THE NOTIONS OF w-NET AND Y-COMPACT SPACE VIEWED UNDER INFINITESIMAL MICROSCOPE

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Abstract. Nonstandard analysis of A. Robinson is used to give a nonstandard description of the notion of w-net introduced by H.J. Wu in [6]. This concept leads to the notion of Y-compact space so that [0,1]-compact spaces are compact in the usual sense while R-compact spaces are E. Hewit's realcompact spaces.

Among typical applications of A.Robinson's infinitesimal (nonstandard) analysis are constructions of various kinds of completions of topological spaces, algebras, Banach spaces etc. If X is the object under consideration, one usually starts with a subset A ⊆ X of the nonstandard picture of the object X, and defines the completion of X as the set c(X) := A/ ~ where ~ is a suitable equivalence relation on the set A. Examples of this kind can be found in almost any book or paper concerning this subject, like A. Robinson [5], W.A.J. Luxemburg and K.D. Stroyan [4], M. Davis [1], W. Henson [3] or J.C. Dyre [2], and they are both standard (e.g. Stone-Čech compactification) and nonstandard (nonstandard hulls of Banach space).

In [6] Hueytzen J. Wu introduced the concept of w-net and used it, among other applications, to state a general form of the Tychonoff Compactness Theorem. Also, this notion leads to the definition of w-complete spaces and w-completions, which seems to be a useful unification and generalization of compactifications and the real-compactification of a Tychonoff space X.

The aim of this note is to give nonstandard characterizations of concepts mentioned above and to use them to discuss some properties of a naturally defined class of Y-compact spaces, for a given Hausdorff space Y. All definitions and concepts of nonstandard analysis used in this paper are standard and can be found in any text on the subject. For a one-page account of the main definitions and principles, it is recommended to the interested reader to see "Non-standard analysis for pedestrians" in [2]. The nonstandard model is assumed to be polysaturated.

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1. w-nets versus w-points

Definition 1. (Wu [6]) Let \( A = \{ f_i \}_{i \in I} \) be a family of continuous functions on a topological space \( X \). A net \( \langle x_m | m \in D \rangle \), where \( (D, \leq) \) is a directed set, is called a w-net induced by \( A \) if \( f_i(x_m) | m \in D \rangle := \langle f_i x_m | m \in D \rangle \) converges for each \( i \in I \).

Definition 2. Let \( A \) be as before. A point \( x \in ^* X \) is called w-point if for each \( i \in I^* f_i(x) \) is near-standard in \( ^* X \). The set of all w-points is denoted by \( W(^* X) \).

Proposition 1. A net \( \langle X_m | m \in D \rangle \) in space \( X \) is a w-net induced by the family \( A \) if and only if \( x_m \) is a w-point for all \( m \in \inf(^* D) \) and \( \forall i \in I \) \( \forall m, n \in \inf(^* D) \) \( f_i x_m \approx^* f_i x_n \) where \( \inf(^* D) := \{ m \in ^* D | (\forall n \in D) n \leq m \} \).

Proof. The following two equivalences are well-known:

(a) \( \langle y_m | m \in D \rangle \) converges iff (\( \forall m, n \in \inf(^* D) \))\( (y_m, y_n \in ns(^* X) \) and \( y_m \approx y_n) \).

(b) \( \langle y_m | m \in D \rangle \) has a cluster point in \( E = cl(E) \subset X \) iff \( \exists m \in \inf(^* D) \) \( (y_m \in ns(^* E)) \).

As usual \( m(x) \) is the monad of \( x, ns(^* X) := \{ y \in X | (\exists x \in X) y \in m(x) \} \) and \( y_1 \approx y_2 \) means that both \( y_1, y_2 \) are in the same monad.

The proposition easily follows from (a).

Proposition 2. (Wu[6]) Let \( A = \{ f_i \}_{i \in I} \) be a family of continuous functions \( f_i \) on \( X \) into Hausdorff spaces \( X_i \) such that the topology on \( X \) is the initial (weak) topology induced by \( A \). Let \( E \subset X \). The following are equivalent:

1. Every w-net in \( E \) induced by \( A \) has a cluster point in the closure \( cl(E) \) of \( E \).
2. Every w-net in \( cl(E) \) induced by \( A \) converges in \( cl(E) \).

Proof (nonstandard). (2)\( \Rightarrow \) (1) is trivial, so let us prove (1)\( \Rightarrow \) (2). By Proposition 1 and the equivalence (a) of its proof, (2) is equivalent with

3. Every w-point \( x \in ^* (cl(E)) \) is near-standard in \( ^* X \).

Assume (1). Let \( x \in ^* (cl(E)) \) be a w-point. Define \( D = \{ O | O \subset X \text{ open and } x \in ^* O \} \), \( (D, \supset) \) is a directed set. Obviously \( O \subset E \) for every \( O \in D \); so let \( \langle y_O | O \in D \rangle \) be a net in \( E \) such that \( y_O \in O \cap E \) for every \( O \in D \). This net is a w-net. Indeed, let for a given \( f_i \in A \), \( z_i = st^* f_i(x) \in X_i \). Then for every open \( V \ni z_i \), \( f_i^{-1}V \) is an open set such that \( x \in (f_i)^{-1}(V) \approx^* (f_i^{-1}V) \). Hence, \( f_i^{-1}V \in D \), which proves that \( \langle y_O | O \in D \rangle \) converges to \( z_i \). By (1) \( \langle y_O | O \in D \rangle \) has a cluster point in \( cl(E) \), hence \( y_m \in ns(cl(E)) \) for some \( m \in \inf(^* D) \). Let \( y \in st y_m \) (we do not assume that \( X \) is Hausdorff). From the definition of \( D \) it follows that \( (\forall f \in A)^* f(x) \approx^* f(y_m) \approx f(y) \). But, since the topology on \( X \) is induced by \( A \), we see that \( x \in m(y) \), which proves that \( x \) is near-standard.

The proposition above yields the following version of the Tychonoff Compactness Theorem proved by Wu in [6].
THEOREM 1. Let $\mathcal{A}$ be a family of continuous functions on a topological space $X$. Then $X$ is compact iff

1. $f(X)$ is contained in a compact subset $C_f$ for each $f \in \mathcal{A}$ and
2. every w-net induced by $\mathcal{A}$ has a cluster point in $X$.

Proof. (nonstandard). Let $x \in^* X$. Condition (1) implies that $^*f(x) \in \text{ns}(^*C_f)$ for every $f \in \mathcal{A}$, hence $x$ is w-point. Condition (2) is by the proof of Proposition 2 equivalent with

3. every w-point $x \in^* X$ is near-standard.

So, $x \in \text{ns}(^*X)$. By the well-known nonstandard characterization of compactness $X$ is compact.

2. w-completing of topological spaces

All definitions in this paragraph are nonstandard versions of definitions taken from the paper of Wu quoted above. Let $W(^*X)$ be the set of all w-points, i.e., $W(^*X) = \{x \in^* X|\forall f \in A|^*f(x) \in \text{ns}(^*X_i)\}$. For $x, y \in W(^*X)$, let $x \sim y$ iff $(\forall f \in A)^*f(x) \approx^* f(y)$. Then, $(Y, e)$, where $Y = W(^*X) / \approx$ and $e : X \to Y$ is the mapping induced by $i : X \to ^*X$ (note that it is not necessarily one-to-one) is called the $w$-transformation of $X$ induced by $A = \{f_i\}_{i \in I}$. It is clear that every function $f_i$ can be extended to $Y$ by $\bar{f_i}(x) = st^*f_i(x)$ for $x \in W(^*X)$ and $[x] \in Y$. It will be assumed that $Y$ is equipped with the initial topology induced by the family $\bar{A} = \{\bar{f_i}\}_{i \in I}$ which makes $e : X \to Y$ continuous and $e(X)$ dense in $Y$. Space $Y$ is a “completion” of $X$ in the following sense.

PROPOSITION 3. (Wu [6]) Every w-net in $Y$ induced by $\bar{A}$ converges in $Y$.

Proof (nonstandard). Let $\langle [y_n]\rangle | n \in D\rangle$ be an w-net in $Y$ where $y_n \in W(^*X)$. By saturation we can assume that there exists an internal function $y : ^*D \to ^*X$ such that $(\forall n \in D)y_n = y(n)$. Let $z_i \in Y_i$ be defined by $\lim_{n \in D} st^*f_i(y_n) = z_i$. Let $H_{i,O} = \{m \in ^*D|f_i(y(m)) \in^* O\}$ and $G_n = \{m \in ^*D|n \leq m\}$ for $i \in I$, $O$ is an open neighborhood of $z_i$ and $n \in D$. Since $\lim \langle st^*f_i(y_n)\rangle | n \in D\rangle = z_i$ the family

$U = \{H_{i,O}|i \in I \text{ and } O \in B(z_i)\} \cup \{G_n|n \in D\}$

has the finite intersection property. By saturation $\cap U \neq \varnothing$. So let $m \in \cap U$. It is easy to see that $y(m)$ must be a w-point in $^*X$ and that $\langle [y_n]\rangle | n \in D\rangle$ converges to $y(m)$.

Definition 3. If $(Y, e)$ is the $w$-transformation of the space $X$ such that $X$ is a Hausdorff space equipped with the initial topology induced by $\mathcal{A}$, then $e : X \to Y$ ia an embedding and the $w$-transformation is called $w$-completion of the space $X$.

The proof of the last proposition in this paragraph is left to the reader.

PROPOSITION 4. The $w$-completion $(Y, e)$ of the Hausdorff space $X$ with respect to the family $\mathcal{A} = \{f_i\}_{i \in I}$ has the following universal property. For any pair
\( (Z, r) \) and family \( B = \{ g_i \}_{i \in I} \) such that \( Z \) is a Hausdorff space with the initial topology induced by \( B \), if \( r : X \to Z \) is an embedding onto a dense subspace of \( Z \) and \((\forall i \in I) g_i \circ r = f_i \), then there exists a unique mapping \( j : Z \to Y \) such that \((\forall i \in I) g_i = f_i \circ j \).

**Definition 4.** Let \( X \) and \( Y \) be two Hausdorff spaces such that the topology on \( X \) is induced by the family \( A = \text{Map}(X, Y) \) of all continuous functions from \( X \) to \( Y \). \((X) \) will be referred to as an \( Y \)-Tychonoff space). If \( X \) coincides with its \( w \)-completion w.r.t. family \( A \), it is called a \( Y \)-compact space. Another characterization is the following. A \( Y \)-Tychonoff space is \( Y \)-compact if \((\forall X \in X)[(\forall f \in \text{Map}(X, Y)) f(x) \in \text{ns}(Y)] \to x \in \text{ns}(X) \).

Let us note that \( R \)-Tychonoff spaces are Tychonoff spaces in the usual sense, while \( R \)-compactness coincides with realcompactness. So it is not surprising that \( Y \)-compact spaces share some properties of realcompact spaces.

**Proposition 5.** Let us assume that the product \( \Pi(X_i | i \in I) \) of a family of \( Y \)-Tychonoff spaces is again a \( Y \)-Tychonoff space. Then if \( X_i \) is \( Y \)-compact for every \( i \in I \), \( \Pi(X_i | i \in I) \) is also a \( Y \)-compact space.

**Proof.** Let \( f \in \text{map}(\Pi(X_i | i \in I), Y) \). This implies that \((\forall i \in I)(f(i) \) is a \( w \)-point in \( X_i \)) \( \text{so, by assumption, let} \ g \in \Pi(Y_i | i \in I) \) has the property \((\forall i \in I) st f(i) = g(i) \). But, the monad of \( g \) is

\[
m(g) = \{ f \in map(\Pi(X_i | i \in I)) | (\forall i \in I) g(i) \approx f(i) \},
\]

hence \( f \) is near-standard.

**Corollary.** Products of realcompact spaces are realcompact.

**REFERENCES**


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