = \\
- [n(n-2) - \varphi^2] R_{ik} + 2 \varphi R_{ik} + [n(n-2) - \varphi^2] K_{ik} - 2 \varphi K^*_{ik} = \\
- [n(n-2) - \varphi^2] R_{ik} + 2 \varphi R_{ik} + [n(n-2) - \varphi^2] K_{ik} - 2 \varphi K^*_{ik} =
\]

\[
2 \varphi R_{ik} - [n(n-2) - \varphi^2] R_{ik} + 2 \varphi K_{ik} + [n(n-2) - \varphi^2] K^*_{ik} = \\
- [n(n-2) - \varphi^2] R_{ik} + 2 \varphi R_{ik} + [n(n-2) - \varphi^2] K_{ik} - 2 \varphi K^*_{ik} = \\
- [n(n-2) - \varphi^2] R_{ik} + 2 \varphi R_{ik} + [n(n-2) - \varphi^2] K_{ik} - 2 \varphi K^*_{ik} =
\]
We multiply (2.18) with \( n(n-2) - \varphi^2 \) and (2.19) with \( 2 \varphi \) and add the obtained relations. Then we have

\[
R_{ik} - K_{ik} = - \frac{[n - 2] R - \varphi R^* [n(n - 2) - \varphi^2] + 2 \varphi \left( [n(n - 2) - \varphi^2] R - \varphi R^* \right)}{4 \varphi^2 - [n(n - 2) - \varphi^2]^2} F_{ik} - \frac{[n - 2] R^* - \varphi R [n(n - 2) - \varphi^2] + 2 \varphi \left( [n(n - 2) - \varphi^2] R - \varphi R^* \right)}{4 \varphi^2 - [n(n - 2) - \varphi^2]^2} F_{ik} + \frac{[n - 2] K - \varphi K^* [n(n - 2) - \varphi^2] + 2 \varphi \left( [n(n - 2) - \varphi^2] K - \varphi K^* \right)}{4 \varphi^2 - [n(n - 2) - \varphi^2]^2} F_{ik} - \frac{[n - 2] K^* - \varphi K [n(n - 2) - \varphi^2] + 2 \varphi \left( [n(n - 2) - \varphi^2] K - \varphi K^* \right)}{4 \varphi^2 - [n(n - 2) - \varphi^2]^2} F_{ik}.
\]

(Since \( p > 2 \), and \( q > 2 \), \( 4 \varphi^2 - [n(n - 2) - \varphi^2]^2 \neq 0 \).)

Substituting \( R_{ik} - K_{ik} \) from this relation into (1.9), we obtain a relation of the form (2.8). Therefore, we have

**Theorem 4.** The connection (1.3) is a PS-concircular connection iff (2.17) holds.

3. **Product coharmonic curvature tensor.** In this section we consider another special product semisymmetric metric F-connection, namely the connection (1.3) satisfying the conditions

\[
(3.1) \quad \rho_a^a = 0, \quad \rho_b^b F_a^b = 0.
\]

Transvecting (1.9) and (1.10) with \( g_{rk} \), we obtain

\[
R = K + 2(n - 2) \rho_a^a + 2 \varphi F_a^b \rho_b^a,
\]

\[
R^* = K^* + 2 \varphi \rho_a^a + 2(n - 2) F_a^b \rho_b^a.
\]

So, if the connection (1.3) satisfies (3.1), we have \( R = K, R^* = K^* \) and the relation (1.12) reduces to

\[
(3.3) \quad R_{r[kj} + b_1 s_{r]kj}^{ij} - a_1 s_{s[kj}^{ij} F_{r]}^{ij} = K_{r[kj} + b_1 s_{r]kj}^{ij} - a_1 s_{s[kj}^{ij} F_{r]}^{ij}.
\]

Conversely, we suppose that for (1.3) we have (3.3). Then, contracting (3.3) with respect to \( i \) and \( j \), we get

\[
(3.4) \quad [n(n - 4) - \varphi^2] R - 4 R^* = [n(n - 4) - \varphi] K - 4 K^*,
\]

\[
(3.5) \quad - 4 R + [n(n - 4) - \varphi^2] R^* = - 4 K + [n(n - 4) - \varphi^2] K^*.
\]

Transvecting this equation with \( g_{rk} \) and \( F^* \), after some calculation, we obtain
We multiply (3.4) with $n(n - 4) - \varphi^2$ and (3.5) with 4 and add the obtained relations. Then we have

$$\{(n(n - 4) - \varphi^2)^2 - 16\} = \{(n(n - 4) - \varphi^2)^2 - 16\}K.$$ 

Since $p > 2$ and $q > 2$, $[n(n - 4) - \varphi^2]^2 - 16 \neq 0$, and therefore $R = K$. Thus (3.2) reduces to

$$(n - 2)\varphi_a^a + \varphi F_a^a \rho_a^b = 0, \quad \varphi \rho_a^a + (n - 2)F_a^a \rho_a^b = 0,$$

from which $\rho_a^a = 0$ and $F_a^a \rho_a^b = 0$.

Therefore, we have

**Theorem 5.** (1.3) satisfies the condition (3.1) if (3.3) holds.

The tensor on the right-hand side of (3.3) is analogous to the conharmonic curvature tensor [2]. Because of that we introduce the following

**Definition.** The tensor on the right-hand side of (3.3) is called a product coharmonic curvature tensor.

The connection (1.3) satisfying (3.1) is called a PS-coharmonic connection.

4. **Holomorphically semisymmetric connections.** The geometrical meaning of the semisymmetric connection was given by E. Bartolotti [1] and it consists in the following. Let $U$ and $V$ be two vectors. The vectors $S_{ij}^k u^i v^j, u^k, v^k$ are, in the general case, linearly independent. But if

$$S_{ij}^k u^i v^j = pu^k + qv^k$$

for every $U$ and $V$, where $p$ and $q$ are scalars, then $S_{ij}^k$ has the form (0.1), and conversely.

To generalize this property in the case of the locally decomposable Riemannian space, we considered in [5] the skew-symmetric tensor $S_{ij}^k$ satisfying the condition $S_{ij}^k u^i F_j^a v^a = pu^k + qF_k^a v^a$ instead of (4.1) and proved the following:

**Theorem.** The skew-symmetric tensor $S_{ij}^k$ satisfies condition (4.1) for every $U$ iff it has the form

$$S_{ij}^k = \delta_j^k S_i - \delta_i^k S_j = F_j^k F_i^a S_a + F_i^k F_j^a S_a +$$

$$+(\delta_j^k \delta_i^b + F_j^k F_i^a) w_{ab} / 2,$$

where $w_{ab}^k$ is an arbitrary skew-symmetric tensor.

One connection whose torsion tensor has the form (4.2) is the connection

$$G_{ij}^k = \left\{ \begin{array}{c}
\frac{k}{ij} + \delta_j^k S_i + \varepsilon g_{ij} S^k - F_j^k F_i^a S_a + \varepsilon F_i^k F_j^a S_a;
\end{array} \right.$$ 

where $\varepsilon = +1$ or $\varepsilon = -1$. (The case $\varepsilon = -1$ resembles more to the classical semisymmetric metric connection, i.e. to the connection $\left\{ \frac{k}{ij} \right\}$.)
the obtained results hold good for the case $\varepsilon = +1$, too.) This connection is an F-connection, i.e. $\partial F/\partial x^k + G^i_{ka} F^a_j - G^i_{kj} F^i_a = 0$, but is not a metric one.

**Definition.** The connection (4.3) is called a *holomorphically semisymmetric $(H S)$-connection*.

The curvature tensor $H^i_{r k j}$ of the connection (4.3) can be expressed in the form [5]

$$ H^i_{r k j} = K^i_{r k j} + \delta^i_{r k j} + F^i_j F^a_k \psi_{a k} - F^i_j F^a_k \psi_{a j} + g_{j r} \psi^i_k - g_{k r} \psi^i_j + F_{j r} F^a_k \psi^i_a - F_{k r} F^a_j \psi^i_a, $$

where $\psi_{a k} = \varepsilon \nabla_a S_j + S_j S_k + F^a_j F^b_k S_b$, $\psi^i_k = g^{i a} \psi_{a k}$.

As in 1, we suppose that $S_i$ is locally a gradient and satisfies the condition (1.4). Then

$$ \psi_{j k} = \psi_{k j}, \quad F^a_j \psi_{a k} = F^a_k \psi_{a j} $$

and the preceding equation reduces to

$$ H^i_{r k j} = K^i_{r k j} + g_{j r} \psi^i_k - g_{k r} \psi^i_j + F_{j r} F^a_k \psi^i_a - F_{k r} F^a_j \psi^i_a. $$

If $H^i_{r k j} = H^a_{r k j}$, $H^* = H^a_{r k j}$, then from (4.5) we obtain

$$ H^i_{r k j} = K^i_{r k j} + g_{j r} \psi^i_k - g_{k r} \psi^i_j + F_{j r} F^a_k \psi^i_a - F_{k r} F^a_j \psi^i_a, $$

(4.6)  

$$ H^*_{r k} = K^*_{r k} + 2 \psi_{r k} - g_{r k} \psi^a_a - g_{r k} \psi^a_a - \psi^a_a, $$

(4.7)  

$$ H^*_{r k} = K^*_{r k} + 2 \psi_{r k} F^a_k F^r_a - g_{r k} F^a_k \psi^a_a. $$

Transvecting (4.5) and (4.6) with $g^{r k}$, we find

$$ H - K = (2 - n) \psi^a_k - \varphi F^a_k \psi^a_a, \quad H^* - K^* = - \psi^a_k + (2 - n) F^a_k \psi^a_a. $$

Taking into account (4.6), (4.7) and (4.8), we can eliminate $\psi^a_k$ from (4.5) and we obtain the relation

$$ H^i_{r k j} - (g_{j r} H^i_k - g_{k r} H^i_j + F_{j r} H^i_k - F_{k r} H^i_j)/2 = $$

$$ - (bH - aH^*) r_{i k j} - (bH^* - aH) r^j_{i k} F^a_j = $$

$$ = K r k i - (g_{j r} K^i_k - g_{k r} K^i_j + F_{j r} K^i_k - F_{k r} K^i_j)/2 - $$

$$ - (bK - Q K^*) r^j_{i k j} - (bK^* - aK) r^j_{i k a}, $$

where we have used the notations (1.14) and (1.15).

**5. HS-concircular connection.** In this section we consider a special HS-connection, namely the connection (4.3) satisfying the condition

$$ \psi^i_k = f \delta^i_k + h F^i_k, $$

where $f$ and $h$ are some scalar functions.
Substituting (5.1) into (4.5), we find
\begin{equation}
H_{rkj}^i = K_{rkj}^i + f_{rjk}^i + h_{arj}^i F_r^a.
\end{equation}

Contracting with respect to \( i \) and \( j \), we get
\[
H_{rk} = K_{rk} + g_{rk}[(2-n)f - \varphi h] + F_{rk}[(2-n)h - \varphi f].
\]

Transvecting this with \( g^F_k \) and \( F^r_k \), we find
\begin{align}
H - K &= n(2-n) - \varphi^2 f + 2\varphi(1-n)h \tag{5.3} \\
H^* - K^* &= 2\varphi(1-n)f + [n(2-n) - \varphi^2]h. \tag{5.4}
\end{align}

We multiply (5.3) with \( n(2-n) - \varphi^2 \) and (5.4) with \( 2(1-n)\varphi \) and subtract the second from the first. Afterward, we multiply (5.3) with \( 2(1-n)\varphi \) and (5.4) with \( n(2-n) - \varphi^2 \) and subtract the first from the second. Then, taking into account the notations (2.6), we have
\begin{equation}
f = \alpha(H - K) + \beta(H^* - K^*), \quad h = \beta(H - K) + \alpha(H^* - K^*) \tag{5.5}
\end{equation}

Substituting (5.5) into (5.2), we find
\begin{align}
H_{rkj}^i - (\alpha H + \beta H^*)r_{rkj}^i - (\beta H + \alpha H^*)r_{arj}^i F_r^a &= \\
= K_{rkj}^i - (\alpha K + \beta K^*)r_{rkj}^i - (\beta K + \alpha K^*)r_{arj}^i F_r^a. 
\end{align}

Conversely, we suppose that for (4.3) we have (5.6). Then, substituting \( H_{rkj}^i - K_{rkj}^i \) from (5.6) into (4.5), we find
\begin{align}
g_{jk}\psi^i_k - g_{kr}\psi^i_r + F_{jr}^i F_r^a\psi^a_k - F_{kr} F_r^i \psi^a_k &= \\
= -(\alpha H + \beta H^* - \alpha K - \beta K^*)r_{rkj}^i + (\beta H + \alpha H^* - \beta K - \alpha K^*)r_{arj}^i F_r^a.
\end{align}

Contracting with respect to \( i \) and \( j \) we obtain
\begin{equation}
2\psi_r^k = g_{rk}[(\alpha(H - K) + \beta(H^* - K^*))(2-n) - [\beta(H - K) + \alpha(H^* - K^*)]\varphi + \psi^a_k] + \\
+ F_{rk}[-[\alpha(H - K) + \beta(H^* - K^*)]\varphi + [\beta(H - K) + (H^* - K^*)](2-n) + F_r^a \psi^a_k]
\end{equation}

and this is an equation of the form (5.1). Therefore, we have

**Theorem 6.** The connection (4.3) satisfies the condition (5.1) iff (5.6) holds.

The tensor on the right-hand side of (5.6) being the product concircular curvature tensor, it is reasonable to introduce the following

**Definition.** The connection (4.3) satisfying (5.1) is called a \( HS - concircular \) connection.
Now, we contract (5.6) with respect to $i$ and $j$ and find
\[
H_{r_k} + [(n - 2)(\alpha H + \beta H^*) + \varphi(\beta H + \alpha H^*)g_{r_k} + \\
+ [\varphi(\alpha H + \beta H^*) + (n - 2)(\beta H + \alpha H^*)F_{r_k} = \\
= K_{r_k} + [(n - 2)(\alpha K + \beta K^*) + \varphi(\beta K + \alpha K^*)g_{r_k} + \\
+ [\varphi(\alpha K + \beta K^*) + (n - 2)(\beta K + \alpha K^*)]F_{r_k}.
\]

Conversely we suppose that for the HS-connection (4.3) we have (5.8). Then substituting $H_{r_k} - K_{r_k}$ from (5.8) into (4.6) we obtain (5.7), i.e. we obtain an equation of the form (5.1). Therefore, we have

**Theorem 7.** An HS-connection is HS-concircular iff (5.8) holds.

In the same way as in section 2, using the relation (5.8) we obtain
\[
H^i_{r_k j} - 2b(\delta^i_j H_{r_k j} - \delta^i_j H_{r_k} + F^i_j H^*_{r_k} - F^i_j H^*_k) + \\
+ 2a(\delta^i_k H^*_{r_k j} - \delta^i_k H^*_r + F^i_k H_{r_k j} - F^i_k H_{r_k}) = \\
= K^i_{r_k j} - 2b(\delta^i_j K_{r_k j} - \delta^i_j K_{r_k} + F^i_j K^*_{r_k} - F^i_j K^*_r) + \\
+ 2a(\delta^i_k K^*_{r_k j} - \delta^i_k K^*_{r_k} + F^i_k K_{r_k j} - F^i_k K_{r_k}),
\]

and therefore we have

**Theorem 8.** The product projective curvature tensor is an invariant of the HS-concircular connection.

6. **Another special HS-connection.** In this section we consider HS-connections satisfying the conditions
\[
(6.1) \\
\psi^a_a = 0, \quad \psi^a_b F^b_a = 0.
\]

Then (4.8) gives $H = K$ and $H^* = K^* st$ and (4.9) reduces to
\[
H^i_{r_k j} - (g_{j r} H^i_k - g_{k r} H^i_j + F_{j r} H^*_{k i} - F_{k r} H^*_{j i})/2 = \\
= K^i_{r_k j} - (g_{j r} K^i_k - g_{k r} K^i_j + F_{j r} K^*_{k i} - F_{k r} K^*_{j i})/2.
\]

Conversely, we suppose that for the HS-connection (4.3) we have (6.2). Then contracting (6.2) with respect to $i$ and $j$, we find
\[
(H - K)g_{r_k} + (H^* - K^*)F_{r_k} = 0.
\]

Transvecting this relation with $g^{r_k}$ and $F^{r_k}$, we obtain
\[
(H - K)n + (H^* - K^*)\varphi = 0, \quad (H - K)\varphi + (H^* - K^*)n = 0.
\]

Consequently $H = K$ and $H^* = K^*$ and (4.8) reduces to
\[
(2 - n)\psi^a_a - \varphi \psi^a_b F^b_a = 0, \quad -\varphi \psi^a_a + (2 - n)F^b_a \psi^b_a = 0,
\]
from which (6.1) follows. Therefore we have

**Theorem 9.** An HS-connection satisfies the condition (6.1) iff (6.1) holds.

7. **Remark concerning HS-connections.** Lowering the index $i$ in (4.5) we have

$$H_{irkj} = 2K_{irkj} + g_{jr}\psi_{ik} - g_{kr}\psi_{ij} + F_{jr}F_{ia}\psi_{ia} - F_{kr}F_{ia}\psi_{ia}.$$

From this we obtain

$$H_{irkj} - H_{rikj} = 2K_{irkj} - g_{jr}\psi_{ik} + g_{kr}\psi_{ij} - F_{jr}F_{ia}\psi_{ak} + F_{kr}F_{ia}\psi_{aj} +$$

$$+ g_{jr}\psi_{ik} - g_{kr}\psi_{ij} + F_{jr}F_{ia}\psi_{ak} - F_{kr}F_{ia}\psi_{aj}.$$

Let us introduce the following notation

$$L_{irkj} = (H_{irkj} - H_{rikj})/2, \quad \rho_{rk} = -\psi_{rk}/2.$$

Then we express the preceding relation in the form

$$L_{irkj} = K_{irkj} + g_{ji}\rho_{rk} - g_{ki}\rho_{rj} + F_{jk}F_{ia}\rho_{ak} - F_{ki}F_{ia}\rho_{aj} -$$

$$- g_{jr}\rho_{ik} + g_{kr}\rho_{ij} - F_{jr}F_{ia}\rho_{ak} + F_{kr}F_{ia}\rho_{aj}.$$

or, raising the index $i$, in the form

$$L_{i r kj} = K_{i r kj} + \delta_{i j}^k \rho_{rk} - \delta_{i j}^k \rho_{rj} + F_{j k}F_{ia}\rho_{ak} - F_{ki}F_{ia}\rho_{aj} -$$

$$- g_{jr}\delta_{i j}^k + g_{kr}\delta_{i j}^j - F_{jr}F_{ia}\rho_{ak} + F_{kr}F_{ia}\rho_{aj}.$$

The right-hand side of this relation has the same form as the right-hand side of (1.6). Therefore, all conclusions of 2 and 3 can be repeated for HS-connections and the tensor $L_{i r kj}$.

**REFERENCES**


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