A NOTE ON $\alpha$-EQUIVALENT TOPOLOGIES

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Abstract. This paper responds to the question of when are two topologies $\alpha$-equivalent by using some recently introduced classes of sets as well as the classes of regular open sets, nowhere dense sets and dense sets.

1. Introduction

In [10] Njåstad gave a characterization of $\alpha$-equivalent topologies on a given set by means of semi-open sets. It is natural to ask whether $\alpha$-equivalent topologies can be characterized by means of some other classes of subsets which are shared by $\alpha$-equivalent topologies, that is by means of classes of regular open, preopen, semi-preopen, nowhere dense and dense sets. We answer that question in the affirmative and show that two topologies have the same collection of $\alpha$-sets if and only if they share both the semi-regularisation topology and the $\gamma$-topology.

We first recall some definitions. Let $A$ be a subset of a topological space $(X,T)$. The closure of $A$ and the interior of $A$ with respect to $T$ are denoted by $\text{cl}A$ and $\text{int}A$, respectively.

Definition. A subset $A$ of $(X,T)$ is called
(i) an $\alpha$-set if $A \subseteq \text{int}(\text{cl}(\text{int}A))$,
(ii) a semi-open set if $A \subseteq \text{cl}(\text{int}A)$,
(iii) a preopen set if $A \subseteq \text{int}(\text{cl}A)$,
(iv) a semi-preopen set if $A \subseteq \text{cl}(\text{int}(\text{cl}A))$.

The first three notions were introduced by Njåstad [10], Levine [8] and Mashhour et al. [9], respectively. The fourth concept was introduced by Abd El-Mousef et al. [1] under the name of $\beta$-open set and it was called semi-preopen set in [3]. The classes of these sets in a space $(X,T)$ are denoted by $T_{\alpha}$, $\text{SO}(T)$, $\text{PO}(T)$ and $\text{SPO}(T)$, respectively. All of these are larger than $T$ and are closed under arbitrary unions. Njåstad [10] showed that $T_{\alpha}$ is a topology on $X$.

For a space $(X,T)$ the family $\{ A \subset X \mid A \cap B \in \text{PO}(T) \text{ whenever } B \in \text{PO}(T) \}$ will be denoted by $T_{\gamma}$. It was shown in [4] that $T_{\gamma}$ is a topology on $X$ larger than $T_{\alpha}$. It will be called the $\gamma$-topology of $T$. 

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The classes of regular open sets, nowhere dense sets and dense sets in \((X, T)\) will be denoted by \(\text{RO}(T)\), \(\text{N}(T)\) and \(\text{D}(T)\) respectively. The complement of a regular open set is called regular closed, and the complement of a dense set is called \textit{codense}.

The following results will be needed in the sequel.

**Proposition 1.** ([10], [2] and [8]) Let \((X, T)\) be a space. Then:

(i) \(T_a = T_{a,a}\),
(ii) \(\text{SO}(T) = \text{SO}(T_a)\),
(iii) \(\text{PO}(T) = \text{PO}(T_a)\),
(iv) \(\text{SPO}(T) = \text{SPO}(T_a)\),
(v) \(\text{RO}(T) = \text{RO}(T_a)\),
(vi) \(\text{D}(T) = \text{D}(T_a)\),
(vii) \(\text{N}(T) = \text{N}(T_a)\).

**Proposition 2.** ([3]) A subset of a space \((X, T)\) is semi-preopen if and only if \(\text{cl} A\) is regular closed. ■

**Proposition 3.** ([3]) In a space \((X, T)\), the intersection of every open and each semi-preopen set is semi-preopen. ■

**Proposition 4.** ([7]) In a space \((X, T)\), a subset \(A\) is semi-open if and only if it is semi-preopen and \(\text{int(cl} A\) \(\subseteq \text{cl(int} A)\). ■

**Proposition 5.** ([7]) If the topologies \(T\) and \(U\) on a set \(X\) have the same \(\gamma\)-topology, then they have the same class of nowhere dense sets. ■

**Proposition 6.** ([7]) Two topologies on a set \(X\) have the same class of semi-preopen sets if and only if their \(\gamma\)-topologies are the same. ■

2. \(\alpha\)-equivalent topologies

**Definition.** ([10]) Two topologies on a set \(X\) are called \(\alpha\)-equivalent if their \(\alpha\)-topologies are the same.

In the same paper Njastad obtained the following result on \(\alpha\)-equivalence in terms of semi-open sets.

**Proposition 7.** Two topologies on a set \(X\) are \(\alpha\)-equivalent if and only if they have the same class of semi-open sets. ■

We consider the same question in relation to the other classes of subsets which are shared by \((X, T)\) and \((X, T_a)\) mentioned in Proposition 1, to obtain analogous characterizations of \(\alpha\)-equivalence. We begin with a consequence of Propositions 7 and 1.

**Proposition 8.** Let \(T\) and \(U\) be two \(\alpha\)-equivalent topologies. Then:

(i) \(\text{PO}(T) = \text{PO}(U)\),
(ii) \(\text{SPO}(T) = \text{SPO}(U)\),
(iii) \(\text{RO}(T) = \text{RO}(U)\),
(iv) \(\text{D}(T) = \text{D}(U)\),
(v) \(\text{N}(T) = \text{N}(U)\). ■

It should be noted that none of these five conditions is equivalent to the condition \(\text{SO}(T) = \text{SO}(U)\). One can easily find the examples. But it is a natural question whether there are two conditions among these five which together imply \(\text{SO}(T) = \text{SO}(U)\). We prove that any pair of independent conditions does so. From Propositions 5 and 6 it follows immediately that for the statements in Proposition
8, (i) $\implies$ (ii) $\implies$ (v) hold in general. The following result shows that (iv) $\implies$ (v) is also true. The closure and the interior of a set $A$ in $(X, U)$ will be denoted by $\text{cl}_U A$ and $\text{int}_U A$ respectively.

**Proposition 9.** Let $T$ and $U$ be topologies on $X$ having the same class of dense sets. Then their classes of nowhere dense sets coincide.

**Proof.** Suppose that $A \in \text{N}(T) - \text{N}(U)$. Then $G = \text{int}(\text{cl}_U A) \neq \emptyset$ because $(X, T)$ and $(X, U)$ have the same class of codense sets. Define $W = \text{int}_U (G - \text{cl} A)$. Since $A$ is $T$-nowhere dense, $G - \text{cl} A$ is $T$-open and non-empty. Hence $W \subseteq \text{cl}_U A - A$ and $W \neq \emptyset$, a contradiction. Therefore $\text{N}(T) \subseteq \text{N}(U)$. The reverse inclusion is shown in an analogous way. ■

In order to prove our main result we first establish two lemmas.

**Lemma 1.** If $T$ and $U$ are topologies on $X$ sharing the classes of semi-preopen sets and regular open sets, then they have the same class of dense sets.

**Proof.** Assume that $\text{cl} A = X$. Then $A \in \text{SPO}(T) = \text{SPO}(U)$ and so $\text{cl}_U A$ is $U$-regular closed by Proposition 2. Hence $\text{cl}_U A$ is $T$-regular closed and thus $\text{cl}_U A = X$, i.e. $A \in \text{D}(U)$. The reverse inclusion is obtained analogously. ■

**Lemma 2.** Let $T$ and $U$ be topologies on $X$ having the same class of dense sets, and let $A$ be a subset of $X$. Then $\text{int}(\text{cl} A) \subseteq \text{cl} \left( \text{int}(A) \right)$ if and only if $\text{int}_U \text{cl}_U A \subseteq \text{cl}_U \text{int}_U A$.

**Proof.** Suppose that $\text{int}(\text{cl} A) \subseteq \text{cl} \left( \text{int}(A) \right)$ and let $W = \text{int}_U (\text{cl}_U A - \text{int}_U A)$ be non-empty. Then $G = \text{int} W \neq \emptyset$ because $(X, T)$ and $(X, U)$ share the class of codense sets. Put $G_1 = G - \text{cl} A$ and $G_2 = G \cap \text{int} A$ and let $W_1 = \text{int}_U G_1$ and $W_2 = \text{int}_U G_2$. We observe that $W_1 \subseteq \text{cl}_U A - \text{cl} A$ which implies $W_1 = \emptyset$ and so $\text{int} G_1 = G_1 = \emptyset$. On the other hand, $W_2 \subseteq A$ and $W_2 \cap \text{int}_U A \subseteq W \cap \text{int}_U A = \emptyset$. Hence $W_2 = \emptyset$ and thus $\text{int} G_2 = G_2 = \emptyset$. Hence both $G_1$ and $G_2$ are empty, $G \subseteq \text{cl} A - \text{int} A$ and so $G \subseteq \text{int}(\text{cl} A) - \text{cl}(\text{int} A) = \emptyset$, a contradiction. Therefore $W = \emptyset$, that is $\text{int}_U \text{cl}_U A \subseteq \text{cl}_U \text{int}_U A$. ■

**Theorem 1.** Let $T$ and $U$ be topologies on a set $X$. Then the following are equivalent:

(a) $T$ and $U$ are $\alpha$-equivalent,

(b) $\text{RO}(T) = \text{RO}(U)$ and $\text{PO}(T) = \text{PO}(U)$.

(c) $\text{RO}(T) = \text{RO}(U)$ and $\text{SPO}(T) = \text{SPO}(U)$.

(d) $\text{RO}(T) = \text{RO}(U)$ and $\text{N}(T) = \text{N}(U)$.

(e) $\text{RO}(T) = \text{RO}(U)$ and $\text{D}(T) = \text{D}(U)$.

(f) $\text{PO}(T) = \text{PO}(U)$ and $\text{D}(T) = \text{D}(U)$.

(g) $\text{SPO}(T) = \text{SPO}(U)$ and $\text{D}(T) = \text{D}(U)$.

**Proof.** (a) $\implies$ (b), (b) $\implies$ (c) and (c) $\implies$ (d) follow from Propositions 8, 5 and 6. Also, (a) $\iff$ (e) and (a) $\implies$ (f) follow from Proposition 8, (e) $\iff$ (d) from Proposition 9 and (f) $\iff$ (g) from Proposition 6.

(d) $\implies$ (e): Suppose $A \in \text{SPO}(T)$ and put $B = A - \text{cl}_U \text{int}_U A$. Then $B \in \text{N}(U) = \text{N}(T)$ by (d). On the other hand, $\text{cl}_U \text{int}_U A$ is $U$-regular closed
and so $T$-regular closed. Hence $B \in \textbf{SPO}(T)$ by Proposition 3 and thus $B = \emptyset$, that is $A \in \textbf{SPO}(U)$.

(c) $\implies$ (a): Suppose that $A \in \textbf{SO}(U)$. According to Lemma 1, $(X,T)$ and $(X,U)$ share the class of codense sets and so $\text{int}_U(A - \text{cl}(\text{int } A)) = \emptyset$, that is $\text{int}_U A \subset \text{cl}_U \text{cl}(\text{int } A)$. Therefore $A \subset \text{cl}_U \text{int}_U A \subset \text{cl}_U \text{cl}(\text{int } A)$. On the other hand, $\text{cl}(\text{int } A)$ is $T$-regular closed and so $U$-regular closed by (c). Hence $A \subset \text{cl}(\text{int } A)$, that is $A \in \textbf{SO}(T)$. The reverse inclusion is obtained analogously.

(g) $\implies$ (a): Suppose that $A \in \textbf{SO}(T)$. Then $A \in \textbf{SPO}(T)$ and $\text{int}(\text{cl } A) \subset \text{cl}(\text{int } A)$ by Proposition 4. According to Lemma 2 and (g) we have that $A \in \textbf{SPO}(U)$ and $\text{int}_U \text{cl}_U A \subset \text{cl}_U \text{int}_U A$ and thus $A \in \textbf{SO}(U)$ again by Proposition 4. The reverse inclusion is obtained analogously. 

For a space $(X,T)$ the topology $T_a$ on $X$ which has as a base the class $\textbf{RO}(T)$ is called the semiregularisation topology of $(X,T)$. Recall that $T_{sa} = T_a$ and $\textbf{RO}(T) = \textbf{RO}(U)$ if and only if $T_s = U_s$ [6]. Theorem 1 and Proposition 6 give the following characterization.

**Theorem 2.** Let $T$ and $U$ be topologies on a set $X$. Then $T_a = U_a$ if and only if $T_\gamma = U_\gamma$ and $T_s = U_s$.

**Corollary 1.** Let $(X,T)$ be a space. Then $T_{sa} = T_a$ if and only if $T_{sr} = T_\gamma$.

It was shown in [5] that the topologies $T$ and $T_\gamma$ on a set $X$ are $\alpha$-equivalent if and only if they share the class of dense sets. Our Theorem 3 will be a slight improvement of this result. For convenience we first establish two simple lemmas.

**Lemma 3.** ([4]) Let $(X,T)$ be a space. Then:

(i) $\text{cl}_T A \subset \text{cl}_T \text{cl}_U A + \text{int}_U A$.

(ii) $\text{int}_U A \subset \text{cl}_T \text{int}_U A$.

**Lemma 4.** Let $T$ and $U$ be topologies on $X$ such that $U \subset T_\gamma$. Then $\text{cl}(\text{int } A) \subset \text{cl}_T \text{int}_U A$ for any subset $A$.

**Proof.** By the assumption and Lemma 3 we have that $\text{cl}(\text{int } A) = \text{cl}_T \text{int}_U A \subset \text{cl}_T \text{int}_U A$.

**Theorem 3.** Let $T$ and $U$ be topologies on a set $X$ satisfying $T \subset U_\gamma$ and $U \subset T_\gamma$. Then $T$ and $U$ are $\alpha$-equivalent if and only if they have the same class of dense sets.

**Proof.** Assume $D(T) = D(U)$ and let $A \in \textbf{SO}(T)$. According to Proposition 4 and Lemma 2 it suffices to show that $A \in \textbf{SPO}(U)$. By Lemma 4 we have $\text{cl}_T A \subset \text{cl}_T \text{cl}(\text{int } A) \subset \text{cl}_U \text{int}_U A$ and so $\text{cl}_U A = \text{cl}_T \text{int}_U A$. Since $\text{int}_A \subset T \subset U_\gamma \subset \textbf{PO}(U)$ we have by Proposition 2 that $\text{cl}_U A$ is $U$-regular closed and hence $A \in \textbf{SPO}(U)$. 

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REFERENCES


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