PROPERTY \((gR)\) UNDER NILPOTENT COMMUTING PERTURBATION

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Abstract. The property \((gR)\), introduced in [Aiena, P., Guillen, J. and Peña, P., Property \((gR)\) and perturbations, to appear in Acta Sci. Math. (Szeged), 2012], is an extension to the context of B-Fredholm theory, of property \((R)\), introduced in [Aiena, P., Guillen, J. and Peña, P., Property \((R)\) for bounded linear operators, Mediterr. J. Math. 8 (4), 491-508, 2011]. In this paper we continue the study of property \((gR)\) and we consider its preservation under perturbations by finite rank and nilpotent operators. We also prove that if \(T\) is left polaroid (resp. right polaroid) and \(N\) is a nilpotent operator which commutes with \(T\) then \(T + N\) is also left polaroid (resp. right polaroid).

1. Introduction and preliminaries

Throughout this paper \(L(X)\) denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space \(X\). For \(T \in L(X)\), we denote by \(N(T)\) the null space of \(T\) and by \(R(T) = T(X)\) the range of \(T\). We denote by \(\alpha(T) := \dim N(T)\) the nullity of \(T\) and by \(\beta(T) := \text{codim} R(T) = \dim X/R(T)\) the defect of \(T\). Other two classical quantities in operator theory are the ascent \(p = p(T)\) of an operator \(T\), defined as the smallest non-negative integer \(p\) such that \(N(T^p) = N(T^{p+1})\) (if such an integer does not exist, we put \(p(T) = \infty\)), and the descent \(q = q(T)\), defined as the smallest non-negative integer \(q\) such that \(R(T^q) = R(T^{q+1})\) (if such an integer does not exist, we put \(q(T) = \infty\)). It is well known that if \(p(T)\) and \(q(T)\) are both finite then \(p(T) = q(T)\). Furthermore, \(0 < p(\lambda I - T) = q(\lambda I - T) < \infty\) if and only if \(\lambda\) is a pole of the resolvent, see [14, Prop. 50.2]. An operator \(T \in L(X)\) is said to be Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm), if \(\alpha(T), \beta(T)\) are both finite (respectively, \(R(T)\) closed and \(\alpha(T) < \infty, \beta(T) < \infty\)). \(T \in L(X)\) is said to be semi-Fredholm if \(T\) is either an upper semi-Fredholm or a lower semi-Fredholm operator. If \(T\) is semi-Fredholm, the index of \(T\) is defined by \(\text{ind} T := \alpha(T) - \beta(T)\). Other two important classes of operators in Fredholm theory are the classes of semi-Browder operators. These classes are defined as follows. \(T \in L(X)\) is said to be Browder...
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A bounded operator \(T \in L(X)\) is said to be upper semi-Weyl (respectively, lower semi-Weyl) if \(T\) is upper Fredholm operator (respectively, lower semi-Fredholm) and index \(\text{ind } T \leq 0\) (respectively, \(\text{ind } T \geq 0\)). A bounded operator \(R \in L(X)\) is said to be Riesz if \(\lambda I - T\) is a Fredholm operator for all \(\lambda \neq 0\), i.e. \(\sigma_f(T) \subseteq \{0\}\). The classical Riesz-Schauder theory of compact operators shows that every compact operator is Riesz. Also quasi-nilpotent operators (in particular nilpotent operators) are Riesz, since \(\sigma_f(Q) \subseteq \sigma(Q) = \{0\}\) for any operator quasi-nilpotent \(Q \in L(X)\). Browder spectra and Weyl spectra are invariant under commuting Riesz perturbations (see [15, 16]), i.e. if \(R\) is a Riesz operator such that \(TR = RT\),

\[
\sigma_{ab}(T) = \sigma_{ab}(T + R) \quad \text{and} \quad \sigma_{uw}(T) = \sigma_{uw}(T + R).
\]

Recall that \(T \in L(X)\) is said to be bounded below if \(T\) is injective and has closed range. Denote by \(\sigma_{ap}(T)\) the classical approximate point spectrum defined by

\[
\sigma_{ap}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below} \}.
\]

Note that if \(\sigma_s(T)\) denotes the surjectivity spectrum

\[
\sigma_s(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not onto} \}.
\]

Obviously, \(\sigma(T) = \sigma_{ap}(T) \cup \sigma_s(T)\). Furthermore \(\sigma_{ap}(T) = \sigma_{ap}(T^*)\) and \(\sigma_s(T) = \sigma_{ap}(T^*)\), where \(T^*\) is the dual of \(T\).

**Theorem 1.1.** [1] If \(T \in L(X)\) and \(Q\) is a quasi-nilpotent operator commuting with \(T\) then

(i) \(\sigma(T) = \sigma(T + Q)\),

(ii) \(\sigma_{ap}(T) = \sigma_{ap}(T + Q)\),

(iii) \(\sigma_s(T) = \sigma_s(T + Q)\).
2. Semi B-Browder spectra under nilpotent perturbations

Given $n \in \mathbb{N}$, we denote by $T_n$ the restriction of $T \in L(X)$ on the subspace $R(T^n) = T^n(X)$. According to [10, 11], $T$ is said to be semi B-Fredholm (respectively, B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and $T_n$, viewed as an operator from the space $R(T^n)$ into itself, is a semi-Fredholm operator (respectively, Fredholm, upper semi-Fredholm, lower semi-Fredholm). Analogously, $T \in L(X)$ is said to be B-Browder (respectively, upper semi B-Browder, lower semi B-Browder), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and $T_n$ is a Browder operator (respectively, upper semi-Browder, lower semi-Browder). If $T_n$ is a semi-Fredholm operator, it follows from [11, Proposition 2.1] that also $T_m$ is semi-Fredholm for every $m \geq n$, and $\text{ind} T_m = \text{ind} T_n$. This enables us to define the index of a semi B-Fredholm operator $T$ as the index of the semi-Fredholm operator $T_n$. Thus, a bounded operator $T \in L(X)$ is said to be a B-Weyl operator if $T$ is a B-Fredholm operator having index 0. $T \in L(X)$ is said to be upper semi B-Weyl if $T$ is upper semi B-Fredholm with index $\text{ind} T \leq 0$, and $T$ is said to be lower semi B-Weyl if $T$ is lower semi B-Fredholm with $\text{ind} T \geq 0$. Note that if $T$ is B-Fredholm then also $T^*$ is B-Fredholm with $\text{ind} T^* = -\text{ind} T$.

The classes of operators defined above motivate the definitions of several spectra. The upper semi B-Browder spectrum is defined by

$$\sigma_{ubb}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Browder} \}.$$ 

The lower semi B-Browder spectrum is defined by

$$\sigma_{lbb}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Browder} \},$$

while the B-Browder spectrum is defined by

$$\sigma_{bb}(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder} \}.$$ 

Clearly, $\sigma_{bb}(T) = \sigma_{ubb}(T) \cup \sigma_{lbb}(T)$. The B-Weyl spectrum is defined by

$$\sigma_{bw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl} \},$$

the upper semi B-Weyl spectrum and lower semi B-Weyl spectrum are defined, respectively, by

$$\sigma_{ubw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl} \},$$

and

$$\sigma_{lbw}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl} \}.$$ 

**Definition 2.1.** $T \in L(X)$ is said to be left (resp. right) Drazin invertible if $p = p(T) < \infty$ (resp. $q = q(T) < \infty$) and $T^{p+1}(X)$ (resp. $T^q(X)$) is closed. $T \in L(X)$ is said to be Drazin invertible if $p(T) = q(T) < \infty$. If $\lambda I - T$ is left (resp. right) Drazin invertible and $\lambda \in \sigma_{ap}(T)$ (resp. $\lambda \in \sigma_s(T)$) then $\lambda$ is said to be a left (resp. right) pole.

Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if $T$ is Drazin invertible. In fact, if $0 < p = p(T) = q(T) < \infty$, then $T^p(X) = T^{p+1}(X)$ is
the kernel of the spectral projection associated with the spectral set \{0\} [14, Prop. 50.2]. The left Drazin spectrum is then defined as

$$\sigma_{ld}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible} \}$$

the right Drazin spectrum is defined as

$$\sigma_{rd}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible} \}$$

and Drazin spectrum is defined as

$$\sigma_{d}(T) := \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible} \}$$

Obviously, $$\sigma_{d}(T) = \sigma_{ld}(T) \cup \sigma_{rd}(T)$$. Furthermore $$\sigma_{ld}(T) = \sigma_{rd}(T^*)$$ and $$\sigma_{d}(T) = \sigma_{d}(T^*)$$, where $$T^*$$ is the dual of $$T$$, see Theorem 2.1 of [3].

**Theorem 2.2.** [13] If $$T \in L(X)$$ then we have

(i) $$T$$ is right Drazin invertible if and only if there exists a $$k \in \mathbb{N}$$ such that $$T^k(X)$$ is closed and $$T_k$$ is onto. In this case $$T^j(X)$$ is closed and $$T_j$$ is onto for all naturals $$j \geq k$$.

(ii) $$T$$ is left Drazin invertible if and only if $$T$$ is upper semi B-Browder.

(iii) $$T$$ is right Drazin invertible if and only if $$T$$ is lower semi B-Browder.

(iv) $$T$$ is Drazin invertible if and only if $$T$$ is B-Browder.

**Corollary 2.3.** If $$T \in L(X)$$ then we have

$$\sigma_{ubb}(T) = \sigma_{ld}(T), \quad \sigma_{lbb}(T) = \sigma_{rd}(T) \quad \text{and} \quad \sigma_{bb}(T) = \sigma_{d}(T).$$

It has been observed in [9], that the B-Browder spectrum is invariant under commuting finite dimensional perturbation. In the next propositions we prove that all Drazin spectra are invariant under nilpotent commuting perturbations.

**Theorem 2.4.** Let $$T \in L(X)$$ and $$N$$ be a nilpotent operator which commutes with $$T$$. Then $$\sigma_{rd}(T + N) = \sigma_{ld}(T + N) = \sigma_{lbb}(T) = \sigma_{d}(T)$$.

**Proof.** Suppose that $$\lambda \notin \sigma_{lbb}(T)$$. By part (iii) of Theorem 2.2, $$\lambda I - T$$ is right Drazin invertible and hence, $$q = q(\lambda I - T) < \infty$$ and $$(\lambda I - T)^q(X)$$ is closed. Let $$n \in \mathbb{N}$$ be such that $$N^n = 0$$ and set $$m_1 = \max\{q, n\}$$. We claim that

$$[(\lambda I - T) + N]^{2k}(X) \subseteq (\lambda I - T)^q(X) \quad \text{for all} \ k \geq m_1. \quad (1)$$

To show this, let $$y \in [(\lambda I - T) + N]^{2k}(X)$$ be arbitrary, so that there exists $$x \in X$$ for which $$[(\lambda I - T) + N]^{2k}(x) = y$$. Then

$$y = \sum_{i=0}^{2k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x))$$

$$= \sum_{i=0}^{k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x)) + \sum_{i=k+1}^{2k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x))$$

$$= \sum_{i=0}^{k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x))$$

$$= (\lambda I - T)^k \left[ \sum_{i=0}^{k} \mu_{i,k} N^i((\lambda I - T)^{k-i}(x)) \right].$$
Therefore \( y \in (\lambda I - T)^k(X) \). Hence, since \( k \geq q \),
\[
[(\lambda I - T) + N]^{2k}(X) \subseteq (\lambda I - T)^k(X) = (\lambda I - T)^q(X).
\]
To prove the opposite inclusion, observe, by using (2), that it also follows that
\[
(\lambda I - T)^q(X) = (\lambda I - T)^{4k}(X) = [(\lambda I - T) + N - N]^{4k}(X)
\]
\[
\subseteq [(\lambda I - T) + N]^{2k}(X),
\]
from which the equality (1) follows. Consequently, \((\lambda I - T)^{2k}(X)\) is closed for all \(k\) sufficiently large. Now, from part (i) of Theorem 2.2, we can choose \(k\) such that the restriction \((\lambda I - T)_{2k}\) of \((\lambda I - T)\) to \(M = (\lambda I - T)^{2k}(X) = [(\lambda I - T) + N]^{2k}(X)\) is onto. If \(N_{2k}\) denotes the restriction of \(N\) to \(M\), then \((\lambda I - T)_{2k} + N_{2k} = [(\lambda I - T) + N]_{2k}\) is onto, so, by Theorem 2.2, part (i), \((\lambda I - T) + N\) is right Drazin invertible, or equivalently, lower semi B-Browder. This shows that \(\sigma_{lbb}(T + N)\) and by symmetry the opposite inclusion holds, so the equality \(\sigma_{lbb}(T + N) = \sigma_{lbb}(T)\).

By duality we have

**Corollary 2.5.** Let \(T \in L(X)\) and \(N\) be a nilpotent operator which commutes with \(T\). Then \(\sigma_{ld}(T + N) = \sigma_{ubb}(T + N) = \sigma_{ubb}(T) = \sigma_{ld}(T)\) and \(\sigma_{d}(T + N) = \sigma_{bb}(T + N) = \sigma_{bb}(T) = \sigma_{d}(T)\).

**Remark 2.6.** Theorem 2.4 and Corollary 2.5 answer positively to a question from [6], in particular it improves Theorem 4.3, where the invariance of the spectrum \(\sigma_{lbb}(T)\), under commuting nilpotent perturbations, was proved assuming that \(T\) has SVEP, while the invariance of \(\sigma_{ubb}(T)\) was proved assuming that \(T^*\) has SVEP.

### 3. Property \((gR)\) under nilpotent perturbations

For an operator \(T \in L(X)\) define
\[
E(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T)\},
\]
\[
E^a(T) = \{\lambda \in \text{iso } \sigma_{ap}(T) : 0 < \alpha(\lambda I - T)\},
\]
\[
\Pi_{90}(T) = \sigma(T) \setminus \sigma_{bb}(T),
\]
\[
\Pi_{90}^a(T) = \sigma_{ap}(T) \setminus \sigma_{ubb}(T).
\]

**Definition 3.1.** A bounded \(T \in L(X)\) is said to satisfy:

(i) property \((gR)\) if \(\sigma_{ap}(T) \setminus \sigma_{ubb}(T) = E(T)\);

(ii) property \((gR^a)\) if \(\sigma_{ap}(T) \setminus \sigma_{ubb}(T) = E^a(T)\);

(iii) property \((gw)\) if \(\sigma(T)_{ap} \setminus \sigma_{ubw}(T) = E(T)\);

(iv) generalized a-Weyl’s theorem if \(\sigma_{ap}(T) \setminus \sigma_{ubw}(T) = E^a(T)\).

Also a-Browder’s theorem admits a generalized version, the generalized a-Browder’s theorem, which means that \(T\) satisfies \(\sigma_{ubw}(T) = \sigma_{ubb}(T)\). However, a-Browder’s theorem and generalized a-Browder’s theorem are equivalent, for a proof see [4].
Theorem 3.2. [7] If $T \in L(X)$, then we have
(i) $T$ satisfies property $(gw)$ if and only if $a$-Browder’s theorem and property $(gR)$ holds for $T$;
(ii) $T$ satisfies generalized $a$-Weyl’s theorem if and only if $a$-Browder’s theorem and property $(gR^n)$ holds for $T$.

Theorem 3.3. Let $T \in L(X)$ and $N$ be a nilpotent operator which commutes with $T$. Then $E(T) = E(T + N)$ and $E^n(T) = E^n(T + N)$.

Proof. Suppose that $N^n = 0$. It is easily seen that
\[ N(\lambda I - T) \subseteq N(\lambda I - T + N)^n. \]  
Indeed, if $x \in N(\lambda I - T)$ then for some suitable binomial coefficients $\mu_{n,j}$, we have
\[ (\lambda I - T + N)^n x = \sum_{j=1}^{n} \mu_{n,j}(\lambda I - T)^j N^{n-j}x = 0, \]
hence $x \in N(\lambda I - T + N)^n$.

Now, let $\lambda \in E(T)$. Then $\lambda \in \text{iso} \sigma(T) = \text{iso} \sigma(T + N)$ and $\alpha(\lambda I - T) > 0$. Suppose that $\alpha(\lambda I - T + N) = 0$. Then $\alpha(\lambda I - T + N)^k = 0$ for all $k \in \mathbb{N}$. From the inclusion (3), we have $\alpha(\lambda I - T) = 0$ and this is impossible. Therefore $\alpha(\lambda I - T + N) > 0$. Consequently, $E(T) \subseteq E(T + N)$ and, again by symmetry, the opposite inclusion holds. Therefore, $E(T) = E(T + N)$. Similarly we can prove that $E^n(T) = E^n(T + N)$.

Theorem 3.4. Let $T \in L(X)$ and $N$ be a nilpotent operator which commutes with $T$. Then $T$ satisfies the property $(gR)$ if and only if $T + N$ satisfies the property $(gR)$.

Proof. By Theorem 3.3 and Theorem 2.4, it follows that
\[ E(T + N) = E(T) = \sigma_{ap}(T) \setminus \sigma_{ubb}(T) = \sigma_{ap}(T + N) \setminus \sigma_{ubb}(T + N), \]
hence $T + N$ satisfies property $(gR)$. By symmetry the reciprocal holds.

Theorem 3.5. Let $T \in L(X)$ and $N$ be a nilpotent operator which commutes with $T$. Then $T$ satisfies the property $(gR^n)$ if and only if $T + N$ satisfies the property $(gR^n)$.

Proof. By Theorem 3.3 and Theorem 2.4, it follows that
\[ E^n(T + N) = E^n(T) = \sigma_{ap}(T) \setminus \sigma_{ubb}(T) = \sigma_{ap}(T + N) \setminus \sigma_{ubb}(T + N), \]
hence $T + N$ satisfies property $(gR^n)$. By symmetry the reciprocal holds.

Definition 3.6. $T \in L(X)$ is said to be left (resp. right) polaroid if $\sigma_{ap}(T)$ is empty or every isolated point of $\sigma_{ap}(T)$ is a left pole (resp. $\sigma_{s}(T)$ is empty or every isolated point of $\sigma_{s}(T)$ is a right pole).

Theorem 3.7. If $T \in L(X)$ is a left polaroid and $N$ is a nilpotent operator commuting with $T$, then $T$ is a left polaroid if only if $T + N$ is a left polaroid.
Proof. Obviously, by Corollary 2.3, we have $\text{iso } \sigma_{ap}(T) = \sigma_{ap}(T) \setminus \sigma_{ubb}(T)$. Therefore,

$$
\text{iso } \sigma_{ap}(T + N) = \text{iso } \sigma_{ap}(T) \\
= \sigma_{ap}(T) \setminus \sigma_{ubb}(T) \\
= \sigma_{ap}(T + N) \setminus \sigma_{ubb}(T + N).
$$

Thus $T + N$ is left polaroid. By symmetry the reciprocal holds.

Remark 3.8. The result of Theorem 3.9 improves Corollary 2.12 of [2], where it was proved that $T + N$ is a left polaroid assuming that $T$ is a left polaroid and $T^*$ has SVEP at the points $\lambda \notin \sigma_{uw}(T)$.

Theorem 3.9. If $T \in L(X)$ is a right polaroid and $N$ is a nilpotent operator commuting with $T$, then $T$ is a right polaroid if only if $T + N$ is a right polaroid.

Proof. Obviously, by Corollary 2.3, we have $\text{iso } \sigma_s(T) = \sigma_s(T) \setminus \sigma_{ubb}(T)$. Therefore,

$$
\text{iso } \sigma_s(T + N) = \text{iso } \sigma_s(T) \\
= \sigma_s(T) \setminus \sigma_{ubb}(T) \\
= \sigma_s(T + N) \setminus \sigma_{ubb}(T + N).
$$

Thus $T + N$ is a right polaroid. By symmetry the reciprocal holds.

Remark 3.10. The result of Theorem 3.9 improves Corollary 2.12 of [2], where it was proved that $T + N$ is a right polaroid assuming that $T$ is a right polaroid and $T^*$ has SVEP at the points $\lambda \notin \sigma_{uw}(T)$.

As in the above theorems, for the $(gw)$ property introduced in [8], we have the following result.

Theorem 3.11. Let $T \in L(X)$ and $N$ be a nilpotent operator which commutes with $T$. Then $T$ satisfies the property $(gw)$ if only if $T + N$ satisfies the property $(gw)$.

Proof. Suppose that $T$ satisfies property $(gw)$. Then $T$ satisfies generalized a-Browder’s theorem, or equivalently a-Browder’s theorem, i.e. $\sigma_{ub}(T) = \sigma_{uw}(T)$. Since these spectra are invariant under $N$, we have that $T + N$ satisfies a-Browder’s theorem. Then, from Theorems 3.4 and 3.2, it follows that $T + N$ satisfies property $(gw)$. By symmetry the reciprocal holds.

As in the above theorems, for the generalized a-Weyl theorem introduced in [12], we have the following result.

Theorem 3.12. Let $T \in L(X)$ and $N$ be a nilpotent operator which commutes with $T$. Then $T$ satisfies the generalized a-Weyl Theorem if only if $T + N$ satisfies the generalized a-Weyl Theorem.

Proof. Suppose that $T$ satisfies generalized a-Weyl’s theorem. Then since a-Browder’s theorem and property $(gR)$ are invariant under $N$, it follows from Theorem 3.2, that $T + N$ satisfies the generalized a-Weyl’s theorem. By symmetry the reciprocal holds.
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