GRAPHS WITH MAXIMUM AND MINIMUM INDEPENDENCE NUMBERS

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Abstract. If \( r(G,k) \) is the number of selections of \( k \) independent vertices in a graph \( G \), and if \( r(G,k) > r(H,k) \), the graph \( G \) is \( i \)-greater than the graph \( H \). The maximal and the minimal graphs w.r.t. the above property are determined in the class of acyclic, unicyclic, connected acyclic and connected unicyclic graphs.

If \( G \) is a graph and \( v_1, v_2, \ldots, v_n \) are its vertices, then the vertices \( v_{i_1}, \ldots, v_{i_k} \) are said to be mutually independent if they pairwise non-adjacent in \( G \). The number of ways in which \( k \) mutually independent vertices \( (k \geq 2) \) can be selected in \( G \) is called the \( k \)-the independence number of \( G \) and is denoted by \( r(G,k) \). In addition, \( r(G,0) = 1 \) and \( r(G,1) = n \) = number of vertices of \( G \).

Let \( G \) and \( H \) be two graphs. \( G \) is \( i \)-greater than \( H \), if \( r(G,k) \geq r(H,k) \) for all values of \( k \). If both \( G > H \) and \( H > G \), then \( G \) and \( H \) are said to be \( i \)-equivalent, \( G \equiv H \).

Let \( G \) be a set of graphs. An element \( G_{\text{max}} \) is \( i \)-maximal in the set \( G \) if \( G_{\text{max}} \geq G \) for all \( G \in G \). Similarly, if \( G_{\text{min}} \in G \) and \( G > G_{\text{min}} \) for all \( G \in G \), then \( G_{\text{min}} \) is called the \( i \)-minimal graph in the set \( G \).

In the present paper we determine the \( i \)-maximal and the \( i \)-minimal graphs for a number of classes of graphs.

Preliminaries

We shall use the following terminology and symbolism. \( G = H \) (resp. \( G \neq H \)) means that the graphs \( G \) and \( H \) are (resp. are not) isomorphic. \( G_1 + G_2 \) is the union of the graphs \( G_1 \) and \( G_2 \).

If the vertices \( v_r \) and \( v_s \) are adjacent in \( G \), then the edge between them is labelled by \( e_{rs} \). If the vertices \( v_r \) and \( v_s \) are not adjacent in \( G \), \( G + e_{rs} \) is the graph obtained from \( G \) by introducing an edge between \( v_r \) and \( v_s \).

AMS Subject Classification (1980): Primary 05C35.
Let $v_r$ be a vertex of $G_1$ and $v_s$ a vertex of $G_2$. Then $(G_1 + G_2) + e_{rs}$ will be denoted by $G_1(r, s)G_2$. The graph obtained by identifying the vertices $v_r$ and $v_s$ is $G_1[r, s]G_2$.

The complete graph, the path, the cycle and the star with $n$ vertices are denoted by $K_n, P_n, C_n$ and $S_n$, respectively. The vertices of $P_n$ are labelled so that $v_r$ and $v_{r+1}$ are adjacent, $r = 1, \ldots, n - 1$. The vertices of $C_n$ are labelled so that $C_n = P_n + e_{1n}$.

The graph obtained by adding $k_r$ pendent edges to the vertex $v_r$ of $G$, for all $r = 1, \ldots, n$ is denoted by $G(k_1, \ldots, k_n)$.

$F_n$, $T_n$, $U_n$ and $U'_n$ will denote the set of all forests, trees, unicyclic graphs and connected unicyclic graphs, respectively, with $n$ vertices.

Some elementary properties of the independence numbers are summarized in the following lemma [1].

**Lemma 1.** Let $G$ be a graph and $v_1, v_2, \ldots, v_n$ be its vertices. Let $A_r$ be the set containing $v_r$ and all vertices adjacent to $v_r$. Then the following statements hold.

(a) $r(G, k) = r(G - v_r, k) + r(G - A_r, k - 1)$.

(b) If the vertices $v_r$ and $v_s$ are not adjacent in $G$, then

$$r(G, k) = r(G + e_{rs}, k) + r(G - A_r - A_s, k - 2).$$

If the vertices $v_r$ and $v_s$ are adjacent in $G$, then

$$r(G, k) = r(G - e_{rs}, k) - r(G - A_r - A_s, k - 2).$$

(c) $r(G_1 \hat{+} G_2, k) = \sum_{j=0}^{k} r(G_1, j) r(G_2, k - j)$.

In order to prove the main results of this paper, namely Propositions 2–8, we need a few auxiliary statements.

**Lemma 2.** If $v$ is a vertex of $G$ then $G > (G - v)$. If $e$ is an edge of $G$, then $(G - e) > G$.

**Lemma 3.** If $G_1 > H_1$, then for all graphs $G_2$, $(G_1 \hat{+} G_2) > (H_1 \hat{+} G_2)$.

**Proof.** Immediate from Lemma 1 c. q.e.d.

**Lemma 4.** If $G$ is a disconnected graph with $n$ vertices and cyclomatic number $c$, then there exists a connected graph $H$ with $n$ vertices and with cyclomatic number $c$, such that $G$ is $i$-greater than $H$.

**Proof.** If $G = G_1 \hat{+} G_2$, $v_r$ is a vertex of $G_1$ and $v_s$ is a vertex of $G_2$, then it is sufficient to choose $H = G_1(r, s)G_2$. q.e.d.

In [1] the following result has been proved.
Proposition 1. $n K_1$ is the i-maximal graph and $P_n$ is the i-minimal graph in $F_n$. $S_n$ is the i-maximal graph and $P_n$ is the i-minimal graph in $T_n$.

Lemma 5. Let $T_n$ be a tree with $n$ vertices and $v_1$ its vertex. If $G$ is a graph and $v_r$ its vertex, then $S_n(1,r)G > T_n(1,r)G > P_n(1,r)G$.

Proof. Applying Lemma 1a to the vertex $v_r$ of $S_n(1,r)G$, $T_n(1,r)G$ and $P_n(1,r)G$, respectively, we obtain

$$r(S_n(1,r)G,k) = r(S_n + (G - v_r), k) + r((n-1)K_1 + (G - A_r), k - 1),$$

$$r(T_n(1,r)G,k) = r(T_n + (G - v_r), k) + r((T - v_1) + (G - A_r), k - 1),$$

$$r(P_n(1,r)G,k) = r(P_n + (G - v_r), k) + r(P_{n-1} + (G - A_r), k - 1).$$

Lemma 5 follows now from Proposition 1 and Lemma 3. q.e.d.

Lemma 6. Using the same notation as in Lemma 5.


Proof. Let in the graph $G$ the vertex $v_r$ be adjacent to the vertices $v_{ri}$, $i = 1, \ldots, d_r$. Then by applying Lemma 1a to the vertices $v_{ri}$, $i = 1, \ldots, d_r$, of $S_n[1,r]G$, $T_n[1,r]G$ and $P_n[1,r]G$, we obtain

$$r(S_n[1,r]G,k) = r(S_n + (G - A_r), k) + \sum_{i=1}^{d_r} r((n-1)K_1 + (G - A_{ri}), k - 1),$$

$$r(T_n[1,r]G,k) = r(T_n + (G - A_r), k) + \sum_{i=1}^{d_r} r((T - v_1) + (G - A_{ri}), k - 1),$$

$$r(P_n[1,r]G,k) = r(P_n + (G - A_r), k) + \sum_{i=1}^{d_r} r(P_{n-1} + (G - A_{ri}), k - 1).$$

Lemma 6 follows again from Proposition 1 and Lemma 3. q.e.d.

Lemma 7. For $j = 1, 2, \ldots, n-1, (P_1 + P_{n-1}) > (P_j + P_{n-j}) > (P_2 + P_{n-2})$.

Proof. We use induction on the number $n$. The validity of Lemma 7 is easily checked for $n \leq 8$.

We suppose now that Lemma 7 holds for $n = h - 2$ and $n = h - 1$, and deduce its validity for $n = h$. It is legitimate to assume that $h > 8$.

By Lemma 1a, $r(P_j + P_{h-j}, k) = r(P_j + P_{h-1-j}, k) + r(P_j + P_{h-2-j}, k - 1)$, which combined with the hypothesis

$$r(P_1 + P_{h-3}, k - 1) \geq r(P_j + P_{h-2-j}, k - 1) \geq r(P_2 + P_{h-4}, k - 1)$$

and $r(P_j + P_{h-2}, k) \geq r(P_j + P_{h-1-j}, k) \geq r(P_2 + P_{h-3}, k)$ yields the required statement for $n = h$. q.e.d.

Lemma 8. For $1 < j < n - 1$, $P_{n-1}(2,1)P_1 > P_{n-1}(j,1)P_1 > P_{n-1}(3,1)P_1$. 


**Proof.** By Lemma 1a, \( r(P_{n-1}(j,1)P_1, k) = r(P_{n-1}, k) + r(P_{j-1} + P_{n-1-j}, k-1) \). Since the first term on the r.h.s. is independent of \( j \), we obtain Lemma 7. q.e.d.

**Lemma 9.** For \( 1 < j < n-2 \), \( P_{n-2}(2,1)P_2 > P_{n-2}(j,1)P_2 > P_{n-2}(3,1)P_2 \).

**Proof.** Analogous, and based on the relation

\[
r(P_{n-2}(j,1)P_2, k) = r(P_{n-2}(j,1), k) + r(P_{n-2}, k-1)
\]

and on Lemma 8. q.e.d.

**Lemma 10.** \( P_{n-1}(3,1)P_1 > P_{n-2}(3,1)P_2 \).

**Proof.** Lemma 10 can be proved by induction on \( n \), using the fact that

\[
r(P_0(i,j)P_0, k) = r(P_{i-1}(i,j), P_0, k) + r(P_{n-2}(i,j)P_0, k-1).
\]

q.e.d.

Let \( v_r \) and \( v_s \) the two adjacent vertices of a graph \( G \). The substitution of the edge \( e_{rs} \) by a path with \( a \) vertices yields the graph \( G(e_{rs} \mid a) \); see Fig. 1.

![Fig. 1](image)

**Lemma 11.** \( G(r,1)P_a > G(e_{rs} \mid a) \).

**Proof.** Let the vertices of the graphs \( G_0, G(e_{rs} \mid a) \) and \( G(r,1)P_a \) be labelled as indicated in Fig. 1. Then \( G(r,1)P_a + e_{as} = G(r_s = G(e_{rs} \mid a) + re_{rs} = G_0 \).

According to Lemma 1b,

\[
\begin{align*}
\text{r}(G(r,1)P_a, k) &= \text{r}(G_0, k) + \text{r}((G - A_s) + P_{n-2}, k - 2), \\
\text{r}(G(e_{rs} \mid a), k) &= \text{r}(G_0, k) + \text{r}((G - A_r - A_s) + P_{n-2}, k - 2).
\end{align*}
\]

\( G - A_r - A_s \) is an induced subgraph of \( G - A_s \). Therefore by Lemma 2, \( r(G - A_s, k) \geq r(G - A_r - A_s, k) \), where as by Lemma 3, \( r((G - A_s) + P_{n-2}, k) \geq r((G - A_r - A_s) + P_{n-2}, k) \) for all values of \( k \). Lemma 11 now follows from the two equalities above q.e.d.

**Lemma 12.** (a) For \( 3 \leq j \leq n \), \( C_j(1,1)P_{n-1} \equiv C_{n-j+3}(1,1)P_{j-3} \).

(b) For \( 3 < j < n \).

\( C_4(1,1)P_{n-4} \equiv C_{n-1}(1,1)P_1 > C_j(1,1)P_{n-j} > C_5(1,1)P_{n-5} \equiv C_{n-2}(1,1)P_2 \).

**Proof.** By Lemma 1a,
\[ r(C_j(1,1)P_{n-j}, k) = r(P_{n-1}, k) + r(P_{j-3} + P_{n-j}, k - 1), \]
from which it follows that \( C_j(1,1)P_{n-j} \) and \( C_{n-j+3}(1,1)P_{j-3} \) are \( i \)-equivalent.

In addition, because of Lemma 7, the r.h.s. of the above equality will be maximal if \( j - 3 = 1 \) or \( n - j = 1 \); the same expression will be minimal for \( j - 3 = 2 \) or \( n - j = 2 \). q.e.d.

The main results

**Proposition 2.** \((n - 2)K + K_2\) is the \( i \)-maximal graph and \( P_{n-2}(3,1)P_2 \) is the \( i \)-minimal graph in the set \( F_n \setminus \{nK_1, P_n\} \).

**Proof.** The first part of Proposition 2 is evident.

Let \( F \) be the \( i \)-minimal graph in \( F_n \setminus \{nK_1, P_n\} \). Then \( F \) must be connected (because of Lemma 4), it must have exactly one vertex of degree greater than two (because of Lemma 5) and this vertex must be of degree three (because of Lemma 6). Let \( v \) be a vertex of \( F \) having degree one and let \( w \) be adjacent to \( v \). Then \( F - v \in T_{n-1} \), \( F - v - w \in F_{n-2} \) and

\[ r(F, k) = r(F - v, k) + r(F - v - w, k - 1). \]

Now because of Proposition 1, \( F \) can be \( i \)-minimal only if (a) \( F - v = P_{n-1} \) and /or (b) \( F - v - w = P_{n-2} \).

In case (a) we have \( F = P_{n-1}(j,1)P_j \). If \( j = 1 \) or \( j = n \), then \( F = P_n \), which is impossible. If \( 1 < j < n \), then \( F \) is not \( i \)-minimal because of Lemmas 8 and 10. Hence case (a) leads to contradictions.

In case (b), \( F = P_{n-2}(j,1)P_2 \). It must be that \( 1 < j < n \); otherwise \( F = P_n \). But then, because of Lemma 9, \( F = P_{n-2}(3,1)P_2 \). q.e.d.

**Proposition 3.** \( P_3(n-3,1) \) is the \( i \)-maximal graph and \( P_{n-2}(3,1)P_2 \) is the \( i \)-minimal graph in the set \( T_n \setminus \{S_n, P_n\} \).

**Proof.** Having in mind Proposition 2, one has to prove only that \( P_2(n-3,1) \) is the \( i \)-maximal graph.

Let \( T \) be any element of \( T_n \setminus \{S_n, P_n\} \), and let \( v_r \) be its vertex of degree one. Then \( r(T, k) = r(T - v_r, k) + r(T - \bar{A}_r, k - 1) \) with \( T - v_r \in T_{n-1} \) and \( T - \bar{A}_r \in F_{n-2} \). In order to have a maximal value for \( r(T, k) \) we have to choose \( T - v_r = S_{n-1} \) (because of Proposition 1) and \( T - \bar{A}_r = (n - 4)K_1 + K_2 \) (because of Proposition 2). This, on the other hand, implies \( T = P_2(n-3,1) \) q.e.d.

Using similar considerations one proves

**Proposition 4.** \((n - 3)K_1 + S_3 \) is the \( i \)-maximal graph in the set \( F_n \setminus \{nK_1, \,(n-2)K_1 + K_2\} \). \( P_2(n-4,2) \) is the \( i \)-maximal graph in the set \( T_n \setminus \{S_n, P_2(n-3,1)\} \).
Proposition 5. \((n-3)K_1 + C_3\) is the i-maximal graph in the set \(U_n\). \(C_n\) and \(C_3(1,1)P_{n-3}\) are two (mutually i-equivalent) i-minimal graphs in the set \(U_n\).

Proof. Every graph \(U \in U_n\) contains a vertex \(v_r\) whose degree is greater than one, such that \(U - v_r\) is a forest with \(n - 1\) vertices and with at least one edge, where as \(U - A_r\) is a forest with at most \(n - 3\) vertices. Now, by Proposition 2, \((n-3)K_1 + C_3\) is the unique graph with the properties \(U - v_r = (n-3)K_1 + C_3\) and \(U - A_r = (n-3)K_1\). This proves the first part of Proposition 5.

The fact that \(C_n\) is i-minimal in \(U_n\) is an immediate consequence of Lemma 11. Then the second part of Proposition 5 follows from Lemma 12. q.e.d.

A similar reasoning leads also to

Proposition 6a. \((n-4)K_1 + C_4\) and \((n-4)K_1 + C_3(1,0,0)\) are the two (mutually i-equivalent) i-maximal graphs in the set \(U_n\)\{\((n-3)K_1 + C_3\}\}.

Proposition 7. \(C_3(n-3,0,0)\) is the i-maximal graph, whereas \(C_n\) and \(C_3(1,1)P_{n-3}\) are the two (mutually i-equivalent) i-minimal graphs in the set \(U_n\).

Proof. Having in mind Proposition 5, only the first part of Proposition 7 remains to be proved. Let \(U\) be the i-maximal element of \(U'_{n-1}\) and let \(q\) be the size of its cycle. By Lemma 6, \(U\) must be of the form \(C_q(k_1,k_2,\ldots ,k_q)\), where \(k_i \geq 0\).

Let \(v_r\) be a vertex of degree one of the graph \(U\). Then \(U - v_r \in U'_{n-1}\) and \(U - A_r \in F_{n-2}\{((n-2)K_1)\}\).

We complete the proof by induction on the number of vertices of \(U\). For \(n = 4,5,6\) it can be established easily that \(U = C_3(n-3,0,0)\). Suppose now that \(C_3(h-4,0,0)\) is i-maximal in \(U'_{h-1}\). Then

\[
r(C_3(h-3,0,0),k) = r(C_3(h-4,0,0),k) + r((h-4)K_1 + K_2,k-1).
\]

Since by Proposition 2, \((h-4)K_1 + K_2\) is i-maximal in \(F_{h-2}\{((h-2)K_1)\}\), we conclude that \(C_3(h-3,0,0)\) is i-maximal in \(U'_{h}\). q.e.d.

Analogous considerations also lead to

Proposition 8a. \(C_3(n-4,1,0)\) is the i-maximal graph in the set \(U'_{n}\{C_3(n-3,0,0)\}\).

In order to complement the results exposed in Proposition 6a and 8a, we determine also the second i-minimal unicyclic graphs.

Propositions 6b and 8b. The two i-equivalent graphs \(C_3(1,1)P_{n-5}\) and \(C_{n-2}(1,1)P_2\) are i-minimal in the set \(U_n \{C_n, C_3(1,1)P_{n-3}\}\) (and therefore also in the set \(U_n \{C_n, C_3(1,1)P_{n-3}\}\)).

Proof. The i-minimal graph in \(U_n \{C_n, C_3(1,1)P_{n-3}\}\) must be connected (because of Lemma 4) and must possess exactly one vertex of degree one (because
of Lemmas 5 and 11). Hence this graph must be of the form $C_j(1,1)P_{n-j}$. The rest of the proof follows from Lemma 12. q.e. d.

Acknowledgment. The author thanks Professor Horst Sachs (limenau, GDR) for useful discussions.

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