THE REGULATION NUMBER OF A GRAPH

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Abstract. The regulation number \( r(G) \) of a graph \( G \) with maximum degree \( d \) is defined as the smallest number of new points in a \( d \)-regular supergraph. It is shown that for \( d \geq 3 \), every possible value of \( r(G) \) between zero and the maximum established by Akiyama, Era and Harary, namely \( d \mod 2 + 1 + d \), is realized by some graph. Also, a characterization is given for \( G \) to have \( r(G) = n \).

1. Introduction. The regulation number \( r(G) \) of a graph \( G \) with maximum degree \( d \geq 3 \),

(1) \[ r(G) \leq d + 2 \text{ when } d \text{ is odd,} \]

(2) \[ r(G) \leq d + 1 \text{ when } d \text{ is even.} \]

Our first purpose is to demonstrate the interpolation theorem that for each \( n \) between 0 and the upper bounds in (1) and (2), there exists a graph with regulation number \( n \). This is accomplished by constructing such a graph.

A necessary and sufficient condition is then derived for a graph to have regulation number \( n \), using the notion of an “\( f \)-factor” due to Tutte, [5].

In general we follow the notation and terminology of [bf 4].

2. Interpolation. We shall show that for each \( d \geq 3 \), every integer \( n \) between zero, the smallest possible value of \( r(G) \), and the maximum value \( d + 1 \) \( \mod d + 2 \) depending on the parity of \( d, n \) is realized as the regulation number of some graph. In the construction of such a graph, it is convenient to use the notation \( G_1 + G_2 + G_3 \) of [2] for the iterated join of three disjoint graphs \( G_i \) defined as the union \( (G_1 + G_2) \cup (G_2 + G_3) \). Similarly, the iterated join of \( n \geq 3 \) disjoint graphs is written \( G_1 + G_2 + G_3 + \cdots + G_n \) and is defined as \( (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n) \). We shall encounter the special case \( K_1 + K_1 + \cdots + K_1 + G_{k+1} + \cdots + G_n \) where \( G_{k+1} \neq K_1 \) and will abbreviate it by \( P_k + G_{k+1} + \cdots + G_n \) (as \( i \) this case the join of the first \( k \) copies of \( K_1 \) gives the path \( P_k \)).
THEOREM 1. Let $d \geq 3$.

1. If $d$ is odd and $0 \leq n \leq d + 2$, then there is a graph $H_n$ with maximum degree $d$ and $r(H_n) = n$.

2. If $d$ is even and $0 \leq n \leq d + 1$, then there is a graph $J_n$ with maximum degree $d$ and $r(J_n) = n$.

Proof. When $n = 0$ and $d \geq 3$ is odd, one can take $H_0$ as a $d$-regular graph or as a spanning subgraph of such a graph. For $n = d + 2$, we have $H_n = K_1 + \overline{K}_2 + K_{d-1}$. (The case $d = 3$ was illustrated in [1]). Now for any positive integer $n$ properly between 0 and $d + 2$, one possible choice is

$$H_n = P_{d-n+3} + \overline{K}_2 + K_{d-1}.$$ 

The proof when $d$ is even is analogous, with $J_n = P_{d-n+2} + \overline{K}_2 + K_{d-1}$. □

Figure 1. Realization graphs for regulation number interpolation

Figure 1 shows the graphs $H_1$ to $H_5$ when $d = 3$. The smallest 3-regular graph containing these $H_n$ is shown in Figure 2. As noted in [bf 4], this is the smallest cubic graph with a bridge.

Figure 2. The smallest cubic graph with a bridge

3. Characterization. Let $G$ be any graph with $p$ points $V = \{1, 2, \ldots, p\}$. Let $f = (f_1, \ldots, f_p)$ be a vector of $p$ non-negative integers. Then an $f$-factor is a spanning subgraph $F$ of $G$ such that the degree of point $i$ in $F$ is $f_i$. We recall the following result of Tutte [5] giving a criterion for the existence of an $f$-factor.

THEOREM B. A graph $G$ has an $f$-factor if and only if for any two disjoint subsets $X$ and $Y$ of $V$, with $o(X,Y)$ the number of odd components of $G - X - Y$, and $d(i, G - X)$ the degree of $i$ in $G - X$ we have

$$o(X,Y) + \sum_{i \in Y} \{f_i - d(i, G - X)\} \leq \sum_{i \in D} f_i.$$ (3)
Let $d_i = d(i, G)$ and let the deficiency of $v_i$ in $G$ be $f_i = d - d_i$. Then it can easily be verified that $G$ has regulation number 0 if and only if $\bar{G}$ has an $f$-factor, where $f = (f_1, \ldots, f_p)$ is the vector of deficiencies. We will extend this observation to obtain a criterion for a graph to have regulation number $n$. Fix $n$ properly between 0 and $d + 2$ and define the join $I_n = \bar{G} + P_n$, with the additional points labelled $p + 1, \ldots, p + n$. Set $I_0 = G$. If $n > 0$, let $f_j = d$ for $j = p + 1, \ldots, p + n$ and set $f = (f_1, \ldots, f_{p+n})$.

**Theorem 2.** Let $0 \leq n \leq d + 2$ and let $G$ be a graph with maximum degree $d$. Then $r(G) = n$ if and only if $n$ is smallest integer such that $I_n$ has an $f$-factor.

**Proof.** Suppose $r(G) = n$ and consider the set of lines added to $G + \bar{K}_n$ to form a $d$-regular graph. These edges form an $f$-factor in $I_n$. Suppose there is some integer $j < n$ such that $I_j$ contains an $f$-factor. Then it is easily verified that these edges would regularize $G + \bar{K}_j$, contradicting the fact that $r(G) = n$. The converse holds by a similar argument. □

Theorem A and Theorem 2 together yield an algorithm which can be used to determine $r(G)$ for a given graph $G$. However, the paper [3] by Erdős and Kelly implicitly contains an $O(n)$ algorithm for this purpose even though they studied and determined the induced regulation number of a graph.

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**REFERENCES**


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