 SOME CHARACTERISTICS OF THE PROCESS MEASURE OF THE AMOUNT OF INFORMATION

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Signs and symbols. $a = a_1 a_2 \ldots a_n$ – binary word of length $n$.
$\Lambda$ – empty word.
$X$ – the space of all finite words over $\{0, 1\}$. ($\Lambda \in X$ by definition)
$l(a)$ – the length of word $a$.
$\mathfrak{r} = a_1 a_2 a_3 \ldots a_n a_0$ – manner of recording the word $a$ required to record two or more words in the form of one word. For example for the words $x$, $y$ and $z$ the record is $\mathfrak{r}_{x y z}$. From the word $\mathfrak{r}_{x y z}$ it is possible to decode the words $x$, $y$ or $z$ by means of general, recursive functions $\pi_1, \pi_2$ and $\pi_3$. (We also have $\mathfrak{r}_{01} = 01$.)
$a \preceq b$ means $b = a w$, $w \in X$ ($aw$ is a concatenation of words $a$ and $w$).
$f(x) \preceq g(x)$ means $\exists C \forall x \in X f(x) \leq g(x) + C$.
$f(x) \succeq g(x)$ means $f(x) \preceq g(x)$ and $g(x) \preceq f(x)$.

The function $F(a_1 a_2 \ldots a_n) = 2^n - 1 + \sum_{i=1}^{n} a_i 2^{n-i}$ gives a one-to-one correspondence of the set $X$ and the set $\{0, 1, 2, \ldots \}$. The symbol $a$ will denote both the word and its corresponding number.

Introduction. The partial recursive function $\mathcal{F} : X^{m+1} \rightarrow X$ of $m + 1$ arguments is called a process according to argument $p$ if the following applies: for a word $p$, $\mathcal{F}(p, y_1, \ldots, y_m)$ exists and if $q \subset p$, then $\mathcal{F}(q, y_1, \ldots, y_m)$ exists and $\mathcal{F}(q, y_1, \ldots, y_m) \subset \mathcal{F}(p, y_1, \ldots, y_m)$.

Definition 1. The conditional process complexity of $(x_1, \ldots, x_n)$, given $(y_1, \ldots, y_m)$, with respect to the processes $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is

$$KP_{\mathcal{F}_1, \ldots, \mathcal{F}_n}(x_1, \ldots, x_n|y_1, \ldots, y_m) = \min_{p \in X} \{ \alpha(p)/\mathcal{F}_1(p, y_1, \ldots, y_m) = x_1, \ldots, \mathcal{F}_n(p, y_1, \ldots, y_m) = x_n \}.$$
The function $\alpha(p)$ is a criterion of complexity and it is usually taken as $\log_2 p$, which in the alphabet $0 - 1$ is equal to $l(p) + C$.

**Theorem 1.** There is a set of optimal $m + 1$ dimensional processes according to argument $p(F_0(p_1, y_1, \ldots, y_m), \ldots, F_n(p_1, y_1, \ldots, y_m))$ such that for any other set of $m + 1$ dimensional processes according to argument $p(G_1(p_1, y_1, \ldots, y_m), \ldots, G_n(p_1, y_1, \ldots, y_m))$ and for any $(x_1, \ldots, x_n)$

$$KP_{2}^{*}, \ldots, F_{n}(x_1, \ldots, x_n/y_1, \ldots, y_m) \preceq KP_{2}^{*}, \ldots, G_{n}(x_1, \ldots, x_n/y_1, \ldots, y_m).$$

The proof of Theorem 1. is standard for this theory and similar with the proof in [2, p. 91 Theorem 1.2].

From now on, the complexity $KP_{2}^{*}, \ldots, F_{n}(x_1, \ldots, x_n/y_1, \ldots, y_m)$ will be designated with $KP(x_1, \ldots, x_n/y_1, \ldots, y_m)$. $KP(x_1, \ldots, x_n)$ means $KP(x_1, \ldots, x_n/\Lambda \ldots, \Lambda)$.

We have the following characteristics of the process complexity:

(i) $KP(x/y) \preceq KP(x) \preceq KP(x/y) + 2KP(y)$

where $K(y)$ is the Kolmogorov complexity of the word $y$. Let $KP(x/y) = l(p)$, that is, $F^z(p, y) = x$. Let us form the function

$$S = \{ F^z(\pi_2(z), F^z(\pi_1(z))), \text{ if } z \text{ has the form } \overline{ab} \}
\begin{cases} 
\Lambda, & \text{otherwise.}
\end{cases}$$

$F^z$ is an optimal two-dimensional process, and $F^z$ an optimal function for Kolmogorov complexity. Let $K(y) = l(p_y)$. The function $S$ is a process by construction. For the program $z = \overline{\pi_1p}$ the results is $x$. Further more, we have

$$KP(x) \preceq KP_0(x) \preceq l(p_y) + KP(x/y) = K(x/y) + 2K(y).$$

**Remark.** The constant 2 may be replaced with $1 + \varepsilon$ by a more appropriate coding of the program $z$.

(ii) $KP(x/y) \preceq K(x/y) + 2\log_2 K(x/y)$

Let us form a process

$$F^2(z, y) = \begin{cases} 
F^z(A(z), y), & \text{if } z \text{ has the form } \overline{ab} \text{ and } l(b) \geq a \text{ otherwise}
\Lambda,
\end{cases}$$

where $A(l(p/q)) = p$ is general recursive $(p, q \in X)$. For $F^z(p_x, y) = x$ and $z = \overline{l(p_x)}$ we have

$$KP(x/y) \preceq KP_2(x/y) \preceq l(z) = l(p_x) + K(x/y) = K(x/y) + 2K(x/y).$$

(iii) If $F(x)$ is a process, then $KP(F(x)) \preceq KP(x)$.

If for $F(x)$ there exists an inverse function that is also a process, then $KP(F(x)) \preceq KP(x)$.

(iv) $KP(x/y) \preceq KP(x/y, z)$

$$KP(x/y) = \min\{l(p)/F^z(p, y) = x\} = \min\{l(p)/G(p, y, z) = x\} \preceq \min\{l(p)/F^z(p, y, z) = x\} = KP(x/y, z).$$
The function $G(p, y, z) = F^c(p, y)$ has $z$ as a fictive argument.

(v) For every partial recursive function $F$ we have

\begin{align*}
KP(y/x, F(x)) &\preceq KP(y/x) \\
KP(y/x, F(x)) &= \min \{l(p)/F^c(p, x, F(x)) = y\} = \\
&= \min \{l(p)G(p, x) = y\} \succeq KP(y/x).
\end{align*}

(vi) If $F$ is an invertible partial recursive function, then

\begin{equation}
KP(x/F(x)) = KP(F(x)/x) \succeq 0
\end{equation}

where $G(p, F(x)) = F^{-1}(F(x))$, which is trivially a process according to $p$.

$KP(F(x)/x) = \min \{l(p)/F^c(p, x) = F(x)\} \succeq 0,$

where $G(p, x) = F(x)$, which is also a process according to $p$.

**Measure of the amount of information.** The process complexity of a word $x$ is very suitable for defining the concept of randomness. Namely, (Shnorr in [4] shows that to a Martin-Löf random binary sequences $\omega$ applies $KP(\omega^n) = n$, where $\omega^n$, is a fragment of the sequences $\omega$ of length $n$. On the other hand, the complexity is also suitable for defining the measure of information. Kolmogorov defines in [1] the measure of information carried by a word $y$ about word $x$ as

\begin{equation}
I(y : x) = K(x) - K(x/y)
\end{equation}

Levin ([5]) also defines the measure of information as $IP(y : x) = KP(x) - KP(x/y)$, where $KP_A(x) = \min \{l(p)/A(p) = x\}$ and $A(p)$ is a function such if $A(p) = x$, then $A(pg) = x$. (Those are the so-called prefix algorithms.)

**Definition 2.** The quantity

\begin{equation}
J(y_1, \ldots, y_m : z_1, \ldots, z_k) = KP(x_1, \ldots, x_n/z_1, \ldots, z_k) - \\
- KP(x_1, \ldots, x_n/y_1, \ldots, y_m, z_1, \ldots, z_k)
\end{equation}

is termed the process measure of the amount of information that $(y_1, \ldots, y_m)$ carries on $(z_1, \ldots, z_k)$ if $(z_1, \ldots, z_k)$ is known. We have the following characteristics of measure $J$:

(i) $J(y : x) \succeq 0$ \hspace{1cm} (2.2)

The property (2.2) follows from the relation (1.1).

(ii) $J(x : x) \succeq KP(x)$ \hspace{1cm} (2.3)

The relation (2.3) is a direct consequence of (1.15). It can be also shown that $J(p_x : x) \succeq KP(x)$, where $p_x$ is such that $F^c(p_x) = x$.

(iii) $J(x, y : z) = J(x : z) + J(y : z/x)$ \hspace{1cm} (2.4)
The proof results directly from the definition of the measure $J$.

(iv) The process measure of information may be compared with measure $I$, introduced by (2.1)

$$I(y : x) - 2 \log_2 K(x/y) \leq J(y : x) \leq I(y : x) + 2 \log_2 K(x)$$

$$J(y : x) = KP(x) - KP(x/y) \leq K(x) + 2 \log_2 K(x) - K(x/y) = I(y : x) + 2 \log_2 K(x).$$

(v) If $F$ is partial recursive and invertible function, $J(F(x) : x) \asymp KP(x)$, $J(x : F(x)) \asymp (F(x))$, $J(F(x) : y) \asymp J(x : y)$.

(vi) It is known that the algorithm measure of the amount of information is not commutative ([2], [3]), that is, it can be shown only as $|J(y : x) - I(x : y)| \leq 12 \cdot I(K(x, y))$. Since $|J(y : x) - I(y : x)| \leq (1 + \varepsilon)I(K(x))$, for the process measure $J$ we have

$$|J(y : x) - J(x : y)| \leq (14 + 2\varepsilon)I(K(x, y)).$$

(vii) For every word $x$ we have $J(\ell(x) : x) \asymp 2 \cdot K(\ell(x))$.

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