ON SOME INEQUALITIES FOR QUASI-MONOTONE SEQUENCES

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0. Let \( p \neq 0 \) be a real constant. The operator \( L_p \) will be defined in the following way (see [1]):

\[
L_p(a_n) = a_{n+1} - p a_n \quad (n \in N).
\]

For a sequence \( a = (a_n) \) we shall say that it is \( p \)-monotone or that it belongs to the class \( K_p \) if the inequality

\[
L_p(a_n) \geq 0
\]
is valid for all \( n \in N \).

The following theorem is given in [1]:

Theorem A. Let for a given sequence \( a = (a_n) \) the sequence \( A = (A_n) \) be defined by

\[
A_n = \frac{p_1 a_1 + \cdots + p_n a_n}{p_1 + \cdots + p_n}.
\]

(i) If we have \( p = q \) then the implication

\[
a \in K_p \Rightarrow A \in K_q
\]

holds true for every sequence of the class \( K_p \) and for arbitrary positive weights \( p = (p_n) \) if and only if \( p = q = 1 \).

(ii) If \( p \) and \( q \) satisfy one of the conditions

\[
p > q > 1; \quad 0 < p < q < 1; \quad \text{or} \quad p < q < 0,
\]

then implication (1) holds true for an arbitrary sequence of the class \( K_p \) if and only if the sequence \( p = (p_n) \) of positive weights is given by

\[
p_n = p^n \frac{q^{n-1} - q^{n-2} \prod_{k=1}^{n-1} p^k - q^k}{p^n - q^n} \quad (n = 2, 3, \ldots)
\]

where the weight \( p_1 \) is an arbitrary given positive number.
1. In this paper we shall shown that some inequalities for monotone sequences are also valid for $p$-monotone sequences, i.e. we shall give the necessary and sufficient conditions for the validity of these inequalities.

First, we shall notice that the following identity follows, from the well-known Abel identity:

\[
\sum_{i=1}^{n} w_i a_i = \sum_{i=1}^{n} p^{i-1} w_i \sum_{k=2}^{n} \left( \sum_{i=k}^{n} p^{i-k} w_i \right) L_p(a_{k-1}).
\]

Using (2), we can easily obtain the following theorem:

**Theorem 1.** Let $w = (w_n)$ be an arbitrary real sequence.

(a) Inequality

\[
\sum_{i=1}^{n} w_i a_i \geq 0
\]

holds for every sequence $a$ from $K_p$ if and only if

\[
\sum_{i=1}^{n} p^{i-1} w_i = 0
\]

and

\[
\sum_{i=k}^{n} p^{i-k} w_i \geq 0 \quad (k = 2, \ldots, n).
\]

(b) Inequality (3) holds for every sequence $a$ from $K_p$ such that $a_1 \geq 0$ if and only if

\[
\sum_{i=k}^{n} p^{i-k} w_i \geq 0 \quad (k = 1, \ldots, n).
\]

2. Let $a \in K_p$, $b \in K_q$ be real sequences, and let $x_{ij}(i = 1, \ldots, n; j = 1, \ldots, m)$ be real numbers. Then necessary and sufficient conditions for the numbers $x_{ij}$, such that the inequality

\[
F(a, b) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} a_i b_j \geq 0
\]

holds: 1° for every $p$-monotone sequence $a$ and for every $q$-monotone sequence $b$, or 2° for every $p$-monotone sequence $a$ and $q$-monotone sequence $b$ such that $a_1 \geq 0$ and $b_1 \geq 0$, are contained in the following theorem:

**Theorem 2.** (a) With the condition 1°, $F(a, b) \geq 0$ if and only if

\[
X_{1,s} = 0, \quad (s = 1, \ldots, m), \quad X_{r,1} = 0 \quad (r = 2, \ldots, n) \quad \text{and} \quad X_{r,s} \geq 0 \quad (r = 2, \ldots, n; \quad s = 2, \ldots, m),
\]
where
\[ X_{r,s} = \sum_{i=1}^{n} \sum_{j=1}^{m} p^{i-r} q^{j-s} x_{ij}. \]

(b) With the condition 2\(\circ\), \(F(a,b) \geq 0\), if and only if
\[ X_{r,s} \geq 0 \quad (r = 1, \ldots, n; \ s = 1, \ldots, m). \]

Proof. (a) (i) Let \(a_i = 0\) (\(i = 1, \ldots, r - 1\)) \(a_i = p^{i-r}(i = r, \ldots, n)\), and let
\(b_j = q^{j-1}(j = 1, \ldots, m)\) or \(b_j = -q^{j-1}(j = 1, \ldots, m)\). Then from (4), we get the condition \(X_{r,1} = 0\). By analogy, we get \(X_{1,s} = 0\). Now, let
\[ a_i = 0 (i = 1, \ldots, r - 1) \quad a_i = p^{i-r}(i = r, \ldots, n), \]
\[ b_j = 0 (j = 1, \ldots, s - 1) \quad b_j = p^{j-s}(j = s, \ldots, m). \]

Then, from (4), we get \(X_{r,s} \geq 0\). So condition (5) is necessary.

(ii) Let be \(s_j = \sum_{i=1}^{n} x_{ij} a_i\). Then
\[ f(a,b) = \sum_{j=1}^{m} s_j b_j = b_1 \sum_{j=1}^{m} q^{j-1} s_j + \sum_{s=2}^{m} \left( \sum_{j=1}^{m} q^{j-s} s_j \right) L_q(b_{s-1}). \]

Now, we write \(x_i = \sum_{j=1}^{m} q^{j-s} x_{ij}\). Then
\[ \sum_{j=1}^{m} q^{j-s} s_j = \sum_{i=1}^{n} \left( \sum_{j=1}^{m} q^{j-s} x_{ij} \right) a_i = \sum_{i=1}^{n} x_i a_i \]
\[ = a_1 \sum_{i=1}^{n} p^{i-1} x_i + \sum_{r=2}^{n} \left( \sum_{i=1}^{n} p^{i-r} x_i \right) L_p(a_{r-1}) \]
\[ = a_1 X_{1,1} + \sum_{r=2}^{n} X_{r,1} L_p(a_{r-1}). \]

For \(s = 1\), we have
\[ \sum_{j=1}^{m} q^{j-1} s_j = a_1 X_{1,1} + \sum_{r=2}^{n} X_{r,1} L_p(a_{r-1}). \]

So,
\[ F(a,b) = a_1 b_1 X_{1,1} + b_1 \sum_{r=2}^{n} X_{r,1} L_p(a_{r-1}) + a_1 \sum_{s=2}^{m} X_{1,s} L_q(b_{s-1}) \]
\[ + \sum_{r=2}^{n} \sum_{s=2}^{m} X_{r,s} L_p(a_{r-1}) L_q(b_{s-1}). \]
Based on (5), it is evident that (4) holds.

Analogously we can prove (b).

**Remark.** For $p = q = 1$, we have the result from [2].

Analogously to the previous proof (see also [3]), we can prove the following theorem:

**Theorem 3.** Let $a_j = (a_{j1}, \ldots, a_{jn})$ ($j = 1, \ldots, m$) be real sequences and let $x_{j_1} \ldots x_{j_m}$ $(j_k = 1, \ldots, n_k, k = 1, \ldots, m)$ be real numbers. Then necessary and sufficient conditions for the numbers $x_{j_1} \ldots x_{j_m}$, for the inequality

$$
\sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} x_{j_1} \ldots x_{j_m} a_{j_1} \cdots a_{j_m} \geq 0
$$

to hold for every $p_j$-monotone sequence $a_j$ such that $a_{j1} \geq 0$ ($j = 1, \ldots, m$) are

$$
\sum_{j_1=s_1}^{n_1} \cdots \sum_{j_m=s_m}^{n_m} p_1^{j_1-s_1} \cdots p_m^{j_m-s_m} x_{j_1} \ldots x_{j_m} \geq 0
$$

for $j_k = 1, \ldots, n_k$, $k = 1, \ldots, m$.

**Remark.** For $p_1 = \cdots = p_m = 1$, we have the result from [3].

3. Now, we shall give a generalization of Theorem A.

Let us consider a triangular matrix of real numbers $(p_{n,i})$ (where $= 0, 1, \ldots, n$; $n = 0, 1, \ldots$). Let us define the sequence $(\sigma_n)$, for a given sequence $(a_n)$ by

$$
\sigma_n = \sum_{j=0}^{n} p_{n,n-j} a_j.
$$

Then the following theorem holds:

**Theorem 4.** A necessary and sufficient condition for the implication

$$(a_n) \in K_p \Rightarrow (\sigma_n) \in K_q$$

to be valid, for every sequence $(a_n)$, where the sequence $(\sigma_n)$ is given by (5), is that the following conditions, for every $n$,

$$
d_{n,n} - q d_{n-1,n-1} = 0, \quad d_{n,n-k} - q d_{n-1,n-k-1} \geq 0 \quad (k = 1, \ldots, n - 1)
$$

$$
d_{n,0} \geq 0,
$$
hold, where

$$
d_{n,k} = \sum_{j=0}^{k} p^{k-j} p_{n,j}.
$$
Proof. We have
\[ L_q(\sigma_{n-1}) = \sigma_n - q\sigma_{n-1} = \sum_{j=0}^{n} p_{n,n-j}a_j - q \sum_{j=0}^{n-1} p_{n-1,n-1-j}a_j = \sum_{j=0}^{n} w_ja_j \]
where \( w_j = p_{n,n-j} - qp_{n-1,n-1-j} \) (\( j = 0, 1, \ldots, n - 1 \)) and \( w_n = p_{n,0} \). Using Theorem 1, we obtain Theorem 4.

REFERENCES

