FORMULAS OF VARIATION OF SOLUTION FOR QUASI-LINEAR
CONTROLLED NEUTRAL DIFFERENTIAL EQUATIONS

(Reported on March 17, 2003)

Let $J = [a, b]$ be a finite interval, $O \subset R^n$, $G \subset R^r$ be open sets. Let the function $f : J \times O^s \times G \rightarrow R^n$ satisfy the following conditions: for almost all $t \in J$ the function $f(t, \cdot) : O^s \times G \rightarrow R^n$ is continuously differentiable; for any $(x_1, \ldots, x_s, u) \in O^s \times G$ the functions $f(t, x_1, \ldots, x_s, u), f_x(\cdot), i = 1, \ldots, s, f_u(\cdot)$ are measurable on $J$; for arbitrary compacts $K \subset O, N \subset G$ there exists a function $m_{K,N}(\cdot) \in L(J, R_+), R_+ = [0, \infty)$, such that for any $(x_1, \ldots, x_s, u) \in K^s \times N$ and for almost all $t \in J$, the following inequality is fulfilled

$$|f(t, x_1, \ldots, x_s, u)| + \sum_{i=1}^s |f_{x_i}(\cdot)| + |f_u(\cdot)| \leq m_{K,N}(t).$$

Let the scalar functions $\tau_i(t), i = 1, \ldots, s, t \in R$, and $\eta_j(t), j = 1, \ldots, k$, be absolutely continuous and continuously differentiable, respectively, and satisfying the conditions: $\tau_i(t) \leq t, \tau_i(t) > 0, i = 1, \ldots, s, \eta_j(t) < t, \eta_j(t) > 0, j = 1, \ldots, k$. Let $\Phi$ be the set of continuously differentiable functions $\varphi : J_1 = [\tau, b) \rightarrow O, \tau = \min(\eta_1(a), \eta_2(a), \eta_3(a), \ldots, \eta_k(a)), \varphi = \sup(\varphi(a) | + | \varphi(t) : t \in J), \Omega$ be the set of measurable functions $u : J \rightarrow G$, satisfying the condition $cl\{u(t) : t \in \Omega\}$ is a compact lying in $G, u \in \sup(\{u(t) : t \in J\); $A_j(t), j \in J, i = 1, \ldots, k$, be continuous matrix functions with dimensions $n \times n$.

To every element $\mu = (t_0, x_0, \varphi, u) \in E = J \times 0 \times \Phi \times \Omega$ let us correspond the differential equation

$$\dot{x}(t) = \sum_{j=1}^k A_j(t) \dot{x}(\eta_j(t)) + f(t, x(\tau_1(t)), \ldots, x(\tau_s(t)), u(t)), \quad (1)$$

with discontinuous initial condition

$$x(t) = \varphi(t), \quad t \in [\tau, t_0], \quad x(t_0) = x_0. \quad (2)$$

**Definition 1.** Let $\mu = (t_0, x_0, \varphi, u) \in E, t_0 < b$. The function $x(t) = x(t; \mu) \in O, t \in [\tau, t_1], t_1 \in (t_0, b]$ is said to be a solution corresponding to the element $\mu \in E$, defined on the interval $[\tau, t_1]$, if on the interval $[\tau, t_0]$ the function $x(t)$ satisfies the condition (2), while on the interval $[t_0, t_1]$ it is absolutely continuous and almost everywhere satisfies the equation (1).

Let us introduce the set of variation:

$$V = \{\delta u : (\delta t_0, \delta x_0, \delta \varphi, \delta u) \in E - \tilde{\mu} : |\delta t_0| \leq c, |\delta x_0| \leq c, \parallel \delta \varphi \parallel \leq c, \parallel \delta u \parallel \leq c\},$$

where $\tilde{\mu} \in E$ is a fixed element, $c > 0$ is a fixed number.

Let $\tilde{x}(t)$ be a solution corresponding to the element $\tilde{\mu} = (t_0, \tilde{x}_0, \tilde{\varphi}, \tilde{u}) \in E$, defined on the interval $[\tau, \tilde{t}_1], \tilde{t}_1 \in (a, b), i = 0, 1$. There exist numbers $\epsilon_1 > 0, \delta_1 > 0$, such that for an arbitrary $(\epsilon, \delta \mu) \in [0, \epsilon_1] \times V$ to the element $\tilde{\mu} + \epsilon \delta \mu \in E$ there corresponds a solution $x(t; \tilde{\mu} + \epsilon \delta \mu)$ defined on $[\tau, \tilde{t}_1 + \delta_1]$.

2000 Mathematics Subject Classification. 34K40.

Key words and phrases. Neutral differential equation, variation of solution.
Due to uniqueness, the solution \( x(t; \tilde{\mu}) \) is a continuation of the solution \( \tilde{x}(t) \) to the interval \([\tau, \bar{t}_1 + \delta_1]\). Therefore the solution \( \tilde{x}(t) \) is assumed to be defined on the interval \([\tau, \bar{t}_1 + \delta_1]\).

Let us define the increment of the solution \( \tilde{x}(t) = x(t; \tilde{\mu}) \)

\[
\Delta x(t; \varepsilon \delta \mu) = x(t; \tilde{\mu} + \varepsilon \delta \mu) - \tilde{x}(t), \quad (t, \varepsilon, \delta \mu) \in [\tau, \bar{t}_1 + \delta_1] \times [0, \varepsilon_1] \times V.
\]

In order to formulate the main results, we will need the following notation:

\[
\sigma_i = (\bar{t}_0, \bar{t}_0, \ldots, \bar{t}_0; \bar{t}_0, \ldots, \bar{t}_0; \bar{t}_0), \ldots, \bar{t}_0; \bar{t}_0; \bar{t}_0); \quad i = 0, \ldots, p;
\]

\[
\sigma_i = (\bar{t}_0, \bar{t}_0, \ldots, \bar{t}_0; \bar{t}_0, \bar{t}_0, \bar{t}_0; \bar{t}_0; \bar{t}_0), \ldots, \bar{t}_0; \bar{t}_0; \bar{t}_0); \quad i = 0, \ldots, p;
\]

\[
\sigma_i^n = (\bar{t}_0, \bar{t}_0, \ldots, \bar{t}_0; \bar{t}_0, \bar{t}_0; \bar{t}_0; \bar{t}_0), \ldots, \bar{t}_0; \bar{t}_0; \bar{t}_0); \quad i = 0, \ldots, p;
\]

\[
\gamma_i = \gamma_i(\bar{t}_0), \quad \rho_i = \rho_i(\bar{t}_0), \quad \gamma_i(t) = \gamma_i^{-1}(t), \quad \rho_i(t) = \gamma_i^{-1}(t); \quad \omega = (t, x_1, \ldots, x_s), \]

\[
\int_0^t \omega = f(\omega, \tilde{\mu}(t)), \quad \int_0^t \omega = f(t, \tilde{x}(t_1), \ldots, \tilde{x}(t_s), \tilde{u}(t)).
\]

**Theorem 1.** Let the following conditions be fulfilled:

1. \( \gamma_i = \gamma_i(t), \quad \bar{t}_i < \gamma_i < \tau_i, \) \( \rho_i(\bar{t}_0) < \tilde{t}_1, \) \( j = 1, \ldots, k; \)
2. there exists a number \( \delta > 0 \) such that

\[
\gamma_1(t) \leq \cdots \leq \gamma_p(t), \quad t \in (\tilde{t}_0 - \delta, \tilde{t}_0];
\]

3. there exist the finite limits:

\[
\lim_{\omega \to \sigma_i} \int_0^t \omega = \int_0^t \omega, \quad \omega \in (\tilde{t}_0 - \delta, \tilde{t}_0) \times O^*, \quad i = 0, \ldots, p,
\]

\[
\lim_{\omega \to \sigma_i} \int_0^t \omega = \int_0^t \omega, \quad \omega_1, \omega_2 \in (\gamma_i - \delta, \gamma_i) \times O^*, \quad i = 0, \ldots, p.
\]

Then there exist numbers \( \epsilon_2 > 0, \delta_2 > 0 \) such that for an arbitrary \( (t, \varepsilon, \delta \mu) \in [\bar{t}_1 - \delta_2, \bar{t}_1 + \delta_2] \times (0, \epsilon_2) \times V^- \), \( V^- = \{\delta \mu \in V : \delta \mu \leq 0\} \) the formula

\[
\Delta x(t; \varepsilon \delta \mu) = \varepsilon \delta x(t; \varepsilon \delta \mu) + o(t; \varepsilon \delta \mu)
\]

is valid, where

\[
\delta x(t; \delta \mu) = \{Y(\tilde{t}_0 - t; \tilde{x}(\tilde{t}_0)) - \sum_{j=1}^k A_j(\tilde{t}_0)\tilde{x}(\tilde{t}_0) + \sum_{i=0}^p (\tilde{t}_i - \tilde{t}^-) f_i^0 \}
\]

\[
- \sum_{i=p+1}^k Y(\gamma_i - t; f_i^-, \gamma_i^-) \delta t_0 + \beta(t; \delta \mu),
\]

\[
\tilde{t}^0 = 1, \quad \tilde{t}^- = \gamma_i^- , \quad i = 1, \ldots, p, \quad \tilde{t}^-_{p+1} = 0;
\]

\[
\beta(t; \delta \mu) = \Phi(\tilde{t}_0; t \delta x_0 - \tilde{x}(\tilde{t}_0) \delta t_0) + \sum_{i=p+1}^k \int_{t_0}^{t_0} Y(\gamma_i(t); t) \tilde{f}_i^0 \gamma_i(t) \delta t_0 \tilde{f}_i^0 \gamma_i(t) d\xi +
\]

\[
+ \sum_{j=1}^k \int_{t_0}^{t_0} Y(\rho_j(t); t) A_j(\rho_j(t)) \tilde{x}(\rho_j(t)) \delta t_0 = \int_{t_0}^{t_0} Y(\rho_j(t); t) A_j(\rho_j(t)) \tilde{x}(\rho_j(t)) d\xi;
\]

\[
\lim_{\varepsilon \to 0} o(t; \varepsilon \delta \mu) / \varepsilon = 0,
\]

uniformly with respect to \( (t, \delta \mu) \in [\bar{t}_1 - \delta_2, \bar{t}_1 + \delta_2] \times V^-; \Phi(\xi, t), Y(\xi, t) \) are matrix functions satisfying the system

\[
\begin{cases}
\delta \frac{d \Phi(\xi, t)}{d t} = - \sum_{i=1}^k Y(\gamma_i(t); t) \tilde{f}_i^0 \gamma_i(t), \\
Y(\xi, t) = \Phi(\xi, t) + \sum_{i=1}^k Y(\rho_j(t); t) A_j(\rho_j(t)) \tilde{x}(\rho_j(t)), \quad \xi \in [\tilde{t}_0, t];
\end{cases}
\]
and the condition

$$\Phi(\xi; t) = Y(\xi; t) = \begin{cases} I, & s = t, \\ \Theta, & \xi > t. \end{cases}$$

Here \( I \) is the identity matrix, \( \Theta \) is the zero matrix.

**Theorem 2.** Let the condition 1) of Theorem 1 and the following conditions be fulfilled:

4) there exists number \( \delta > 0 \) such that

$$\gamma_1(t) \leq \cdots \leq \gamma_p(t), \quad t \in [l_0, l_0 + \delta];$$

5) there exists the finite limits:

$$\lim_{\omega \to 0^+} f(\omega) = f_i^+, \quad \omega \in [l_0, l_0 + \delta) \times O^s, \quad i = 0, \ldots, p,$$

$$\lim_{(\omega_1, \omega_2) \to (\sigma_1, \sigma_2)} \left[ f(\omega_1) - f(\omega_2) \right] = f_i^+, \quad \omega_1, \omega_2 \in [\gamma_i, \gamma_i + \delta) \times O^s.$$

Then there exist numbers \( \varepsilon_2 > 0, \delta_2 > 0 \) such that for an arbitrary \((t, \varepsilon, \mu) \in [l_1 - \delta_2, l_1 + \delta_2] \times [0, \varepsilon_2] \times V^+, V^+ = \{ \delta \mu \in V : \delta t_0 \geq 0 \} \) the formula (3) is valid, where \( \delta x(t; \delta \mu) \) has the form

$$\delta x(t; \delta \mu) = \left\{ Y(\tilde{l}_0 + t) \tilde{\varphi}(\tilde{l}_0) - \sum_{j=1}^{k} A_j(\tilde{l}_0) \tilde{\varphi}(\eta_j(\tilde{l}_0)) + \sum_{i=0}^{p} (\hat{\gamma}_{i+1} - \hat{\gamma}_{i}) f_i^+ \right\} -$$

$$- \sum_{i=p+1}^{s} Y(\gamma_{i+1}(t) f_i^+) \delta t_0 + \beta(t; \delta \mu),$$

$$\hat{\gamma}_0^+ = 1, \quad \hat{\gamma}_i^+ = \hat{\gamma}_i^+, \quad i = 1, \ldots, p, \quad \hat{\gamma}_{p+1}^+ = 0.$$  

**Theorem 3.** Let the assumptions of Theorems 1, 2 be fulfilled and

$$\gamma_i, l_0 \notin \{ \eta_1, (\eta_2, \ldots, \eta_{k}, \eta_{k+1}, \ldots) \} \in (a, b):$$

$$e = 1, 2, \ldots, m = 1, \ldots, e, \quad k_m = 1, \ldots, k, \quad i = p + 1, \ldots, s;$$

$$\sum_{i=0}^{p} (\hat{\gamma}_{i+1} - \hat{\gamma}_{i}) f_i^+ = \sum_{i=0}^{p} (\hat{\gamma}_{i+1} - \hat{\gamma}_{i}) f_i^+ = \delta t_0, \quad \hat{\gamma}_i = \hat{\gamma}_i, \quad i = p + 1, \ldots, s.$$

Then there exist numbers \( \varepsilon_2 > 0, \delta_2 > 0 \) such that for an arbitrary \((t, \varepsilon, \delta \mu) \in [l_1 - \delta_2, l_1 + \delta_2] \times [0, \varepsilon_2] \times V^+ = \{ \delta \mu \in V : \delta t_0 \geq 0 \} \) the formula (3) is valid, where \( \delta x(t; \delta \mu) \) has the form

$$\delta x(t; \delta \mu) = \left\{ Y(l_0 + t) \tilde{\varphi}(l_0) - \sum_{j=1}^{k} A_j(l_0) \tilde{\varphi}(\eta_j(l_0)) - f_0 \right\} + \sum_{i=p+1}^{s} Y(\gamma_i(t) f_i) \delta t_0 + \beta(t; \delta \mu).$$

Finally we note that the formulas of variation of solution for various classes of delay and neutral differential equations are given in [1–6].

**References**


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