A REMARK ON SUPRA-ADDITIVE AND SUPRA-MULTIPLICATIVE OPERATORS ON $C(X)$

Z. Ercan, Ankara

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Abstract. M. Radulescu proved the following result: Let $X$ be a compact Hausdorff topological space and $\pi: C(X) \to C(X)$ a supra-additive and supra-multiplicative operator. Then $\pi$ is linear and multiplicative. We generalize this result to arbitrary topological spaces.

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1. The result

We follow the terminology of [1]. As usual for a topological space $X$, the space of real valued continuous (bounded) functions on $K$ is denoted by $C(X)$ ($C_b(X)$). For each $x \in X$, $\delta_x: C(X) \to \mathbb{R}$ is defined by $\delta_x(f) = f(x)$. For $B \subset X$, $\chi_B$ denotes the characteristic function of $B$. For each $n \in \mathbb{R}$, $n$ denotes the constant function with value $n$. A map $\pi: C(X) \to C(Y)$ is called

(i) supra-additive if $\pi(f + g) \geq \pi(f) + \pi(g)$ for each $f, g \in C(X)$,
(ii) supra-multiplicative if $\pi(fg) \geq \pi(f)\pi(g)$ for each $f, g \in C(X)$.

The following theorem is the main result of [4].

**Theorem 1.** Let $X$ be a compact Hausdorff space and $\pi: C(X) \to C(X)$ a supra-additive and supra-multiplicative map. Then $\pi$ is multiplicative and linear.

The main result of this note is to generalize the above theorem as follows.
Theorem 2. Let $X$ and $Y$ be topological spaces and $\pi: C(X) \to C(Y)$ a supra-additive and supra-multiplicative map. Then the following statements are equivalent.

(i) $\pi(f^+ \wedge n - f^- \wedge n)(y) \to \pi(f)(y)$ for each $f \in C(X)$ and $y \in Y$.

(ii) $\pi$ is linear and multiplicative.

Proof. (ii) $\implies$ (i): For each $y \in T$, $\delta_y \circ \pi$ is a Riesz homomorphism, so

\[ \pi(f \wedge n)(y) = \delta_y \circ \pi(f \wedge n) = \delta_y \circ \pi(f) \wedge n \to \delta_y \circ \pi(f) = \pi(f)(y) \]

(i) $\implies$ (ii):

Claim 1. Let $K$ be a compact Hausdorff space and let $T: C(K) \to \mathbb{R}$ be supra-additive and supra-multiplicative. Then $T$ is linear and multiplicative.

Indeed, let $T^\sim: C(K) \to C(K)$ be defined by $T^\sim(f) = T(f)1$. Then $T^\sim$ is supra-additive and supra-multiplicative, so by Theorem 1, $T^\sim$ is linear and multiplicative, so $T$ is linear and multiplicative.

Claim 2. For each topological space $M$ there exists a compact Hausdorff space $K_M$ such that $C(K_M)$ and $C_b(M)$ are Riesz and algebraic isomorphic spaces.

As $C_b(M)$ is an AM-space with order unit $1$, this follows from the Kakutani-Krein Representation Theorem (see [1]).

Claim 3. Let $\pi^\sim = \pi|_{C_b(X)}$. Then for each $y \in Y$, $\delta_y \circ \pi^\sim: C_b(X) \to \mathbb{R}$ is linear and multiplicative.

This follows from Theorem 1 and from the above claims.

Claim 4. $\pi$ is linear.

To see this we use the linearity of $\delta_y \circ \pi^\sim$ as follows. Let $f, g \geq 0$ be given. Then

\[ \pi(f + g)(y) = \lim \delta_y \circ \pi^\sim((f + g) \wedge n) \leq \lim \delta_y \circ \pi^\sim(f \wedge n + g \wedge n). \]

Since $\delta_y \circ \pi^\sim$ is linear and $\pi$ is supra-additive we have

\[ \pi(f + g) \leq \pi(f) + \pi(g) \leq \pi(f + g), \]

so $\pi$ is additive on $C(X)^+$. Now by the Kantorovic Theorem (see Theorem 1.7. [1]), $\varphi: C(X) \to C(Y)$ defined by $\varphi(f) = \pi(f^+) - \pi(f^-)$ is linear and from the second assumption it is clear that $\varphi = \pi$, so $\pi$ is linear.

Claim 5. $\pi$ is multiplicative.

Indeed, let $0 \leq f \in C(X)$ be given. As for each $y \in Y$, $\delta_y \circ \pi^\sim$ is multiplicative, we have

\[ \pi(f^2)(y) = \delta_y \circ \pi(f^2) = \lim \delta_y \circ \pi^\sim((f \wedge n)^2) = \lim \delta_y \circ \pi^\sim((f \wedge n^2)^2) = (\lim \delta_y \circ \pi^\sim((f \wedge n^2)^2))^2 = \pi(f)^2(y). \]
so \( \pi(f^2) = \pi(f)^2 \). Let \( f \in C(X) \) be given. As \( \pi(f^+)\pi(f^-) = 0 \), due to the linearity of \( \pi \) we have \( \pi(f^2) = \pi(f)^2 \). Now the multiplicativity follows from the equality

\[
fg = \frac{1}{4}((f + g)^2 - (f - g)^2).
\]

Recall that a topological space \( X \) is called pseudocompact if \( C(X) = C_b(X) \) ([3]). It is clear that any countable compact space is pseudocompact. Now the following corollary immediately follows from the above theorem.

**Corollary 3.** Let \( X \) be a pseudocompact space and \( Y \) a topological space. A map \( \pi: C(X) \to C(Y) \) is supra-additive and supra-multiplicative if and only if it is linear and multiplicative.

Recall that a topological space is called realcompact if it is homeomorphic to a closed subspace of the product space of \( \mathbb{R} \). It is well known that a Hausdorff space is compact if and only if it is realcompact and pseudocompact (see [3]). If \( K \) is a realcompact space and \( T: \ C(K) \to \mathbb{R} \) is nonzero linear and multiplicative then there exists \( k \in K \) such that \( T(f) = f(k) \) for each \( f \in C(K) \) (see [2] for a simple proof). By using this fact we have the following theorem.

**Theorem 4.** Let \( X \) be a realcompact space and let \( Y \) be an arbitrary topological space. Let \( \pi: C(X) \to C(Y) \) be a supra-additive and supra-multiplicative map. Then the following assertions are equivalent.

(i) \( \pi(f^+ \wedge n - f^- \wedge n)(y) \to \pi(f)(y) \) for each \( f \in C(X) \) and \( y \in Y \)

(ii) There exists a clopen subset \( B \subset Y \) and a continuous function \( \sigma \colon Y \to X \) such that

\[
\pi(f)(y) = \chi_B(y)f(\sigma(y))
\]

for each \( y \in Y \), \( f \in C(X) \).

**Proof.** It is clear that (ii) \( \implies \) (i). Suppose that (i) holds. Then from Theorem 2, \( \pi \) is linear and multiplicative. The fact that \( \pi(1)^2 = \pi(1) \) for each \( y \in Y \) implies that either \( \pi(1)(y) = 0 \) or \( \pi(1)(y) = 1 \), so \( B = \{ y \in Y : \pi(1)(y) = 1 \} \) is clopen in \( Y \). Let \( y \in Y \) be given. As \( X \) is realcompact and \( \delta_y \circ \pi: C(X) \to \mathbb{R} \) is linear and multiplicative there exists \( \alpha(y) \) such that

\[
\pi(f)(y) = \pi(1)(y)f(\alpha(y)) = \chi_B(y)f(\alpha(y)).
\]

Since \( X \) is completely regular Hausdorff space, \( \alpha(y) \) must be unique for each \( y \in B \). Let \( x_0 \in Y \) be fixed and let \( \sigma: Y \to X \) be defined by \( \sigma(y) = \alpha(y) \) when \( y \in B \) and \( \sigma(y) = x_0 \) otherwise. It is clear that \( \sigma\vert_B: B \to X \) is continuous. Since \( B \) is clopen, actually \( \sigma \) itself is continuous. This completes the proof.

\[\square\]
References

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Author’s address: Z. Ercan, Middle East Technical University, Department of Mathematics, 06531 Ankara, Turkey, e-mail: zercan@metu.edu.tr.