ASYMPTOTIC PLANARITY OF DRESHER MEAN VALUES

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Abstract. A family of Dresher mean values is asymptotically planar with respect to its two parameters. An asymptotic formula presenting this property holds if: (a) all variables converge to the same value; and, equivalently, because of means homogeneity, (b) for variables with same additive increment converging to infinity.

Suppose $x = (x_1, x_2, \ldots, x_n)$, $a = (a, a, \ldots, a)$, and $q = (q_1, q_2, \ldots, q_n)$ are sequences of nonnegative reals and $a > 0$. Without loss of generality let the weights $q_i$ be normalized by $q_1 + q_2 + \cdots + q_n = 1$. The geometric, the harmonic, and the quadratic mean values respectively are

$$G_q(x) = \prod_{i=1}^{n} x_i^{q_i}, \quad A_q(x) = \sum_{i=1}^{n} q_i x_i, \quad Q_q(x) = \sqrt[n]{\sum_{i=1}^{n} q_i x_i^2}.$$ 

Note that $\sigma^2_q(x) = Q_q^2(x) - A_q^2(x)$ is a weighted variance of $x$, which satisfies $\sigma^2_q(x + a) = \sigma^2_q(x)$. Dresher mean values [2] are a two-parameter family of means that increase with each parameter

$$D_{s,t}(x) = \begin{cases} 
\left(\frac{\sum_{i=1}^{n} q_i x_i^s}{\sum_{j=1}^{n} q_j x_j^t}\right)^{1/(s-t)}, & \text{if } s \neq t \\
\exp\left(\frac{\sum_{i=1}^{n} q_i x_i^t \log x_i}{\sum_{j=1}^{n} q_j x_j^t}\right), & \text{if } s = t.
\end{cases}$$

Theorem. Dresher mean values for both cases $s \neq t$ and $s = t$ have the unique asymptotic formulas

$$D_{s,t}(x) = A_q(x) + \frac{s + t - 1}{2a} (Q_q^2(x) - A_q^2(x)) + o(Q_q^2(x - a))$$

$$= A_q(x) + (s + t - 1)(Q_q(x) - A_q(x)) + o(Q_q^2(x - a))$$

$$= G_q(x) + (s + t)(A_q(x) - G_q(x)) + o(Q_q^2(x - a)), \quad x \to a,$$
and if \( a \to \infty \), then

\[
D_{s,t}(x + a) = a + A_q(x) + \frac{s + t - 1}{2a^2} (Q^2_q(x) - A^2_q(x)) + o(1/a)
\]

\[
= A_q(x + a) + (s + t - 1)(Q_q(x + a) - A_q(x + a)) + o(1/a)
\]

\[
= G_q(x + a) + (s + t)(A_q(x + a) - G_q(x + a)) + o(1/a).
\]

Asymptotic planarity implies Hoehn and Niven property for Dresher mean values

\[
D_{s,t}(x + a) - a \to A_q(x), \quad a \to \infty.
\]

**Proof.** Suppose \( s \neq t \) and \( h = x - a \). Then

\[
x^s_i = a^s \left(1 + \frac{h_i}{a}\right)^s = a^s \left(1 + \frac{s}{a}h_i + \frac{s(s-1)}{2a^2}h_i^2 + o\right), \quad h_i \to 0,
\]

\[
\sum_{i=1}^{n} q_i x^s_i = a^s \left(1 + \frac{s}{a}A_q(h) + \frac{s(s-1)}{2a^2}Q^2_q(h) + o\right),
\]

where \( o = o(h^2) \) and \( o = o(Q^2_q(h)) \), respectively. Therefore

\[
\log D_{s,t}(x) = \frac{1}{s-t} \left[ \log \sum_{i=1}^{n} q_i x^s_i - \log \sum_{j=1}^{n} q_j x^t_j \right]
\]

\[
= \frac{1}{s-t} \left[ \log a^s + \log \left(1 + \frac{s}{a}A_q(h) + \frac{s(s-1)}{2a^2}Q^2_q(h) + o\right) \right.
\]

\[
- \log a^t - \log \left(1 + \frac{t}{a}A_q(h) + \frac{t(t-1)}{2a^2}Q^2_q(h) + o\right) \left. \right]
\]

\[
= \log a + \frac{1}{s-t} \left[ \frac{s}{a}A_q(h) + \frac{s(s-1)}{2a^2}Q^2_q(h) - \frac{s^2}{2a^2}A^2_q(h) \right.
\]

\[
- \frac{t}{a}A_q(h) - \frac{t(t-1)}{2a^2}Q^2_q(h) + \frac{t^2}{2a^2}A^2_q(h) + o \left. \right]
\]

\[
= \log a + \frac{1}{a}A_q(h) - \frac{1}{2a^2}A^2_q(h) + \frac{s + t - 1}{2a^2} (Q^2_q(h) - A^2_q(h)) + o
\]

\[
= \log \left[a \left(1 + \frac{1}{a}A_q(h)\right) \left(1 + \frac{s + t - 1}{2a^2} (Q^2_q(h) - A^2_q(h))\right)\right] + o.
\]

Since the obtained expression is well defined and continuous at \( s = t \), for both cases \( s \neq t \) and \( s = t \) we have

\[
D_{s,t}(x) = a \exp \left(\frac{1}{a}A_q(h) - \frac{1}{2a^2}A^2_q(h) + \frac{s + t - 1}{2a^2} (Q^2_q(h) - A^2_q(h)) + o\right)
\]

\[
= a \left(1 + \frac{1}{a}A_q(h) - \frac{1}{2a^2}A^2_q(h) + \frac{s + t - 1}{2a^2} (Q^2_q(h) - A^2_q(h)) \right.
\]

\[
+ \frac{1}{2a^2}A^2_q(h) + o \right)
\]

\[
= a + A_q(h) + \frac{s + t - 1}{2a^2} (Q^2_q(h) - A^2_q(h)) + o (Q^2_q(h) - A^2_q(h)).
\]
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This gives the first line of the first formula. The third line follows from asymptotic linearity of power mean values [1], particularly

\[(Q_q(x) - A_q(x))/(A_q(x) - G_q(x)) \to 1, \quad x \to a.\] (1)

In the second formula the first line follows from the above proof with \(a \to \infty\), \(o = o(1/a^2)\), and \(o = o(1/a)\) in the last unspecified appearance of \(o\). Hoehn and Niven property [2], which is a consequence of asymptotic linearity property [1], states

\[M_q(x + a) - A_q(x + a) \to 0, \quad a \to \infty,\]

where \(M\) is any power mean value. Therefore

\[Q_q(x + a) + A_q(x + a)/2a \to 1, \quad a \to \infty,\]

what implies the second line. The third line follows from the asymptotic linearity formula at infinity, i.e. (1) for the argument \(x + a\) and \(a \to \infty\). (The second formula also follows from the first one and from homogeneity of involved mean values.)

Conjecture. Let \(x\) be a sequence of reals and \(a > 0\). The unified asymptotic formula for Dresher mean values holds for convergent variables, as well as for an additive infinitely increasing parameter

\[D_{s,t}(a + x) = a + A_q(x) + \frac{s + t - 1}{2a} \sigma_q^2(x) + o\left(\frac{\sigma_q^2(x)}{2a}\right)\]

\[= A_q(a + x) + (s + t - 1)(Q_q(a + x) - A_q(a + x)) + o\left(\frac{\sigma_q^2(x)}{2a}\right)\]

\[= G_q(a + x) + (s + t)(A_q(a + x) - G_q(a + x)) + o\left(\frac{\sigma_q^2(x)}{2a}\right),\]

where either \(x \to 0\) or \(a \to \infty\). Infinitesimals \(\sigma_q^2(x)/2a, Q_q(a + x) - A_q(a + x)\), and \(A_q(a + x) - G_q(a + x)\) are equivalent.

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References


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