ON PERFECT AND UNIQUE MAXIMUM INDEPENDENT SETS IN GRAPHS

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Abstract. A perfect independent set \( I \) of a graph \( G \) is defined to be an independent set with the property that any vertex not in \( I \) has at least two neighbors in \( I \). For a nonnegative integer \( k \), a subset \( I \) of the vertex set \( V(G) \) of a graph \( G \) is said to be \( k \)-independent, if \( I \) is independent and every independent subset \( I' \) of \( G \) with \( |I'| \geq |I| - (k - 1) \) is a subset of \( I \). A set \( I \) of vertices of \( G \) is a super \( k \)-independent set of \( G \) if \( I \) is \( k \)-independent in the graph \( G[I, V(G) - I] \), where \( G[I, V(G) - I] \) is the bipartite graph obtained from \( G \) by deleting all edges which are not incident with vertices of \( I \). It is easy to see that a set \( I \) is 0-independent if and only if it is a maximum independent set and 1-independent if and only if it is a unique maximum independent set of \( G \).

In this paper we mainly investigate connections between perfect independent sets and \( k \)-independent as well as super \( k \)-independent sets for \( k = 0 \) and \( k = 1 \).

Keywords: independent sets, perfect independent sets, unique independent sets, strong unique independent sets

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1. TERMINOLOGY AND INTRODUCTION

We will assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2] or Lovász and Plummer [11]). In this paper, all graphs are finite, undirected, and simple. The vertex set and edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. The neighborhood \( N_G(x) \) of a vertex \( x \) is the set of vertices adjacent to \( x \), and the number \( d_G(x) = |N_G(x)| \) is the degree of \( x \). If \( S \subseteq V(G) \), then we define the neighborhood of \( S \) by \( N_G(S) = \bigcup_{x \in S} N_G(x) \).

If \( S \) and \( T \) are two disjoint subsets of \( V(G) \), then let \( G[S, T] \) be the bipartite graph consisting of the partite sets \( S \) and \( T \) and all edges of \( G \) with one end in \( S \) and the other one in \( T \), and we define \( e_G(S, T) = |E(G[S, T])| \). A graph without any cycle is called a forest.
A set $I$ of vertices is *independent* if no two vertices of $I$ are adjacent. The *independence number* $\alpha(G)$ of a graph $G$ is the maximum cardinality among the independent sets of vertices of $G$. Croitoru and Suditu [3] call an independent set $I$ of a graph $G$ a *perfect independent set* if any vertex not in $I$ has at least two neighbors in $I$.

For a nonnegative integer $k$, by Siemes, Topp, Volkmann [12], an independent set $I$ of the vertex set $V(G)$ of a graph $G$ is said to be $k$-*independent*, if every independent subset $I'$ of $G$ with $|I'| \geq |I| - (k - 1)$ is a subset of $I$. Furthermore, a set $I$ of vertices of $G$ is *super $k$-independent* if $I$ is $k$-independent in the bipartite graph $G[I, V(G) - I]$. Obviously, a set $I$ is $0$-independent if and only if it is maximum independent and $1$-independent if and only if it is a unique maximum independent set of $G$. In this paper we mainly deal with super $k$-independent sets for $k = 0, 1$. We call a super $0$-independent and super $1$-independent set also a *super independent* and *super unique independent* set, respectively.

If a bipartite graph $G$ has partite sets $A$ and $B$ such that $B$ is a unique maximum independent set of $G$, then Hopkins and Staton [5] speak of a *strong unique independence graph*. If a bipartite graph $G$ has partite sets $A$ and $B$ such that $B$ is a maximum independent set of $G$, then $G$ will be called a *strong maximum independence graph*.

A *vertex cover* in $G$ is a set of vertices that are incident with all edges of $G$. The minimum cardinality of a vertex cover in a graph $G$ is called the *covering number* and is denoted by $\tau(G)$. A set of edges in a graph is called a *matching* if no two edges are incident. The size of any largest matching in $G$ is called the *matching number* of $G$ and is denoted by $\nu(G)$. It is easy to see and well-known that $\nu(G) \leq \tau(G)$ and $\alpha(G) + \tau(G) = |V(G)|$ for any graph $G$.

A *block* of a graph is a maximal connected subgraph having no cut-vertex. A *block-cactus* graph is a graph whose blocks are either complete graphs or cycles.

In this paper we investigate connections between perfect independent sets and $k$-independent as well as super $k$-independent sets for $k = 0$ and $k = 1$. In addition, we present various families of graphs with a strong unique (or maximum) independence spanning forest.

## 2. Preliminary results

In [1], p. 272, Berge proved that an independent set $I$ in a graph $G$ is $0$-independent if and only if $|N_G(J) \cap I| \geq |J|$ for every independent subset $J$ of $V(G) - I$. In [12], the authors presented the following extensions of Berge’s result.

**Theorem 2.1** (Siemes, Topp, Volkmann [12] 1994). *For a nonnegative integer $k$, an independent set $I$ of vertices of a graph $G$ is a $k$-independent set in $G$ if and only*
Corollary 2.2. For a nonnegative integer $k$, an independent set $I$ of vertices of a graph $G$ is a super $k$-independent set in $G$ if and only if

$$|N_G(J) \cap I| \geq |J| + k$$

for every subset $J$ of $V(G) - I$ with $J \neq \emptyset$ when $k \geq 1$.

Proof. In view of the definition, $I$ is a super $k$-independent set in $G$ if and only if $I$ is $k$-independent in the bipartite graph $G^* = G[I, V(G) - I]$. According to Theorem 2.1, this is equivalent to

$$|N_{G^*}(J) \cap I| \geq |J| + k$$

for every independent subset $J$ of $V(G^*) - I$ with $J \neq \emptyset$ when $k \geq 1$. However, this is equivalent to

$$|N_G(J) \cap I| \geq |J| + k$$

for every subset $J$ of $V(G) - I$ with $J \neq \emptyset$ when $k \geq 1$, and the proof is complete. ☐

Theorem 2.1 as well as Corollary 2.2 play an important role in our investigations.

Observation 2.3. If $G$ is a claw-free graph, then every perfect independent set is also a maximum independent set.

Proof. If $I \subseteq V(G)$ is a perfect independent set and $J \subseteq V(G) - I$ an independent set, then $e_G(J, I) \geq 2|J|$. Since $G$ is claw-free, we observe that

$$2|J| \leq e_G(J, I) = e_G(J, I \cap N_G(J)) \leq 2|I \cap N_G(J)|$$

and hence $|J| \leq |I \cap N_G(J)|$. Theorem 2.1 with $k = 0$ yields the desired result. ☐

Theorem 2.4 (Listing [9] 1862, König [8] 1936). A graph $G$ is a forest if and only if $|E(G)| - |V(G)| + \sigma(G) = 0$, where $\sigma(G)$ denotes the number of components of $G$. 

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Theorem 2.5 (König [6] 1916). A graph is bipartite if and only if it contains no cycle of odd length.

3. Perfect and super unique independent sets

Clearly, a super unique independent set is a unique maximum independent set, and a unique maximum independent set is a perfect independent set. In this section we will present some classes of graphs with the property that each perfect independent set is also a super unique independent set.

Proposition 3.1. Let $G$ be a graph with a perfect independent set $I$. If $I$ is not a super unique independent set, then the bipartite graph $G[I, V(G) - I]$ contains a cycle.

Proof. Since $I$ is not a super unique independent set, there exists, in view of Corollary 2.2 with $k = 1$, a set $\emptyset \neq J \subseteq V(G) - I$ such that $|N_G(J) \cap I| \leq |J|$. Let $H = G[N_G(J) \cap I, J]$ be the induced bipartite subgraph of $G[I, V(G) - I]$. Since $I$ is a perfect independent set, it follows that $|E(H)| \geq 2|J|$, and this leads to

$$ |V(H)| = |N_G(J) \cap I| + |J| \leq 2|J| \leq |E(H)|. $$

Therefore, Theorem 2.4 implies that the graph $H$ and hence also the bipartite graph $G[I, V(G) - I]$ contains a cycle. \qed

Proposition 3.1 and Theorem 2.5 immediately yield the following corollary.

Corollary 3.2. Let $G$ be a graph without any even cycle, and let $I$ be an independent set. Then $I$ is a perfect independent set if and only if $I$ is a super unique independent set.

Theorem 3.3. If $G$ is a graph, then every even cycle of $G$ induces a complete subgraph of $G$ if and only if the bipartite graph $G[I, V(G) - I]$ is a forest for each independent set $I \subseteq V(G)$.

Proof. Assume that every even cycle of $G$ induces a complete graph. Suppose that there exists an independent set $I \subseteq V(G)$ such that $G[I, V(G) - I]$ contains a cycle $C$. This implies $|I \cap V(C)| \geq 2$. Since $C$ induces a complete graph, we arrive at the contradiction that $I$ is an independent set.

Conversely, let $G[I, V(G) - I]$ be a forest for each independent set $I \subseteq V(G)$. Let $C = v_1v_2 \ldots v_pv_1$ be an even cycle of length $p \geq 4$. We will prove by induction on $p$ that $C$ induces a complete subgraph. Let $A = \{v_1, v_3, \ldots, v_{p-1}\}$ and $B = \{v_2, v_4, \ldots, v_p\}$. Since $C$ induces a complete subgraph of $G[A, B]$, the induced subgraph $G[I, V(A) - I]$ is a forest for each independent set $I \subseteq V(A)$. Therefore, Theorem 2.4 implies that the graph $H$ and hence also the bipartite graph $G[I, V(A) - I]$ contains a cycle.
\{v_2, v_4, \ldots, v_p\}. Neither \(G[A, V(G) - A]\) nor \(G[B, V(G) - B]\) is a forest and thus, neither \(A\) nor \(B\) is an independent set in \(G\). Hence, there exist odd integers \(1 \leq i < j \leq p - 1\) and even integers \(2 \leq k < l \leq p\) such that \(v_i\) and \(v_j\) as well as \(v_k\) and \(v_l\) are adjacent. In the case that \(p = 4\), it follows that \(C\) induces a complete graph. Let now \(p \geq 6\) and assume, without loss of generality, that \(i < k\). Then there are the two possibilities, namely \(1 \leq i < k < l < j \leq p - 1\) or \(1 \leq i < k < j < l \leq p\). In both cases we will show that \(C\) has a chord \(uw\) with \(u \in A\) and \(w \in B\).

If \(1 \leq i < k < l < j \leq p - 1\), then

\[
C_0 = v_i v_{i+1} \ldots v_k v_l v_{l+1} \ldots v_j v_i
\]
is an even cycle with \(|V(C_0)| < |V(C)|\). Therefore, by the induction hypothesis, \(C_0\) induces a complete graph. In particular, \(v_i v_l\) is a chord of \(C\).

If \(1 \leq i < k < j < l \leq p\), then

\[
C_1 = v_i v_{i+1} \ldots v_k v_l v_{l-1} \ldots v_j v_i,
\]
\[
C_2 = v_i v_j v_{j-1} \ldots v_k v_l v_{l+1} \ldots v_i
\]
are even cycles such that \(|V(C_1)| + |V(C_2)| = |V(C)| + 4\) and hence \(|V(C_1)| = |V(C_2)| = |V(C)|\) if and only if \(|V(C)| = 4\). Since \(|V(C)| \geq 6\), we conclude that \(|V(C_1)| < |V(C)|\) or \(|V(C_2)| < |V(C)|\). According to the induction hypothesis, the cycle \(C_1\) or \(C_2\) induces a complete graph. In particular, \(v_i v_k, v_k v_j, v_j v_l, v_l v_i \in E(G)\). Since \(|V(C)| \geq 6\), at least one of these four edges is a chord of \(C\).

If \(C\) has a chord \(uw\) with \(u \in A\) and \(w \in B\), then we will finally show that \(C\) induces a complete graph. Let, without loss of generality, \(u = v_1\) and \(w = v_q\) with an even integer \(4 \leq q \leq p - 2\). The cycles

\[
C_3 = v_1 v_2 \ldots v_{q-1} v_q v_1, \quad C_4 = v_1 v_q v_{q+1} \ldots v_{p-1} v_p v_1
\]
are even and such that \(|V(C_3)|, |V(C_4)| < |V(C)|\). By the induction hypothesis, the cycles \(C_3\) and \(C_4\) induce complete graphs. Now let \(x\) and \(y\) be two arbitrary vertices in \(V(C)\). If \(x, y \in V(C_3)\) or \(x, y \in V(C_4)\), then they are adjacent. If not, then \(v_1 x v_y v_1\) is a cycle of length four, and by the induction hypothesis, the vertices \(x\) and \(y\) are adjacent. Consequently, \(C\) induces a complete subgraph, and the proof is complete. \(\square\)

Proposition 3.1 and Theorem 3.3 immediately lead to the following results.

**Corollary 3.4.** Let \(G\) be a graph with the property that every even cycle induces a complete subgraph, and let \(I\) be an independent set. Then \(I\) is a perfect independent set if and only if \(I\) is a super unique independent set.
Corollary 3.5. Let $G$ be a block-cactus graph such that every even block is a complete subgraph, and let $I$ be an independent set. Then $I$ is a perfect independent set if and only if $I$ is a super unique independent set.

Theorem 3.6. Let $G$ be a bipartite graph, and let $I \subseteq V(G)$ be an independent set. Then $I$ is a unique maximum independent set if and only if $I$ is a super unique independent set.

Proof. Let $I$ be a unique maximum independent set. Theorem 2.1 implies that $|N_G(J) \cap I| > |J|$ for all independent sets $\emptyset \neq J \subseteq V(G) - I$. Let $A$ and $B$ be the partite sets of $G$ and let $L \neq \emptyset$ be an arbitrary subset of $V(G) - I$. It follows that $L \cap A$ and $L \cap B$ are independent sets such that, without loss of generality, $L \cap A \neq \emptyset$. We deduce from Theorem 2.1 that

$$|N_G(L \cap A) \cap I| > |L \cap A|, \quad |N_G(L \cap B) \cap I| \geq |L \cap B|.$$ 

Therefore, we obtain

$$|N_G(L) \cap I| = |N_G(L \cap A) \cap I| + |N_G(L \cap B) \cap I| > |L \cap A| + |L \cap B| = |L|.$$ 

Thus, with respect to Corollary 2.2, $I$ is a super unique independent set, and the proof is complete.

4. Perfect and unique independent sets

Proposition 4.1. Let $G$ be a graph with a perfect independent set $I$. If $I$ is not a unique maximum independent set, then there exists an induced bipartite subgraph of $G$ which is not a forest.

Proof. Since $I$ is not a unique maximum independent set, there exists, in view of Theorem 2.1 with $k = 1$, an independent set $\emptyset \neq J \subseteq V(G) - I$ such that $|N_G(J) \cap I| \leq |J|$. If we define the induced bipartite graph $H = G[N_G(J) \cap I, J]$, then, since $I$ is a perfect independent set, it follows that $|E(H)| \geq 2|J|$. This yields

$$|V(H)| = |N_G(J) \cap I| + |J| \leq 2|J| \leq |E(H)|.$$ 

Therefore, Theorem 2.4 implies that the induced bipartite subgraph $H$ is not a forest.
Observation 4.2. If $G$ is a graph, then every even cycle of $G$ contains a chord if and only if every induced bipartite subgraph of $G$ is a forest.

Proof. Assume that every even cycle contains a chord. Suppose that there exists an induced bipartite subgraph $H$ with a cycle. Let $C$ be a shortest cycle in $H$. Since $C$ has a chord in $G$, this chord also belongs to $H$, a contradiction to the minimum length of $C$.

Conversely, assume that every induced bipartite subgraph of $G$ is a forest. Let $C$ be an even cycle in $G$. Suppose that $C$ has no chord. Then $C$ is an induced bipartite subgraph of $G$ but no forest. This contradiction completes the proof. □

Proposition 4.1 and Observation 4.1 immediately lead to the next result.

Corollary 4.3. Let $G$ be a graph with the property that every even cycle contains a chord, and let $I$ be an independent set. Then $I$ is a perfect independent set if and only if $I$ is a unique maximum independent set.

5. Strong (unique) maximum independence spanning forests

In view of Theorem 2.1, we establish easily the following facts.

Corollary 5.1. Let $G$ be a bipartite graph.

The graph $G$ is a strong maximum independence graph if and only if there exist partite sets $A$ and $B$ such that $|N_G(S)| \geq |S|$ for all $S \subseteq A$.

The graph $G$ is a strong unique independence graph if and only if there exist partite sets $A$ and $B$ such that $|N_G(S)| > |S|$ for all $\emptyset \neq S \subseteq A$.

Theorem 5.2 (König [7] 1931). If $G$ is a bipartite graph, then

$$\tau(G) = \nu(G).$$

Theorem 5.3 (König-Hall, König [7] 1931, Hall [4] 1935). Let $G$ be a bipartite graph with partite sets $A$ and $B$. Then $G$ contains a matching $M$ with the property that every vertex in $A$ is incident with an edge in $M$ if and only if $|N_G(S)| \geq |S|$ for all $S \subseteq A$.

Theorem 5.4 (Lovász [10] 1970). Let $G$ be a bipartite graph with partite sets $A$ and $B$. Then $G$ contains a spanning forest $F$ such that $d_F(v) = 2$ for all $v \in A$ if and only if $|N_G(S)| > |S|$ for all $\emptyset \neq S \subseteq A$.

A proof of Theorem 5.4 can also be find in [11] on p. 20. Corollary 5.1 shows that Theorem 5.3 and Theorem 5.4 characterize the strong maximum and the strong unique independence graphs, respectively.
Theorem 5.5. If $G$ is a graph, then the following statements are equivalent.

(a) $\nu(G) = \tau(G)$.
(b) There exists a super independent set in $G$.
(c) Every maximum independent set in $G$ is a super independent set.

Proof. (a) $\Rightarrow$ (c): Let $I$ be a maximum independent set, and let $M$ be a maximum matching in $G$. This leads to

$$|V(G) - I| = \tau(G) = \nu(G) = |M|.$$  

This implies that $M$ is a matching in the bipartite graph $G[I, V(G) - I]$ with the property that every vertex in $V(G) - I$ is incident with an edge in $M$. It follows that $|N_G(S) \cap I| \geq |S|$ for all $S \subseteq V(G) - I$. Hence, by Corollary 2.2, $I$ is a super independent set in $G$.

(b) $\Rightarrow$ (a): Let $I$ be a super independent set in $G$. As a consequence of Corollary 2.2 we obtain $|N_G(S) \cap I| \geq |S|$ for all $S \subseteq V(G) - I$. Hence, by Theorem 5.3, there exists a matching $M$ in the bipartite graph $G[I, V(G) - I]$ with the property that every vertex in $V(G) - I$ is incident with an edge in $M$. It follows that $\tau(G) = |V(G) - I| = |M| \leq \nu(G)$. Because of $\nu(G) \leq \tau(G)$, we deduce that $\nu(G) = \tau(G)$.

Since (c) $\Rightarrow$ (b) is immediate, the proof is complete. 

For reason of completeness, we will give a short proof of the next theorem by Hopkins and Staton [5].

Theorem 5.6 (Hopkins, Staton [5] 1985). Let $G$ be a connected bipartite graph. The graph $G$ is a strong unique independence graph if and only if $G$ has a strong unique independence spanning tree $T$. In addition, the unique maximum independent sets of $G$ and $T$ coincide.

Proof. Assume that $G$ is a strong unique independence graph. Let $A$ and $B$ be the partite sets such that $B$ is a unique maximum independent set of $G$. Combining Corollary 5.1 and Theorem 5.4, we find that $G$ contains a spanning forest $F$ such that $d_F(v) = 2$ for all $v \in A$. We now extend $F$ to a spanning tree $T$ of $G$ by adding as many edges as necessary. This yields $d_T(v) \geq 2$ for all $v \in A$. Hence, $B$ is a perfect independent set in $T$, and Corollary 3.2 implies that $B$ is a unique independent set in $T$.

Conversely, assume that $G$ has a strong unique independence spanning tree $T$ with the partite sets $A$ and $B$ such that $B$ is the unique maximum independent set of $T$. It follows easily from Theorem 2.5 that $A$ and $B$ are also independent sets in $G$. Obviously, $B$ is also a unique maximum independent set in $G$. 

Using Theorem 5.3 instead of Theorem 5.4, one can prove the next result similar to Theorem 5.6. Its proof is therefore omitted.
Theorem 5.7 (Volkmann [13] 1988). Let $G$ be a connected bipartite graph. The graph $G$ is a strong maximum independence graph if and only if $G$ has a strong maximum independence spanning tree $T$. In addition, the maximum independent sets of $G$ and $T$ coincide.

Theorem 5.8. If $G$ is a graph, then the following statements are valid.
(a) If $G$ has a super unique independent set, then $G$ has a strong unique independence spanning forest $T$ with $\alpha(T) = \alpha(G)$.
(b) If $G$ is a bipartite graph with a unique maximum independent set, then $G$ has a strong unique independence spanning forest $T$ with $\alpha(T) = \alpha(G)$.
(c) If $\nu(G) = \tau(G)$, then $G$ has a strong maximum independence spanning forest $T$ with $\alpha(T) = \alpha(G)$.
(d) If $G$ is a bipartite graph, then $G$ has a strong maximum independence spanning forest $T$ with $\alpha(T) = \alpha(G)$.

Proof. (a) Let $I$ be a super unique independent set in $G$. This means that $I$ is a unique maximum independent set in the bipartite graph $H = G[I, V(G) - I]$, and thus $H$ is a strong unique independence graph. If $H_1, H_2, \ldots, H_p$ are the components of $H$, then $I \cap V(H_i)$ are strong unique independent sets in $H_i$ for $i = 1, 2, \ldots, p$. In view of Theorem 5.6, each component $H_i$ has a strong maximum independence spanning tree $T_i$ with a unique maximum independent set $I \cap V(H_i)$ for $i = 1, 2, \ldots, p$. Obviously, $T = \bigcup_{i=1}^{p} T_i$ is a strong maximum independence spanning forest of $G$ with $\alpha(T) = \alpha(G) = |I|$.

(b) Let $I$ be a unique maximum independent set in the bipartite graph $G$. According to Theorem 3.6, $I$ is a super unique independent set in $G$ and (a) yields the desired result.

(c) Let $\nu(G) = \tau(G)$. In view of Theorem 5.5, $G$ has a super independent set. Using Theorem 5.7 instead of Theorem 5.6, the proof is analogous to the proof of (a) and is therefore omitted.

(d) If $G$ is bipartite, then Theorem 5.2 yields $\nu(G) = \tau(G)$. Now (c) leads to the desired result. □

Theorem 5.9. Let $G$ be a block-cactus graph such that every even block is a complete subgraph. If $I \subseteq V(G)$ is a perfect independent set, then $F = G[I, V(G) - I]$ is a strong unique independence spanning forest of $G$.

Proof. In view of Theorem 3.3, $F$ is a spanning forest of $G$. According to Corollary 3.5, $I$ is a super unique independent set in $G$. Altogether, we see that $F$ is a strong unique independence spanning forest of $G$ with the unique maximum independent set $I$. □
Theorem 5.8 (b) and Theorem 5.9 are generalizations of the following result by Hopkins and Staton [5].

**Corollary 5.10** (Hopkins, Staton [5] 1985). A tree $T$ has a unique maximum independent set $I$ if and only if $T$ has a spanning forest $F$ such that each component of $F$ is a strong unique independence tree and each edge in $T - E(F)$ joins two vertices not in $I$.

**References**


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