ON OPTIMAL DECAY RATES FOR WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$

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Dedicated to Professor Jindřich Nečas on his 70th birthday

Abstract. This paper is concerned with optimal lower bounds of decay rates for solutions to the Navier-Stokes equations in $\mathbb{R}^n$. Necessary and sufficient conditions are given such that the corresponding Navier-Stokes solutions are shown to satisfy the algebraic bound

$$\|u(t)\| \geq (t + 1)^{-\frac{n+4}{2}}.$$

Keywords: decay rates, Navier-Stokes equations

MSC 2000: 35Q10

1. INTRODUCTION AND THE RESULTS

Consider the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$, which will be treated in this paper in the form of the integral equation

$$u(t) = e^{-tA}a - \int_0^t \nabla \cdot e^{-(t-s)A}P(u \otimes u)(s) \, ds,$$

for prescribed initial velocity $a(x) = (a_1(x), \ldots, a_n(x))$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and unknown velocity $u(x, t) = (u_1(x, t), \ldots, u_n(x, t))$. Here, $A = -\Delta$ is the Laplacian on $\mathbb{R}^n$; $\{e^{-tA}\}_{t \geq 0}$ is the heat semigroup; $P = (P_{jk})$ is the bounded projection onto divergence-free vector fields; $u \otimes v$ is the matrix with entries $(u \otimes v)_{jk} = u_jv_k$; $\nabla = (\partial_1, \ldots, \partial_n)$ with $\partial_j = \partial/\partial x_j$; and

$$(\nabla \cdot e^{-tA}P(u \otimes u))_j = \sum_{k, \ell = 1}^n \partial_\ell e^{-tA}P_{jk}(u_\ell u_k), \quad j = 1, \ldots, n.$$
It is well known that for each \( a \in L^2 \) with \( \nabla \cdot a = 0 \), (NS) has a weak solution \( u \) defined for all \( t \geq 0 \), satisfying the energy inequality

\[
\|u(t)\|^2 + 2 \int_0^t \|\nabla u\|^2 \, ds \leq \|a\|^2 \quad \text{for all } t \geq 0.
\]

Hereafter \( \| \cdot \|_r \) denotes the \( L^r \)-norm.

As shown in [10], there exists a weak solution \( u \) such that

\[
(1.1) \quad \|u(t)\|_2 \leq C(1 + t)^{-\frac{n+2}{4}} \tag{1.1}
\]

whenever

\[
(1.2) \quad a \in L^2, \quad \nabla \cdot a = 0 \quad \text{and} \quad \int (1 + |y|)|a(y)| \, dy < \infty. \tag{1.2}
\]

Assumption (1.2) implies \( a \in L^1 \); so the divergence-free condition gives (see [4])

\[
(1.3) \quad \int a(y) \, dy = 0. \tag{1.3}
\]

Furthermore, it is shown in [2] that in this case the solution \( u \) satisfies

\[
(1.4) \quad \lim_{t \to \infty} t^{\frac{n+2}{4}} \left\| u_j(t) + (\partial_t E_t)(\cdot) \int y a_j(y) \, dy \right. \\
+ F_{\ell, jk}(\cdot, t) \int_0^\infty (u_\ell u_k)(y, s) \, dy \, ds \right\|_2 = 0
\]

for \( j = 1, \ldots, n \), where

\[
E_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \quad F_{\ell, jk}(x, t) = \partial_\ell E_t(x) \delta_{jk} + \int_t^\infty \partial_\ell \partial_j \partial_k E_s(x) \, ds.
\]

(Hereafter, we use the summation convention). Equation (NS) is then written in the form

\[
u_j(x, t) = \int E_t(x-y)a_j(y) \, dy - \int_0^t \int F_{\ell, jk}(x-y, t-s)(u_\ell u_k)(y, s) \, dy \, ds, \quad j = 1, \ldots, n,
\]

as proved in [2]; and the integrals in (1.4) are finite, due to (1.1) and (1.2). Assertion (1.4) was first proved in [1] for smooth solutions when \( n = 3 \), and then extended in [2] to the case of weak solutions in all space dimensions by applying the spectral method as given in [3, 5].

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The argument of [10] suggests that the decay property (1.1) will be optimal in general. So we are interested in finding a class of weak solutions \( u \) satisfying the reverse estimate
\[
\|u(t)\|_2 \geq Ct^{-\frac{n+2}{4}}
\]
at least for large \( t \).

In this paper we discuss this kind of lower bound problem.

**Theorem A.** Under the assumption (1.2), let
\[
b_k \ell = \int y e a_k(y) \, dy, \quad c_k \ell = \int_0^\infty \int (u_k u_k)(y, s) \, dy \, ds.
\]

(i) We have
\[
\lim_{t \to \infty} t^{\frac{n+2}{4}} \|u(t)\|_2 = 0
\]
if and only if \( (b_k \ell) = 0 \) and \( (c_k \ell) = (c_\delta k \ell) \) for some constant \( c \geq 0 \).

(ii) There exists \( c' > 0 \) such that
\[
\|u(t)\|_2 \geq c't^{-\frac{n+2}{4}} \quad \text{for large } t > 0,
\]
if and only if \( (b_k \ell) \neq 0 \) or \( (c_k \ell) \neq (c_\delta k \ell) \). In particular, \( u \) satisfies (1.6) whenever \( (b_k \ell) \neq 0 \).

**Remark.** Theorem A (i) implies only that
\[
\limsup_{t \to \infty} t^{\frac{n+2}{4}} \|u(t)\|_2 > 0
\]
if and only if \( (b_k \ell) \neq 0 \) or \( (c_k \ell) \neq (c_\delta k \ell) \). Note, however, that our second assertion (1.6) is more stringent than (1.5'). Moreover, (1.6) holds for all large \( t > 0 \) and for all space dimensions, although \( \|u(t)\|_2 \) is only known to be lower semicontinuous when \( n \geq 3 \). We know nothing about the characterization of solutions satisfying \( (c_k \ell) = (c_\delta k \ell) \).

We next consider weak solutions \( u \) satisfying
\[
\|u(t)\|_2 \leq C(1 + t)^{-\frac{n}{4}}.
\]

As shown in [3, 6, 10], such solutions exist for all \( a \in L^2 \) satisfying
\[
\nabla \cdot a = 0, \quad \|e^{tA}a\|_2 \leq C(1 + t)^{-\frac{n}{4}}.
\]
Theorem B. Suppose $a$ satisfies (1.8) and let $u$ be a weak solution satisfying (1.7). Then

\begin{equation}
\|u(t)\|_2 \geq ct^{-\frac{n}{2}} \text{ for large } t > 0,
\end{equation}

if and only if

\begin{equation}
\|e^{-tA}a\|_2 \geq ct^{-\frac{n}{2}} \text{ for large } t > 0.
\end{equation}

The lemma below gives simple examples of $a$ satisfying (1.10).

Lemma. Let $a \in L^2$, $\nabla \cdot a = 0$, and suppose that

\begin{equation}
\int_{S^{n-1}} |\hat{a}(r,\omega)|^2 \, d\omega \in L^\infty(\mathbb{R}_+), \quad \liminf_{r \to 0} \int_{S^{n-1}} |\hat{a}(r,\omega)|^2 \, d\omega > 0,
\end{equation}

where the Fourier transform $\hat{a}$ is defined by

$$
\hat{a}(\xi) = \int e^{-ix \cdot \xi} a(x) \, dx, \quad i = \sqrt{-1},
$$

$S^{n-1}$ is the unit sphere of $\mathbb{R}^n$, and $\xi = (r,\omega)$ in polar coordinates. Then,

\begin{equation}
\|e^{-tA}a\|_2 \leq C(1 + t)^{-\frac{n}{2}} \text{ for all } t > 0; \quad \|e^{-tA}a\|_2 \geq c' t^{-\frac{n}{2}} \text{ for large } t > 0,
\end{equation}

with constants $C > 0$ and $c' > 0$ independent of $t$.

Proof. Parseval’s relation gives

$$
\|e^{-tA}a\|_2^2 = (2\pi)^{-n} \int e^{-2|\xi|^2} |\hat{a}(\xi)|^2 \, d\xi = (8\pi^2 t)^{-\frac{n}{2}} \int e^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}))|^2 \, d\eta
$$

so that

$$(8\pi^2 t)^{-\frac{n}{2}} \|e^{-tA}a\|_2^2 = \int e^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}))|^2 \, d\eta.$$

The assumption and Fatou’s lemma together imply

$$
\liminf_{t \to \infty} (8\pi^2 t)^{-\frac{n}{2}} \|e^{-tA}a\|_2^2 = \liminf_{t \to \infty} \int e^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}))|^2 \, d\eta
$$

$$
\geq \int_{0}^{\infty} e^{-r^2} \left( \liminf_{t \to \infty} \int_{S^{n-1}} |\hat{a}(r(2t)^{-\frac{1}{2}},\omega)|^2 \, d\omega \right) r^{n-1} \, dr > 0.
$$

This proves the second estimate of (1.12). The first estimate follows from $\|e^{-tA}a\|_2 \leq \|a\|_2$ and

$$
\|e^{-tA}a\|_2 = (8\pi^2 t)^{-\frac{n}{2}} \int e^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}))|^2 \, d\eta
$$

$$
\leq Ct^{-\frac{n}{2}} \int_{S^{n-1}} |\hat{a}(\cdot,\omega)|^2 \, d\omega \int_{0}^{\infty} e^{-r^2} r^{n-1} \, dr.
$$

The proof is complete. □
Remarks. (i) Condition (1.11) implies that \( \hat{a} \) is discontinuous at \( \xi = 0 \). Indeed, since \( \nabla \cdot a = 0 \), we have \( \xi \cdot \hat{a}(\xi) = 0 \); so if \( \hat{a} \) is continuous at \( \xi = 0 \), we get \( \omega \cdot \hat{a}(0) = 0 \) for all unit vectors \( \omega \), and \( \hat{a}(0) = 0 \). (For this reason, \( a \in L^1 \) implies (1.3)).

(ii) The assumption of Lemma is not vacuous. Indeed, suppose \( \hat{a} \) is written in the form

\[
\hat{a}(\xi) = f(|\xi|)g(\xi/|\xi|),
\]

in terms of functions \( f(r) \) and \( g(\omega) \) such that

\[
g \in L^2(S^{n-1}), \quad g \neq 0, \quad \omega \cdot g(\omega) \equiv 0 \quad (\omega \in S^{n-1})
\]

and

\[
f \in BC([0, \infty)), \quad \int_0^\infty |f(r)|^2 r^{n-1} \, dr < \infty, \quad f(0) \neq 0.
\]

Then, \( \hat{a} \) satisfies condition (1.11).

(iii) In this connection, we note that under condition (1.2) we have

\[
(1.10') \quad \| e^{-tA}a \|_2 \geq c t^{-\frac{n+2}{4}} \quad \text{for large } t > 0
\]

if and only if \((b_{k\ell}) \neq 0\). Indeed, using (1.2) and (1.3), we have (see Section 4)

\[
(1.4') \quad \lim_{t \to \infty} t^{\frac{n-2}{2}} \| e^{-tA}a_k + \partial_t E_t b_{k\ell} \|_2 = 0, \quad k = 1, \ldots, n.
\]

Suppose \((b_{k\ell}) \neq 0\). Then \((\sum_k \| \partial_t E_t b_{k\ell} \|_2^2)^{1/2} = Ct^{-\frac{n-2}{4}}\) with \( C > 0 \); so we get

\[
\| e^{-tA}a \|_2 \geq \left( \sum_k \| \partial_t E_t b_{k\ell} \|_2^2 \right)^{1/2} - \left( \sum_k \| e^{-tA}a_k + \partial_t E_t b_{k\ell} \|_2^2 \right)^{1/2} \geq c t^{-\frac{n+2}{4}}
\]

for large \( t > 0 \). Conversely, if we assume (1.10'), then (1.4') implies

\[
\left( \sum_k \| \partial_t E_t b_{k\ell} \|_2^2 \right)^{1/2} \geq \| e^{-tA}a \|_2 - \left( \sum_k \| e^{-tA}a_k + \partial_t E_t b_{k\ell} \|_2^2 \right)^{1/2} \geq c t^{-\frac{n+2}{4}}
\]

for large \( t > 0 \). Hence \( \sum_k \| \partial_t E_t b_{k\ell} \|_2^2 \geq 0 \) for large \( t > 0 \), which implies \((b_{k\ell}) \neq 0\).

The \( L^2 \) decay problem for weak solutions of the Navier-Stokes equations was successfully studied for the first time by [5] and the result was then systematically developed by [3, 6, 10]. Estimates (1.6) and (1.9) are studied in [6]–[9] in case \( n = 2, 3 \), and some sufficient conditions are obtained. Our Theorems A and B provide necessary and sufficient conditions for those estimates to hold. We further note that our lower bound estimates (1.6) and (1.9) hold in all space dimensions \( n \geq 2 \), although the
function $\|u(t)\|_2$ is known only to be lower semicontinuous when $n \geq 3$. As will be seen in the proof below, this is due to (1.4) and the fact that the functions $\partial_t E_t(x)$ and $F_{\ell,jk}(x,t)$ are written in the form $t^{-\frac{n}{2}}K(xt^{-\frac{1}{2}})$ in terms of some bounded, integrable and uniformly continuous functions $K$.

We finally consider an example of two-dimensional flows $u$ with $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, which was first treated by [7].

**Theorem C.** When $n = 2$, there is a smooth weak solution $u$ such that $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, and, with some constant $\gamma > 0$,

$$\|u(t)\|_q \leq C_q e^{-\gamma t} \quad \text{and} \quad |u(x,t)| \leq C_m e^{-\gamma t}(1 + |x|)^{-m}$$

for all $1 \leq q \leq \infty$ and all integers $m \geq 0$.

The above example was studied by [7, 8, 9], in which is given the exponential decay of $\|u(t)\|_q$ for $2 \leq q \leq \infty$. Our estimates (1.13) include the case $1 \leq q < 2$ as well as the decay estimates in the spatial direction. Theorem C is proved in [2].

In what follows we prove Theorems A and B, and conclude the paper with the proof of (1.4) which was given also in [2].

2. PROOF OF THEOREM A

We begin with the following

**Proposition 2.1.** Let $(b_{k\ell})$ and $(c_{k\ell})$ be real $n \times n$ matrices and let $(c_{k\ell})$ be symmetric. Then

$$b_{k\ell} \partial_t E_t(x) \delta_{jk} + c_{k\ell} F_{\ell,jk}(x,t) = 0, \quad j = 1, \ldots, n,$$

for all $x \in \mathbb{R}^n$ and for some $t > 0$, if and only if

$$(b_{k\ell}) = 0 \quad \text{and} \quad (c_{k\ell}) = (c\delta_{k\ell}) \quad \text{for some } c \in \mathbb{R}. \quad (2.2)$$

Furthermore, (2.2) implies that (2.1) holds for all $x$ and for all $t > 0$.

**Proof.** Assumption (2.1) implies, via the Fourier transformation,

$$b_{k\ell} \xi e^{-t|\xi|^2} \delta_{jk} = -c_{k\ell} \xi_j \left( e^{-t|\xi|^2} \delta_{jk} - \xi_j \xi_k \int_t^\infty e^{-s|\xi|^2} ds \right) = -(c_{k\ell} - |\xi|^{-2} c_{k\ell} \xi_j \xi_k) \xi e^{-t|\xi|^2}$$
for some \( t > 0 \), and we get \( |\xi|^2(b_{jj} + c_{jj})\xi_j = \xi_j c_{k\ell}\xi_k \xi_{\ell} \). Taking \( \xi_j = 0 \) for any fixed \( j \), \( \xi_{\ell} = 1 \) for any fixed \( \ell \neq j \), and \( \xi_k = 0 \) for all \( k \) such that \( k \neq j \) and \( k \neq \ell \), we easily obtain \( b_{jj} + c_{jj} = 0 \) whenever \( j \neq \ell \), and so

\[
|\xi|^2(b_{jj} + c_{jj})\xi_j = \xi_j c_{k\ell}\xi_k \xi_{\ell}, \quad j = 1, \ldots, n.
\]

We let \( \xi_j = 1 \) and \( \xi_k = 0 \) for \( k \neq j \), to get \( b_{jj} + c_{jj} = c_{jj} \); so \( b_{jj} = 0 \). This implies

\[
|\xi|^2 c_{jj} \xi_j = \xi_j c_{k\ell}\xi_k \xi_{\ell}, \quad j = 1, \ldots, n.
\]

Hence, \( c_{11} = \ldots = c_{nn} = c_{k\ell}\xi_k \xi_{\ell}\xi_k \xi_{\ell} \). We then set \( j = 1, \xi_1 = \xi_2 = 1 \) and \( \xi_k = 0 \) for \( k \geq 3 \) in (2.3), to get \( 2c_{11} = c_{11} + c_{22} + c_{12} + c_{21} = 2(c_{11} + c_{12}) \) since \( c_{k\ell} = c_{k\ell} \) by assumption. Therefore, \( c_{12} = 0 \). We thus obtain \( c_{j\ell} = 0 = -b_{j\ell} \) whenever \( j \neq \ell \); so \( (b_{k\ell}) = 0 \) and \( (c_{k\ell}) = (c_{k\ell}) \). That (2.2) implies (2.1) for all \( t > 0 \) is easily seen from

\[
F_{k,jk} = \partial_j E_{\ell} + \int_0^\infty \partial_j \Delta E_s \, ds = \partial_j E_\ell + \int_0^\infty \partial_j \partial_s E_s \, ds = \partial_j E_\ell - \partial_j E_t = 0,
\]

where \( \partial_s = \partial/\partial s \). The proof of Proposition 2.1 is complete. \( \square \)

To establish Theorem A, it suffices in view of (1.4) to prove the following

**Proposition 2.2.** Let \( a \) satisfy (1.2) and define

\[
b_{k\ell} = \int y_1 a_k(y) \, dy, \quad c_{k\ell} = \frac{1}{\int_0^\infty (u_c u_k)(y, s) \, dy \, ds}.
\]

Then we have

\[
(2.4) \quad \text{either } (b_{k\ell}) \neq 0 \text{ or } (c_{k\ell}) \neq (c_{k\ell}),
\]

if and only if a corresponding weak solution \( u \) satisfies

\[
(2.5) \quad \|u(t)\|_2 \geq c' t^{-\frac{n+2}{2}} \text{ for large } t > 0
\]

with a constant \( c' > 0 \) independent of \( t \).

**Proof.** In what follows we write

\[
b_{k\ell} = (b_{1k\ell}, \ldots, b_{nk\ell}), \quad F_{k,k} = (F_{1k,k}, \ldots, F_{nk,k}).
\]

Assume first (2.4). By Proposition 2.1, we have \( \|\partial_t E_{\ell} b_{k\ell} + F_{k,k} c_{k\ell}\|_2 = Ct^{-\frac{n+2}{2}} \) for all \( t > 0 \) with some \( C > 0 \), and so (1.4) implies

\[
\|u(t)\|_2 \geq \|\partial_t E_{\ell} b_{k\ell} + F_{k,k} c_{k\ell}\|_2 - \|u(t)\| + \|\partial_t E_{\ell} b_{k\ell} + F_{k,k} c_{k\ell}\|_2
\]

\[
= Ct^{-\frac{n+2}{2}} - o(t^{-\frac{n+2}{2}}) \geq c' t^{-\frac{n+2}{2}}
\]

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for large $t > 0$. Assume next (2.5). By (1.4) we have
\[
\|\partial_t E_t b_t + F_{\ell,k} c_{k\ell}\|_2 \geq \|u(t)\|_2 - \|u(t) + \partial_t E_t b_t + F_{\ell,k} c_{k\ell}\|_2 
\geq c' t^{-\frac{14}{14 + 2}} - o(t^{-\frac{14}{14 + 2}}),
\]
and so
\[
\|\partial_t E_t b_t + F_{\ell,k} c_{k\ell}\|_2 > 0 \quad \text{for large } t > 0.
\]
We thus obtain (2.4) by Proposition 2.1. This proves Proposition 2.2. \hfill \Box

3. Proof of Theorem B

Suppose that $n \geq 3$. We have
\[
c_{k\ell} = \int_0^\infty \int (u_{\ell} u_k)(y,s) \, dy \, ds < \infty;
\]
so the argument given in [2, Sect. 5] applies to our present situation, implying
\[
(3.1) \quad \lim_{t \to \infty} t^{\frac{2 + 2}{14 + 2}} \|u(t) - e^{-tA} a + F_{\ell,k} c_{k\ell}\|_2 = 0.
\]
Suppose (1.9) holds. Since $\|F_{\ell,k} c_{k\ell}\|_2 = Ct^{-\frac{14 + 2}{14}}$, it follows from (3.1) that
\[
\|e^{-tA} a\|_2 \geq \|u(t)\|_2 - \|u(t) - e^{-tA} a - F_{\ell,k} c_{k\ell}\|_2
\geq \|u(t)\|_2 - \|u(t) - e^{-tA} a + F_{\ell,k} c_{k\ell}\|_2 - \|F_{\ell,k} c_{k\ell}\|_2
\geq ct^{-\frac{14 + 2}{14 + 2}} - c' t^{-\frac{14 + 2}{14 + 2}}
\]
for large $t > 0$. This proves (1.10). Conversely, if (1.10) holds, then (3.1) implies
\[
\|u(t)\|_2 \geq \|e^{-tA} a\|_2 - \|F_{\ell,k} c_{k\ell}\|_2 - \|u(t) - e^{-tA} a + F_{\ell,k} c_{k\ell}\|_2
\geq ct^{-\frac{14 + 2}{14 + 2}} - c' t^{-\frac{14 + 2}{14 + 2}}
\]
for large $t > 0$. This proves (1.9) in case $n \geq 3$.

When $n = 2$, we introduce
\[
c_{k\ell}(t) = \int_0^{t/2} \int (u_{\ell} u_k)(y,s) \, dy \, ds
\]
instead of $c_{k\ell}$. The argument of [2, Sect. 5] is then modified to yield
\[
(3.1') \quad \|u(t) - e^{-tA} a + F_{\ell,k} c_{k\ell}(t)\|_2 \leq Ct^{-1} \log(1 + t).
\]
See also Section 4 below. Since
\[ \|F_{t,k}\|_2 \leq Ct^{-1} \int_0^{t/2} \|u(s)\|_2^2 ds \leq Ct^{-1} \log(1 + t), \]
this implies \( \|u(t) - e^{-tA}a\|_2 \leq Ct^{-1} \log(1 + t) \). Now we can prove the result in the same way as in the case \( n \geq 3 \). Indeed, (1.10) implies
\[ \|u(t)\|_2 \geq \|e^{-tA}a\|_2 - \|u(t) - e^{-tA}a\|_2 \geq ct^{-1/2} - Ct^{-1} \log(1 + t) \geq c't^{-1/2} \]
for large \( t > 0 \), while (1.9) yields
\[ \|e^{-tA}a\|_2 \geq \|u(t)\|_2 - \|u(t)e^{-tA}\|_2 \geq ct^{-1/2} - Ct^{-1} \log(1 + t) \geq c't^{-1/2} \]
for large \( t > 0 \). The proof of Theorem B is complete.

4. Proof of (1.4)

Here we present the proof of (1.4) given in [2]. The same method can be applied to the proof of (3.1) and (3.1') with no essential change. Let \( a \) satisfy (1.2) and so (1.3). We first prove
\[ (4.1) \lim_{t \to \infty} t^{n/2} \left\| e^{-tA}a + (\partial_k E_t)(\cdot) \int y_k a(y) dy \right\|_2 = 0. \]
Direct calculation gives
\[
e^{-tA}a = \int [E_t(x - y) - E_t(x)]a(y) dy = \int \int_0^1 (\partial_k E_t)(x - y\theta) y_k a(y) d\theta dy
\]
\[ = - (\partial_k E_t)(x) \int y_k a(y) dy - \int \int_0^1 [(\partial_k E_t)(x - y\theta) - (\partial_k E_t)(x)] y_k a(y) d\theta dy, \]
so
\[
e^{-tA}a + (\partial_k E_t)(x) \int y_k a(y) dy = - \int \int_0^1 [(\partial_k E_t)(x - y\theta) - (\partial_k E_t)(x)] y_k a(y) d\theta dy. \]
We can write \( (\partial_k E_t)(x) = t^{-n/2} (\partial_k E_1)(x t^{-1/2}) \), to obtain
\[
\left\| e^{-tA}a + (\partial_k E_t)(\cdot) \int y_k a(y) dy \right\|_2 \leq Ct^{-n/2} \int_0^1 \varphi_1(y, \theta) ||a(y)|| d\theta dy.
\]
Here \( \varphi_t(y, \theta) = \| (\nabla E_1)(\cdot - y\theta t^{-\frac{1}{2}}) - (\nabla E_1)(\cdot) \|_2 \) is bounded and \( \lim_{t \to \infty} \varphi_t(y, \theta) = 0 \) for any fixed \( (y, \theta) \). Since \( |y| |a(y)| \) is integrable by (1.2), the dominated convergence theorem yields

\[
\lim_{t \to \infty} \int_0^1 \varphi_t(y, \theta) |y| a(y) \, d\theta \, dy = 0.
\]

This proves (4.1). Now let \( u \) satisfy (1.1). We next show that the function

\[
w(t) = u(t) - e^{-tA}a = - \int_0^t \int F_{t,k}(x - y, t - s)(u_{t}u_k)(y, s) \, dy \, ds
\]
satisfies

\[
\lim_{t \to \infty} t^{\frac{n+2}{2}} \left\| w(t) + F_{t,k}(\cdot, t) \int_0^\infty \int (u_{t}u_k)(y, s) \, dy \, ds \right\|_2 = 0.
\]

Indeed, we have

\[
w(t) + F_{t,k}(x, t) \int_0^\infty \int (u_{t}u_k)(y, s) \, dy \, ds
\]
\[
= F_{t,k}(x, t) \int_{t/2}^\infty \int (u_{t}u_k)(y, s) \, dy \, ds
\]
\[
- \int_0^{t/2} \int (F_{t,k}(x - y, t - s) - F_{t,k}(x, t - s))(u_{t}u_k)(y, s) \, dy \, ds
\]
\[
- \int_0^{t/2} \int (F_{t,k}(x, t) - F_{t,k}(x, t - s))(u_{t}u_k)(y, s) \, dy \, ds
\]
\[
- \int_{t/2}^t \int F_{t,k}(x - y, t - s)(u_{t}u_k)(y, s) \, dy \, ds
\]
\[
\equiv I_1 + I_2 + I_3 + I_4.
\]

It is easy to see that

\[
\lim_{t \to \infty} t^{\frac{n+2}{2}} \left\| I_1 \right\|_2 \leq C \int_{t/2}^\infty (1 + s)^{-1 - \frac{n}{2}} \, ds \to 0 \quad \text{as } t \to \infty.
\]

We write \( I_3 \) in the form

\[
I_3 = \int_0^{t/2} \int_0^1 s(\partial_t F_{t,k})(x, t - s\theta)(u_{t}u_k)(y, s) \, d\theta \, dy \, ds
\]
to get

\[
\left\| I_3 \right\|_2 \leq C \int_0^{t/2} \int_0^1 s(t - s\theta)^{-1 - \frac{n}{2}} |u(y, s)|^2 \, d\theta \, dy \, ds
\]
\[
\leq Ct^{-1 - \frac{n+2}{2}} \int_0^{t/2} s |u(s)|^2 \, ds
\]
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and so

$$t^{n/2} \|I_3\|_2 \leq C t^{-1} \int_0^t (1 + s)^{-n/2} ds \to 0 \quad \text{as } t \to \infty.$$  

To estimate $I_2$, note that we can write $F_{\ell,k}(x, t) = t^{-n/2} K(x t^{-1/2})$, to get

$$\|I_2\| \leq C t^{-n/2} \int_0^{t/2} \int \|K(\cdot - y(t-s)^{-1/2}) - K(\cdot)\|_2 |u(y, s)|^2 dy ds \equiv C t^{-n/2} \int_0^{t/2} \psi_t(s) ds.$$  

Since $\psi_t(s) \leq C \|u(s)\|_2^2$, the dominated convergence theorem implies

$$\lim_{t \to \infty} \int_0^M \psi_t(s) ds = 0 \quad \text{for any fixed } M > 0.$$  

Given $\varepsilon > 0$, choose $M > 0$ so that $\int_M^\infty \|u(s)\|_2^2 ds < \varepsilon$. Then for $t > 2M$,

$$\int_0^{t/2} \psi_t(s) ds \leq \int_0^M \psi_t(s) ds + C \int_M^\infty \|u(s)\|_2^2 ds \leq \int_0^M \psi_t(s) ds + C \varepsilon.$$  

This implies that

$$\lim_{t \to \infty} t^{n/2} \|I_2\|_2 = 0.$$  

It remains to prove

$$\lim_{t \to \infty} t^{n/2} \|I_4\|_2 = 0.$$  

To do so, we follow the arguments of [3, 5]. The function

$$v(t) = -\int_\tau^t \int F_{\ell,k}(x-y, t-s)(u_{\ell} u_k)(y, s) dy ds = u(t) - e^{-(t-\tau)} A u(\tau)$$  

defined for $t \geq \tau > 0$ satisfies

$$\partial_t v + Av = -P(u \cdot \nabla u) \quad (t > \tau), \quad v(\tau) = 0.$$  

(We may assume $v$ is smooth, replacing $u$ by the approximate solutions $u_N$ given in [3]). Since $(P(u \cdot \nabla v), v) = (u \cdot \nabla v, v) = 0$, the standard energy integral method gives

$$\partial_t \|v\|_2^2 + 2 \|A^{1/2} v\|_2^2 = -2(u \cdot \nabla u, v) = 2(u \cdot \nabla v, u) = 2(u \cdot \nabla u, u_0)$$

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and

\[ 2|u \cdot \nabla v, u_0| \leq 2\|u\|_2\|A^{1/2}v\|_2\|u_0\|_\infty \leq C\|u\|_2\|A^{1/2}v\|_2(t - \tau)^{-\frac{n+2}{4}}, \]

where \(u_0(t) = e^{-(t-\tau)A}u(\tau)\). We thus obtain

\[ \partial_t \|v\|_2^2 + \|A^{1/2}v\|_2^2 \leq C(t - \tau)^{-n-1}\tau^{-\frac{n+2}{2}}. \]

Let \(\{E_\lambda\}_{\lambda > 0}\) be the spectral measure associated to \(A\). Since \(\|A^{1/2}v\|_2 \geq \varrho(\|v\|_2^2 - \|E_\varrho v\|_2^2)\) for any \(\varrho > 0\), the above estimate yields

\[ \partial_t \|v\|_2^2 + \varrho\|v\|_2^2 \leq \varrho\|E_\varrho v\|_2^2 + C(t - \tau)^{-n-1}\tau^{-\frac{n+2}{2}}. \]

But, \(\|E_\varrho v\|_2^2 \leq C \varrho \frac{n+2}{2} \left( \int_\tau^t \|u\|_2^2 ds \right)^2\) as shown in [3, 5]; so

\[ \partial_t \|v\|_2^2 + \varrho\|v\|_2^2 \leq C \varrho \frac{n+2}{2} \left( \int_\tau^t \|u\|_2^2 ds \right)^2 + C(t - \tau)^{-n-1}\tau^{-\frac{n+2}{2}}. \]

Here we set \(\varrho = m/(t - \tau)\), \(m > 0\), and multiply both sides by \((t - \tau)^m\), to obtain

\[ \partial_t ((t - \tau)^m\|v\|_2^2) \leq C_m(t - \tau)^{m-\frac{n+2}{2}} \left( \int_\tau^t \|u\|_2^2 ds \right)^2 + C(t - \tau)^{m-n-1}\tau^{-\frac{n+2}{2}}. \]

Now fix \(m\) so that \(m > n/2 + 2\) and \(m > n + 1\), and integrate the above inequality, to get

\[ \|v(t)\|_2^2 \leq C(t - \tau)^{-2-\frac{n+2}{2}} \left( \int_\tau^t \|u\|_2^2 ds \right)^2 + C(t - \tau)^{-n-1}\tau^{-\frac{n+2}{2}}. \]

Inserting \(t = t/2\) yields \(v(t) = I_4\), so

\[ t^{n+\frac{2}{2}}\|I_4\|_2^2 \leq Ct^{n-1} \left( \int_{t/2}^\infty \|u\|_2^2 ds \right)^2 + Ct^{-1} \leq Ct^{-1} \to 0 \]

as \(t \to \infty\). This proves (4.6).

**Acknowledgment.** We are grateful to Professor Jindřich Nečas for valuable discussions on the problem treated in this paper. We also thank Dr. Kenji Nakanishi for showing us the examples given in Lemma. The first author is partially supported by Grant-in-Aid for Scientific Research, No. 10440045, from the Japan Ministry of Education.
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