QUASILINEAR AND QUADRATIC SINGULARLY PERTURBED NEUMANN’S PROBLEM

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Abstract. The problem of existence and asymptotic behaviour of solutions of the quasilinear and quadratic singularly perturbed Neumann’s problem as a small parameter at the highest derivative tends to zero is studied.

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1. Introduction

In the paper [2] the author established sufficient conditions for the existence and uniform convergence of the solutions of a semilinear singularly perturbed differential equation $\varepsilon y'' + ky = f(t, y)$ to a solution of the reduced problem $ku = f(t, u)$ as the small positive parameter $\varepsilon$ tends to zero. The purpose of this paper is an extension of Theorem 1 of the above cited paper to more general cases. We will consider Neumann’s problem

\[ (\text{NP}_\varepsilon) \begin{align*}
\varepsilon y'' &= F(t, y, y'), \quad a < t < b, \\
y'(a, \varepsilon) &= 0, \quad y'(b, \varepsilon) = 0,
\end{align*} \]

where $F \in C^1([a, b] \times \mathbb{R}^2)$ and $\varepsilon$ is a small positive parameter. The proofs of the theorems are based upon the method of lower and upper solutions.

As usual, we say that $\alpha \in C^2([a, b])$ is a lower solution for (NP$_\varepsilon$) if $\alpha'(a, \varepsilon) \geq 0$, $\alpha'(b, \varepsilon) \leq 0$, and $\varepsilon \alpha''(t, \varepsilon) \geq F(t, \alpha(t, \varepsilon), \alpha'(t, \varepsilon))$ for every $t \in [a, b]$. An upper solution $\beta \in C^2([a, b])$ satisfies $\beta'(a, \varepsilon) \leq 0$, $\beta'(b, \varepsilon) \geq 0$, and $\varepsilon \beta''(t, \varepsilon) \leq F(t, \beta(t, \varepsilon), \beta'(t, \varepsilon))$ for every $t \in [a, b]$. 

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Definition 1. We say that a function $F$ satisfies the Bernstein-Nagumo condition if for each $M > 0$ there exists a continuous function $h_M: [0, \infty) \to [a_M, \infty)$ with $a_M > 0$ and $\int_0^\infty \frac{s}{h_M(s)} ds = \infty$ such that for all $y, |y| \leq M$, all $t \in [a, b]$ and all $z \in \mathbb{R}$ we have

$$|F(t, y, z)| \leq h_M(|z|).$$

Remark. As a remark we conclude that the functions of the form $F(t, y, y') = f(t, y)y' + g(t, y)$ and $F(t, y, y') = f(t, y)y'^2 + g(t, y)$ satisfy the Bernstein-Nagumo condition.

Lemma 1. If $\alpha, \beta$ are lower and upper solutions for $(\text{NP}_0)$ such that $\alpha(t, \varepsilon) \leq \beta(t, \varepsilon)$ on $[a, b]$ and $F$ satisfies the Bernstein-Nagumo condition, then there exists a solution $y$ of $(\text{NP}_0)$ with $\alpha(t, \varepsilon) \leq y(t, \varepsilon) \leq \beta(t, \varepsilon)$, $a \leq t \leq b$.

Notation. Let

$$D_\delta(u) = \{(t, y) \in \mathbb{R}^2 : a \leq t \leq b, |y - u(t)| < \delta\},$$
$$D_{\delta, a}(u) = \{(t, y) \in \mathbb{R}^2 : a \leq t \leq b, y \in \mathbb{R} \cap D_\delta(u),$$

and

$$D_{\delta, b}(u) = \{(t, y) \in \mathbb{R}^2 : a \leq t \leq b, y \in \mathbb{R} \cap D_\delta(u),$$

where $\delta \leq b - a$ is a positive constant and $u = u(t)$ is a solution of the reduced problem $F(t, u, u'') = 0$ defined on $[a, b]$ such that $u \in C^2([a, b])$.

Let $h(t, y)$ denote $F(t, y, u'(t))$.

2. Quasilinear Neumann’s problem

In this section we consider the quasilinear Neumann’s problem

$$(\text{NP}_1)$$

$$\varepsilon y'' = f(t, y)y' + g(t, y), \quad a < t < b,$$
$$y'(a, \varepsilon) = 0, \quad y'(b, \varepsilon) = 0,$$

where $f, g \in C^1(D_\delta(u))$. Concerning the behaviour of solutions of $(\text{NP}_1)$ for $\varepsilon \to 0^+$ we have the following result.

Theorem 1. Consider the problem $(\text{NP}_1)$. Let there exist a solution $u \in C^2([a, b])$ of the reduced problem. Let $\delta, m$ be positive constants such that $\frac{\partial^2 h(t, y)}{\partial y} \geq m$ for every $(t, y) \in D_\delta(u)$. Let $f(t, y) \leq 0$ and $f(t, y) \geq 0$ for every $(t, y) \in D_{\delta, a}(u)$.
and \((t, y) \in D_{b, \rho(u)}\), respectively. Then there exists \(\varepsilon_0\) such that for any \(\varepsilon \in (0, \varepsilon_0]\) the problem \((NP_1)\) has a solution satisfying the inequality
\[
|y(t, \varepsilon) - u(t)| \leq v_1(t, \varepsilon) + v_2(t, \varepsilon) + C\varepsilon
\]
on \([a, b]\), where
\[
v_1(t, \varepsilon) = |u'(a)| \frac{\exp\left(-\sqrt{\frac{\varepsilon}{t}}(b - t)\right) + \exp\left(-\sqrt{\frac{\varepsilon}{t}}(t - b)\right)}{\sqrt{\frac{\varepsilon}{t}} (\exp\left[\sqrt{\frac{\varepsilon}{t}}(b - a)\right] - \exp\left[-\sqrt{\frac{\varepsilon}{t}}(b - a)\right])},
\]
\[
v_2(t, \varepsilon) = |u''(a)| \frac{\exp\left(-\sqrt{\frac{\varepsilon}{t}}(a - t)\right) + \exp\left(-\sqrt{\frac{\varepsilon}{t}}(t - a)\right)}{\sqrt{\frac{\varepsilon}{t}} (\exp\left[\sqrt{\frac{\varepsilon}{t}}(b - a)\right] - \exp\left[-\sqrt{\frac{\varepsilon}{t}}(b - a)\right])}
\]
and \(C\) is a positive constant.

**Proof.** We define the lower solutions by
\[
\alpha(t, \varepsilon) = u(t) - v_1(t, \varepsilon) - v_2(t, \varepsilon) - \Gamma(\varepsilon)
\]
and the upper solutions by
\[
\beta(t, \varepsilon) = u(t) + v_1(t, \varepsilon) + v_2(t, \varepsilon) + \Gamma(\varepsilon).
\]
Here \(\Gamma(\varepsilon) = \frac{\varepsilon}{m}\), where \(\gamma\) is a constant which will be defined below. One can easily check that the functions \(\alpha, \beta\) satisfy the boundary conditions required for the lower and upper solutions of \((NP_1)\) and \(\alpha \leq \beta\) on \([a, b]\). Now we show that \(\varepsilon \alpha''(t, \varepsilon) \geq f(t, \alpha(t, \varepsilon))\alpha'(t, \varepsilon) + g(t, \alpha(t, \varepsilon))\) and \(\varepsilon \beta''(t, \varepsilon) \leq f(t, \beta(t, \varepsilon))\beta'(t, \varepsilon) + g(t, \beta(t, \varepsilon))\) on \([a, b]\). By the Taylor theorem we obtain
\[
\varepsilon \alpha'' - F(t, \alpha, \alpha')
\]
\[
= \varepsilon \alpha'' - (F(t, \alpha, \alpha') - F(t, u, u'))
= \varepsilon \alpha'' - \left[(F(t, \alpha, u') - F(t, u, u')) + (F(t, \alpha, \alpha') - F(t, \alpha, u'))\right]
= \varepsilon \alpha'' - \left[\frac{\partial h(t, \eta(t, \varepsilon))}{\partial y}(\alpha - u) + f(t, \alpha)(\alpha' - u')\right]
= \varepsilon u'' - \varepsilon v'' + \frac{\partial h(t, \eta)}{\partial y} (v_1 + v_2 + \Gamma) + f(t, \alpha)(v_1' + v_2')
\geq \varepsilon u'' - \varepsilon v'' + m (v_1 + v_2 + \Gamma) + f(t, \alpha)(v_1' + v_2')
= \varepsilon u'' + \varepsilon \gamma + f(t, \alpha)(v_1' + v_2')
\geq -\varepsilon |u''| + \varepsilon \gamma + f(t, \alpha)(v_1' + v_2')
\]
and
\[
F(t, \beta, \beta') - \varepsilon \beta'' \geq -\varepsilon |u''| + \varepsilon \gamma + f(t, \beta)(v_1' + v_2'),
\]
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where \((t, \eta(t, \varepsilon))\) is a point between \((t, \alpha(t, \varepsilon))\) and \((t, u(t))\), \((t, \eta(t, \varepsilon)) \in D_\delta(u)\) for sufficiently small \(\varepsilon\).

Let \(u'(a) \neq 0\), \(u'(b) \neq 0\) if \(u'(a) = 0\) or \(u'(b) = 0\), we proceed analogously. From the above assumptions we obtain that \(f(t, \alpha)(v'_1 + v'_2) \geq 0\) and \(f(t, \beta)(v'_1 + v'_2) \geq 0\) on \([a, a + \delta] \cup [b - \delta, b]\) for \(\varepsilon \in (0, \varepsilon_1]\) where \(\delta = \min\{\delta_1, \delta_2\}\), and \(\delta_1, \varepsilon_1\) are such that \(v'_1 + v'_2 < 0\) \((v'_1 + v'_2 > 0)\) on \([a, a + \delta_1]\) \([b - \delta_1, b]\) and \((t, \alpha) \in D_\delta(u), (t, \beta) \in D_\delta(u)\) for \(\varepsilon \in (0, \varepsilon_1]\). On the interval \([a + \delta_1, b - \delta_1]\) we have \(|f(t, \alpha)(v'_1 + v'_2)| \leq c_1 \varepsilon\) and \(|f(t, \beta)(v'_1 + v'_2)| \leq c_1 \varepsilon\) for sufficiently small \(\varepsilon\), for instance if \(\varepsilon \in (0, \varepsilon_0], \varepsilon_0 \leq \varepsilon_1\) and \(c_1\) is a suitable positive constant (if \(u'(a) = 0\) \((u'(b) = 0)\) then \(|f(t, \alpha)(v'_1 + v'_2)| \leq c_1 \varepsilon\) and \(|f(t, \beta)(v'_1 + v'_2)| \leq c_1 \varepsilon\) on \([a, b - \delta]\) \([a + \delta, b]\).

Thus if we choose a constant \(\gamma \geq \gamma + \max\{|u''(t)|, t \in [a, b]\}\) then \(\varepsilon y''(t, \varepsilon) \geq f(t, \alpha(t, \varepsilon))a''(t, \varepsilon) + g(t, \alpha(t, \varepsilon))\) \(\varepsilon y''(t, \varepsilon) \leq f(t, \beta(t, \varepsilon))\beta''(t, \varepsilon) + g(t, \beta(t, \varepsilon))\) on \([a, b]\). The existence of a solution of \((NP_1)\) satisfying the above inequalities follows from Lemma. This completes the proof.

**Example 1.** As an illustrative example we consider the \((NP_1)\) for the differential equation \(\varepsilon y'' = y' - \left(\frac{1}{2}\right)\) on \([0, 1]\). General solution of the reduced problem \(uu' - \left(\frac{1}{2}\right) = 0\) is \(u^2 = t^2 - t + k, k \in \mathbb{R}\); however, only \(u(t) = t - \frac{1}{2}\) satisfies the assumptions asked on the solution of the reduced problem. On the basis of Theorem 1, there is \(\varepsilon_0\) such that for every \(\varepsilon \in (0, \varepsilon_0]\) the problem has a solution satisfying \(|y(t, \varepsilon) - \left(\frac{1}{2}\right)| \leq v_1 + v_2 + c_1 \varepsilon\) on \([0, 1]\).

3. Quadratic Neumann’s problem

Now we will consider the quadratic Neumann’s problem

\[
(NP_2) \quad \varepsilon y'' = f(t, y)y' + g(t, y), \quad a < t < b, \\
y'(a, \varepsilon) = 0, \quad y'(b, \varepsilon) = 0,
\]

where \(f, g \in C^1(D_\delta(u)).\)

**Theorem 2.** Consider the problem \((NP_2)\). Let there exist a solution \(u \in C^2([a, b])\) of the reduced problem. Let \(\delta, m\) be positive constants such that \(\frac{\partial h(t, u)}{\partial y} \geq m\) for every \((t, y) \in D_\delta(u)\). Let \(f(t, y) \leq 0\) \((f(t, y) \geq 0)\) for \((t, y) \in D_{\delta, a}(u)\) when \(u'(a) > 0\) \((u'(a) < 0)\) and \(f(t, y) \leq 0\) \((f(t, y) \geq 0)\) for \((t, y) \in D_{\delta, b}(u)\) when \(u'(b) < 0\) \((u'(b) > 0)\). Then there exists \(\varepsilon_0\) such that for any \(\varepsilon \in (0, \varepsilon_0]\) the problem \((NP_2)\) has a solution satisfying the inequality

\[
|y(t, \varepsilon) - u(t)| \leq v_1(t, \varepsilon) + v_2(t, \varepsilon) + C \varepsilon
\]

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on \([a, b]\) where \(v_1, v_2\) are the functions from Theorem 1 and \(C\) is a positive constant.

**Proof.** The idea of the proof is essentially the same as in the proof of Theorem 1. Let us define the lower solutions by

\[
\alpha(t, \varepsilon) = u(t) - v_1(t, \varepsilon) - v_2(t, \varepsilon) - \Gamma(\varepsilon)
\]

and the upper solutions by

\[
\beta(t, \varepsilon) = u(t) + v_1(t, \varepsilon) + v_2(t, \varepsilon) + \Gamma(\varepsilon).
\]

Analogously as in Theorem 1 we obtain

\[
\varepsilon \alpha'' - F(t, \alpha, \alpha') \geq -\varepsilon |u''| + \gamma \varepsilon - f(t, \alpha)(\alpha'^2 - u'^2)
\]

\[
= -\varepsilon |u''| + \gamma \varepsilon + f(t, \alpha)\left(v_1' + v_2'\right)(2u' - v_1' - v_2')
\]

and

\[
F(t, \beta, \beta') - \varepsilon \beta'' \geq -\varepsilon |u''| + \gamma \varepsilon + f(t, \beta)(\beta'^2 - u'^2)
\]

\[
= -\varepsilon |u''| + \gamma \varepsilon + f(t, \beta)\left(v_1' + v_2'\right)(2u' + v_1' + v_2').
\]

Similarly as in the previous theorem we conclude (for \(u'(a) \neq 0, u'(b) \neq 0\)) that

\[
f(t, \alpha)\left(v_1' + v_2'\right)(2u' - v_1' - v_2') \geq 0, \quad f(t, \beta)\left(v_1' + v_2'\right)(2u' + v_1' + v_2') \geq 0
\]

on \([a, a + \delta] \cup [b - \delta, b]\) and

\[
|f(t, \alpha)\left(v_1' + v_2'\right)(2u' - v_1' - v_2')| \leq c_2 \varepsilon,
\]

\[
|f(t, \beta)\left(v_1' + v_2'\right)(2u' + v_1' + v_2')| \leq c_2 \varepsilon
\]

on \([a + \delta, b - \delta]\) for \(\varepsilon \in (0, \varepsilon_0]\), sufficiently small \(\delta > 0\) and a suitable positive constant \(c_2\). Therefore, for \(\gamma \geq c_2 + \max\{u''(t), t \in [a, b]\}\) we have

\[
\varepsilon \alpha''(t, \varepsilon) \geq f(t, \alpha(t, \varepsilon))\alpha'^2(t, \varepsilon) + g(t, \alpha(t, \varepsilon))
\]

and

\[
\varepsilon \beta''(t, \varepsilon) \leq f(t, \beta(t, \varepsilon))\beta'^2(t, \varepsilon) + g(t, \beta(t, \varepsilon))
\]

on \([a, b]\). Hence Theorem 2 is proved. \(\square\)

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Example 2. Consider problem (NP₂) for differential equation \( \varepsilon y'' = y y^2 - (t + 1) \) on \([-2, 1]\). Obviously \( u(t) = t + 1 \) is the only solution of the reduced problem \( uu'' - (t + 1) = 0 \) satisfying the assumptions of Theorem 2. Hence, there is \( \varepsilon_0 \) such that for every \( \varepsilon \in (0, \varepsilon_0] \) the problem has a solution satisfying

\[ |y(t, \varepsilon) - (t + 1)| \leq v_1 + v_2 + c_2 \varepsilon \]

on \([-2, 1]\).

References


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