ON FIXED POINT THEOREMS OF MAIA TYPE

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1. In this note we present some variants of the following result of Maia [10]: Let $X$ be a non-empty set endowed in with two metrics $\rho$, $\sigma$, and let $f$ be a mapping of $X$ into itself. Suppose that $\rho(x,y) \leq \sigma(x,y)$ in $X$, $X$ is a complete space and $f$ is continuous with respect to $\rho$, and $\sigma(fx,fy) \leq k \cdot \sigma(x,y)$ for all $x, y$ in $X$, where $0 \leq k < 1$. Then, $f$ has a unique fixed point in $X$.

This theorem (cf. also [18], [11], [4], [12], [17]) generalizes the Banach fixed-point principle and is connected with Bielecki’s method [1] of changing the norm in the theory of differential equations. Our results follow as a consequence of two metrics, of two transformations [3] and of the generalized metric space concept ([8], [9]).

2. Let $(E, \| \cdot \|)$ be a Banach space, let $S$ be a normal cone in $E$ (see e.g. [6]) and let $\preceq$ denote the partial order in $E$ generated by the cone $S$. Suppose that $X$ is a non-empty set and a function $d_E : X \times X \to S$ satisfying for arbitrary elements $x, y, z$ in $X$ the following conditions:

(A 1) $d_E(x,y) = \theta$ if and only if $x = y$ ($\theta$ denotes the zero of the space $E$);
(A 2) $d_E(x,y) = d_E(y,x)$;
(A 3) $d_E(x,y) \preceq d_E(x,z) + d_E(z,y)$

Then, this function $d_E$ is called the generalized metric in $X$.

Further, let us put $d^+(x,y) = \|d_E(x,y)\|$ for $x$ and $y$ in $X$. If every $d^+$-Cauchy sequence in $X$ is $d^+$-convergent (i.e., $\lim_{p,q \to \infty} d^+(x_p,x_q) = 0$ for a sequence $(x_n)$ in $X$, implies the existence of an element $x_0$ in $X$ such that $\lim_{n \to \infty} d^+(x_n,x_0) = 0$), then $(X, d_E)$ is called [6] a generalized complete metric space.

Moreover, in this paper we shall use the notations of $\mathcal{L}^*$-space, the $\mathcal{L}^*$-product of $\mathcal{L}^*$-spaces and a continuous mapping of $\mathcal{L}^*$-space into $\mathcal{L}^*$-space (see e.g. [7]).

3. Let $E$, $S$ and $\preceq$ be as above. In this section suppose we are given:

$L$ - a bounded positive linear operator of $E$ into itself with the spectral radius $r(L)$ less than one (see e.g. [6]);
$X$, $A$ - two non-empty sets;

$\rho_E, \sigma_E$ - two generalized metrics in $X$ such that $\rho_E(x, y) \leq C \cdot \sigma_E(x, y)$ for all $x, y$ in $X$, where $C$ is a positive constant;

$T$ - a transformation from $A$ to $X$ such that $(T[A], \rho_E)$ is a generalized complete metric space$^1$.

Modifying the reasoning from [6, Th. II. 6. 2], we obtain the following result:

**Proposition 1.** Let $(X, \rho_E)$ be a generalized complete metric space, let $f: X \to X$ be a continuous mapping with respect to $\rho^+$, and let $\sigma_E(f(x, y)) \leq L(\sigma_E(x, y))$ for all $x, y$ in $X$. Then $f$ has a unique fixed point $\xi$ in $X$. Moreover, if $x_0 \in X$ and $x_n = f x_{n-1}$ for $n \geq 1$, then:

(i) $\lim_{n \to \infty} \|\rho_E(x_n, \xi)\| = 0$,

(ii) $\|\rho_E(x_n, \xi)\| \leq N \cdot C \cdot \|L^m u\|$ for all $m \geq 0$, where $N$ is same constant and $u$ is a solution of equation $u = \sigma_E(x_0, fx_0) + Lu$ in the space $E$ (see [6, Th. I. 2. 2]).

Now, we shall prove

**Proposition 2.** Let $(X, \rho_E)$ be a generalized complete metric space, let $f_m: X \to X$ ($m = 0, 1, \ldots$) be continuous mappings with respect to $\rho^+$, and let $\sigma_E(f_m(x, y)) \leq L(\sigma_E(x, y))$ for all $x, y$ in $X$. Denote by $\xi_n$ ($m = 0, 1, \ldots$) a unique fixed point of $f_m$, and suppose that $\lim_{n \to \infty} \|\sigma_E(f, x_n, f_0 x)\| = 0$ for every $x$ in $X$. Then $\lim_{n \to \infty} \|\rho_E(\xi_n, \xi_0)\| = 0$.

**Proof.** Consider the linear equation $u = \sigma_E(\xi_0, f_0 \xi_0) + Lu$ ($n = 1, 2, \ldots$) with the unique solution $u_n$ in $E$ (see [6, Th. I. 2. 2]). By Proposition 1 we obtain $\|\rho_E(\xi_n, \xi_0)\| \leq N \cdot C \cdot \|u_n\|$ for $n \geq 1$, where $N$ is constant.

Let $\varepsilon > 0$ by such that $r(L) + \varepsilon < 1$. Further, let us denote by $\|\cdot\|_e$ the norm equivalent to $\|\cdot\|$ such that $\|L\|_e \leq \varepsilon + r(L)$ (see [6, p. 15]) $\|L\|_e$ is the norm of $E$ generated by $\|\cdot\|_e$. We have

$$
\|u_n\|_e \leq \|\sigma_E(f_n \xi_0, f_0 \xi_0)\|_e + \|Lu_n\|_e \leq \|\sigma_E(f_n \xi_0, f_0 \xi_0)\|_e + (r(L) + \varepsilon) \|u_n\|_e
$$

for $n \geq 1$. Since $\lim_{n \to \infty} \|\sigma_E(f_n \xi_0, f_0 \xi_0)\|_e = 0$, so $\lim_{n \to \infty} \|u_n\|_e \leq (\varepsilon + r(L)) \cdot \lim_{n \to \infty} \|u_n\|_e$, and consequently $\lim_{n \to \infty} \|\rho_E(\xi_n, \xi_0)\| = 0$.

**Theorem 1.** Let $H: A \to X$ be a mapping such that $H[A] \subset T[A]$ and $\sigma_E(Hx, Hy) \leq L(\sigma_E(Tx, Ty))$ for all $x, y$ in $A$. Suppose that $\lim_{n \to \infty} \|\rho_E(Hx_n, Hx)\| = 0$ for every sequence $(x_n)$ in $A$ with $\lim_{n \to \infty} \|\rho_E(Tx_n, Tx)\| = 0$ Then:

(i) for every $u$ in $T[A]$ the set $H[T^{-1} u]$ contains only one element$^2$;

(ii) there exists a unique element $\xi$ in $T[A]$ such that $H[T^{-1} \xi] = \xi$, and every sequence of successive approximations $u_{n+1} = H[T^{-1} u_n]$ ($n = 1, 2, \ldots$) is \(\rho^+\)-convergent to $\xi$;

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$^1$T[A] denotes the image of the set $A$ by the transformation $T$

$^2$T\(^{-1}\)u denotes the inverse image of $u$ under $T$
(iii) \(Hx = Tx\) for all \(x\) in \(T^{-1} \xi\);

(iv) if \(Hx_i = Tx_i\) \((i = 1, 2)\), then \(Tx_1 = Tx_2\).

**Proof.** Let us put \(fz = H[T^{-1} z]\) for \(z\) in \(T[A]\). Obviously, \(fz \in T[A]\) for all \(z\) in \(T[A]\). If \(v_i \in f(z) \ (i = 1, 2)\), then \(v_i = Hx_i\) with \(Tx_i = z\). Hence \(\theta \leq \sigma_E(v_1, v_2) \leq L(\sigma_E(Tx_1, Tx_2)) = \theta\) and \(v_1 = v_2\). Therefore, \(H[T^{-1} z]\) contains only one element.

It can be easily seen that the mapping \(f\) of \(T[A]\) into itself is continuous with respect to \(\rho^+\). Indeed, let \(z_n \in T[A]\) for \(n \geq 1\) and let \(\lim_{n \to \infty} ||\rho E(z_n, z_0)|| = 0\). Then there exist \(x_m \in T^{-1} z_m\) \((m = 0, 1, \ldots)\) such that \(fz_m = Hx_m\). We have ||\(\rho_E(Hx_n, Hx_0)\)|| = ||\(\rho_E(fz_n, fz_0)\)|| for \(n \geq 1\), and consequently \(\lim_{n \to \infty} ||\rho_E(Hx_n, Hx_0)\|| = 0\).

Further, it is easy to verify that \(\sigma_E(fu, fv) \leq L(\sigma_E(u, v))\) for all \(u, v\) in \(T[A]\). Consequently, applying Proposition 1 the proof of (ii) is completed.

Obviously, (iii) holds and we omit the proof. Now, we prove (iv): Suppose that \(Hx_i = Tx_i\) \((i = 1, 2)\) and \(Tx_1 \neq Tx_2\). Then, \(\sigma_E(Tx_1, Tx_2) \leq L(\sigma_E(Tx_1, Tx_2))\) and \(\sigma_E(Tx_1, Tx_2) \not\subset S\). Therefore, by theorem II. 5. 4 from [6, p. 81], we obtain \(r(L) \geq 1\). This contradiction completes our proof.

Using Theorem 1 and Proposition 2 we obtain the following

**Theorem 2.** Let \(H_m : A \to X (m = 0, 1, \ldots)\) be mappings with \(H_m[A] \subset T[A]\) and \(\sigma_E(H_m x, H_m y) \leq L(\sigma_E(Tx, Ty))\) for all \(x, y\) in \(A\). Further, suppose that \(\lim_{n \to \infty} ||\rho E(H_m x_n, H_m x)|| = 0\) for every sequence \((x_n)\) in \(A\) with \(\lim_{n \to \infty} ||\rho E(Tx_n, Tx)|| = 0\).

Let \(\xi_m(m = 0, 1, \ldots)\) be an element in \(T[A]\) such that \(H_m[T^{-1} \xi_m] = \xi_m\). Assume that \(\lim_{n \to \infty} ||\sigma_E(H_n x, H_n y)|| = 0\) for every \(x\) in \(A\). Then \(\lim_{n \to \infty} ||\rho E(Ty_n, Ty_0)|| = 0\), where \(y_m \in T^{-1} \xi_m\) for \(m \geq 0\).

4. M. Krasnoselskii [5] has given the following version of well-known result of Schauder: If \(W\) is a non-empty bounded closed convex subset of a Banach space, \(f\) is a contraction and \(g\) is completely continuous on \(W\) with \(f x + gy \in W\) for all \(x, y\) in \(W\), then the equation \(fx + gy = x\) has a solution in \(W\).

Now, we give a modification and some generalization of this Krasnoselskii’s result.

Let \((E, ||||)\) be a Banach space, let \(S\) be a cone in \(E\) with the partial order \(\preceq\) such that if \(\theta \preceq x \preceq y\) then \(||x|| \preceq ||y||\), and let \(L\) be as in Sec. 3. Further, let \(X\) be a vector space endowed with two generalized norms \(||\cdot||_1, ||\cdot||_2\) (see [6, p. 94]) such that \(||x||_1 \preceq C \cdot ||x||_2\) for all \(x\) in \(X\). Denote: \(\rho_E, \sigma_E\)-generalized metrics in \(X\) generated by \(||\cdot||_1\) and \(||\cdot||_2\), respectively.

**Theorem 3.** Let \(K\) be a non-empty convex subset of \(X\), let \((K, \rho^+)\) be a complete space and let \(Q, F\) be transformations with the values in \(K\) defined on \(K\) and \(K \times K\) respectively. Assume, moreover, that the following condition holds:

(i) \(Q : (K, \rho^+) \to (K, \rho^+)\) is continuous, \(Q[K]\) is a conditionally compact set with respect to \(\sigma^+\) and \(||Q(u, y) - F(v, y)||_2 \preceq ||Qu - Qv||_2\) for all \(u, v, y\) in \(K\);
(ii) \( |||F(x, y) - F(x, z)||_2 \leq L(|||y - z||_2) \) for all \( x, y, z \) in \( K \);

(iii) for every \( x \) in \( K \) the function \( y \mapsto F(x, y) \) of \( K \) into itself is continuous with respect to \( \rho^+ \).

Then there exists a point \( x \) in \( K \) such that \( F(x, x) = x \).

**Proof.** Consider the mapping \( y \mapsto F(x, y) \) (\( x \) is fixed in \( K \)) of \( K \) into itself. By Proposition 1, there exists exactly one \( u_x \) in \( K \) such that \( F(x, u_x) = u_x \). Now define an operator \( V \) as \( x \mapsto u_x \).

This operator \( V \) maps continuously \((K, \rho^+)\) into itself. Indeed, let \( (x_n) \) be a sequence in \( K \) such that \( \rho^+(x_n, x_0) \to 0 \) as \( n \to \infty \). Let us put \( f_n x = F(x_n, x) \) \((m = 0, 1, \ldots)\) for \( x \) in \( K \). The conditions (i) and (ii) imply that all the assumptions of the Proposition 2 are satisfied. Therefore, \( f_m \) has a unique fixed point \( \xi_m \) and \( \rho^+(\xi_m, \xi_0) \to 0 \) as \( n \to \infty \), so we are done.

Now we are going to show that \( V[K] \) is conditionally compact with respect to \( \rho^+ \): Let \( (x_n) \) be a sequence in \( K \), and let \( y_n = F(x_n, u_{x_n}) \) for \( n \geq 1 \). Let \( \varepsilon > 0 \) be such that \( r(L) + \varepsilon < 1 \), let \( \|\cdot\| \) be the norm equivalent to \( |||\cdot||| \) with \( |||L||| \leq r(L) + \varepsilon \), and let us put \( \sigma^+_\varepsilon(x, y) = |||x - y|||_\varepsilon \) for \( x, y \) in \( K \). We have

\[
|||y_i - y_j|||_\varepsilon \leq |||L(y_i - y_j)|||_\varepsilon + |||Qx_i - Qx_j|||_\varepsilon \leq (r(L) + \varepsilon) |||y_i - y_j|||_\varepsilon + |||Qx_i - Qx_j|||_\varepsilon,
\]

hence

\[
(1 - (r(L) + \varepsilon)) \cdot |||y_i - y_j|||_\varepsilon \leq |||Qx_i - Qx_j|||_\varepsilon
\]

for every \( i, j \leq 1 \). Suppose that \((Qx_n)\) is a \( \sigma^+\)-Cauchy sequence. Then, \((Qx_n)\) is a \( \sigma^+_\varepsilon \)-Cauchy sequence and consequently \((y_n)\) is \( \rho^+\)-convergent in \( K \).

By application of the Schauder fixed point theorem, our proof is completed.

**Remark.** The above theorem will remain true if (i) is replaced by the following condition: \( Q \) is continuous and \( Q[K] \) is a conditionally compact set with respect to \( \rho^+ \), and \( |||F(u, y) - F(v, y)|||_2 \leq |||Qu - Qv|||_1 \) for all \( u, v, y \) in \( K \).

5. Let us remark applications and further results can be obtained if the concept of a generalized metric space in the Luxemburg sense [9] (not every two points have necessarily a finite distance) will be used. Cf. [13]–[17]. How, we give some application of Theorem 2 (in the case of) to functional equations.

In this section, let \((\mathbb{R}^k, ||\cdot||)\) denote the \( k \)-dimensional Euclidean space, let \( E = \mathbb{R}^k \), and let \( S = \{(t_1, t_2, \ldots, t_k) \in \mathbb{R}^k; t_i \geq 0 \text{ for } 1 \leq i \leq k\} \). Then, \((x_1, x_2, \ldots, x_k) \leq (y_1, y_2, \ldots, y_k)\) if we have \( x_i \leq y_i \) for every \( 1 \leq i \leq k \).

Suppose that \( J = [0, \infty), K_{ij} \geq 0 \) \((i, j = 1, 2, \ldots, k)\) are constants, and \( p: J \to J \) is a locally bounded function. Let us denote by:

- \( A \) - the set of continuous functions \((x_1, x_2, \ldots, x_k)\) from \( J \) to \( \mathbb{R}^k \) such that \( x_1(t) = 0(\exp(p(t))) \) \((1 \leq i \leq k)\) for every \( t \) in \( J \);
- \( X \) - the set of bounded continuous functions from \( J \) to \( \mathbb{R}^k \); 
- \( \Lambda \) - the metric space with the metric \( \delta \).
\[ \mathcal{F} - \text{the set of continuous functions } (f_1, f_2, \ldots, f_k) \text{ from } J \times \mathbb{R}^k \times \Lambda \text{ into } \mathbb{R}^k \]
satisfying the following conditions:

\[ |f_i(t, t_1, \ldots, t_k, \lambda) - f_i(t, s_1, s_2, \ldots, s_k, \lambda)| \leq \sum_{j=1}^{k} K_{ij} |t_j - s_j| \]

\((1 \leq i \leq k)\) for every \(t \in J, t_j, s_j \in \mathbb{R}^k\) and \(\lambda \in \Lambda; f_i(t, \theta, \lambda) = 0(\exp(p(t)))\) \((1 \leq i \leq k)\) for fixed \(\lambda \in \Lambda\) and every \(t \in J\) \((\theta \text{ denotes the zero of space } \mathbb{R}^k)\).

The set \(\Lambda\) admits a norm \(\|\cdot\|\) defined as \(\|x\| = \sup \{|\exp(-p(t))| \cdot |x(t)|; t \geq 0\}\). In \(X\) we define the generalized metric \(d_E\) as follows: for each \(x = (x_1, \ldots, x_k)\) and \(y = (y_1, \ldots, y_k)\) write \(d_E(x, y) = (\|x_1 - y_1\|, \|x_2 - y_2\|, \ldots, \|x_k - y_k\|)\), where \(\|\cdot\|\) denotes the usual supremum norm in the space of bounded continuous functions on \(J\). Obviously, \((X, d_E)\) is a generalized complete metric space.

We shall deal with the set \(\mathcal{F}\) as an \(L^*\)-space endowed with convergence:

\[ \lim_{n \to \infty} (f_1^n, f_2^n, \ldots, f_k^n) = (f_1^0, f_2^0, \ldots, f_k^0) \]

if and only if

\[ \lim_{n \to \infty} \sup \{\exp(-p(t)) \cdot |f_i^n(t, u, \lambda) - f_i^0(t, u, \lambda)|; (t, u) \in J \times \mathbb{R}^k\} = 0 \]

for every \(\lambda \in \Lambda\) and every \(1 \leq i \leq k\). Moreover, \(\mathcal{F} \times \Lambda\) be the \(L^*\)-product of the \(L^*\)-spaces \(\mathcal{F}, \Lambda\).

Further, suppose that \(h: J \to J\) is a continuous function, there exists a constant \(q > 0\) such that \(\exp(p(h(t))) \leq q \cdot \exp(p(t))\) for all \(t \in J\), and \([q \cdot K_{ij}]\) \((1 \leq i, j \leq k)\) is a non-zero matrix with

\[
\begin{bmatrix}
1 - qK_{11} & -qK_{12} & \cdots & -qK_{1k} \\
-qK_{21} & 1 - qK_{22} & \cdots & -qK_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
-qK_{k1} & -qK_{k2} & \cdots & 1 - qK_{kk}
\end{bmatrix} > 0
\]

for every \(i = 1, 2, \ldots, k\).

Under these conditions we have the following theorem:

For an arbitrary \(F\) in \(\mathcal{F}\) and \(\lambda \in \Lambda\) there exists a unique function \(x_{(F, \lambda)}\) in \(A\) such that

\[ x_{(F, \lambda)}(t) = F(t, x_{(F, \lambda)}(h(t)), \lambda) \]

for every \(t \geq 0\). Moreover, if there exists functions \(\alpha, \beta\) from \(J\) to \(J\) such that \(\alpha(t) = 0(\exp(p(t)))\) for \(t \geq 0\), \(\beta(t) \to 0\) as \(t \to 0^+\) and

\[ |f_i(t, u, \lambda) - f_i(t, u, \mu)| \leq \alpha(t) \cdot \beta(\delta(\lambda, \mu)) \quad (1 \leq i \leq k) \]

for all \((f_1, f_2, \ldots, f_k) \in \mathcal{F}, t \geq 0, u \in \mathbb{R}^k\) and \(\lambda, \mu \in \Lambda\), then the function

\[ (F, \lambda) \mapsto x_{(F, \lambda)} \]
maps continuously $L^*$-space $\mathcal{F} \times \Lambda$ into Banach space $A$.

Proof. Let $m = 0,1,\ldots$ Let $F^{(m)} = (f_1^{(m)}, \ldots, f_k^{(m)}) \in \mathcal{F}$ and $\lambda_m \in \Lambda$ be such that $\lim_{n \to \infty} F^{(n)} = F^{(0)}$ and $\lim_{n \to \infty} \delta(\lambda_n, \lambda_0) = 0$. For each $x$ in $A$, define:

$$(T_x)(t) = \exp(-p(t)) \cdot x(t),$$

$$(H_m x)(t) = \exp(-p(t)) \cdot F^{(m)}(t, x(h(t)), \lambda_m)$$

on $J$.

For $x = (x_1, x_2, \ldots, x_k) \in A$ and $t \geq 0$ we obtain

$$| (H_m x)(t) | \leq | F^{(m)}(t, x(h(t)), \lambda_m) - F^{(m)}(t, \theta, \lambda_m) | + | F^{(m)}(t, \theta, \lambda_m) | \cdot \exp(-p(t)) \leq$$

$$\leq \left( \sum_{j=1}^{k} \sum_{j=1}^{k} K_{ij} | x_j(h(t)) | + | F^{(m)}(t, \theta, \lambda_m) | \right) \cdot \exp(-p(t)) \leq$$

$$\leq (c_1 \cdot \exp(p(h(t)))) + c_2 \cdot \exp(p(t)) \cdot \exp(-p(t)) \leq c_1 q + c_2$$

with some constants $c_1$, $c_2$, and therefore $H_m$ maps $A$ into $X$. Further, it can be easily seen that $T[A] = X$ and $H_m[A] \subset T[A]$.

We observe [2] that the operator $L$ generated by the matrix $[g \cdot K_{ij}]$ is a bounded positive linear operator with the spectral radius less than 1. For $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ in $A$ and $t \geq 0$ we have

$$\exp(-p(t)) \cdot | f^{(m)}_i(t, x(h(t)), \lambda_m) - f^{(m)}_i(t, y(h(t)), \lambda_m) | \leq$$

$$\leq \left( \sum_{j=1}^{k} K_{ij} \cdot \sup_{t \geq 0} \exp(-p(t)) | x_j(t) - y_j(t) | \right) \cdot \exp(-p(t)) \cdot \exp(p(h(t))) \leq$$

$$\leq q \cdot \sum_{j=1}^{k} K_{ij} \cdot \sup_{t \geq 0} \exp(-p(t)) \cdot | x_j(t) - y_j(t) |,$$

$$d_E(H_m x, H_m y) = (\sup_{t \geq 0} \exp(-p(t)) \cdot | f^{(m)}_1(t, x(h(t)), \lambda_m) - f^{(m)}_1(t, y(h(t)), \lambda_m) |, \ldots, \sup_{t \geq 0} \exp(-p(t)) \cdot | f^{(m)}_k(t, x(h(t)), \lambda_m) - f^{(m)}_k(t, y(h(t)), \lambda_m) |),$$

$$L(d_E(Tx, Ty)) = \left( q \cdot \sum_{j=1}^{k} K_{ij} \cdot \sup_{t \geq 0} \exp(-p(t)) \cdot | x_j(t) - y_j(t) |, \ldots, q \cdot \sum_{j=1}^{k} K_{kj} \cdot \sup_{t \geq 0} \exp(-p(t)) \cdot | x_j(t) - y_j(t) | \right)$$
and therefore $d_E(H_m x, H_n y) \leq L(d_E(Tx, Ty))$.

Let us fix $x$ in $A$. For $t \geq 0$, $1 \leq i \leq k$ and $n \geq 1$ we get

$$|f_i^{(n)}(t, x(h(t)), \lambda_n) - f_i^{(0)}(t, x(h(t)), \lambda_0)| \leq \alpha(t) \cdot \beta(\delta(\lambda_n, \lambda_0)) +$$

$$+ |f_i^{(n)}(t, x(h(t)), \lambda_0) - f_i^{(0)}(t, x(h(t)), \lambda_0)|$$

hence

$$\sup_{t \geq 0} \exp(-p(t)) |f_i^{(n)}(t, x(h(t)), \lambda_n) - f_i^{(0)}(t, x(h(t)), \lambda_0)| \leq c \cdot \beta(\delta(\lambda_n, \lambda_0)) +$$

$$+ \sup \{\exp(-p(t)) |f_i^{(n)}(t, u, \lambda_0) - f_i^{(0)}(t, u, \lambda_0)| \mid (t, u) \in J \times \mathbb{R}^k \}$$

with some constant $c$, and it follows

$$\lim_{n \to \infty} \sup_{t \geq 0} \exp(-p(t)) |f_i^{(n)}(t, x(h(t)), \lambda_n) - f_i^{(0)}(t, x(h(t)), \lambda_0)| = 0.$$ 

Finally, $||d_E(H_n x, H_0 x)|| \to 0$ as $n \to \infty$.

This proves that the theorem 1 and 2 is applicable to the mappings $T$, $H_m(m = 0, 1, \ldots)$, and the proof is finished.

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