PAIRWISE FUZZY CONNECTEDNESS BETWEEN FUZZY SETS

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Abstract. In this paper the concept of fuzzy connectedness between fuzzy sets [6] is generalized to fuzzy bitopological spaces and some of its properties are studied.

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1. Preliminaries

Let X and Y be non-empty sets. A fuzzy set λ in X is a mapping from X to the unit interval [0,1]. The null fuzzy set 0 (resp. the whole fuzzy set 1) is the mapping from X to the unit interval [0,1] which takes the only value 0 (resp. 1) in that interval. The basic operations on fuzzy sets are defined as follows:

\[
\bigcup_{\alpha \in \Lambda} \lambda_\alpha (x) = \sup_{\alpha \in \Lambda} \lambda_\alpha (x), \quad \forall x \in X,
\]

\[
\bigcap_{\alpha \in \Lambda} \lambda_\alpha (x) = \inf_{\alpha \in \Lambda} \lambda_\alpha (x), \quad \forall x \in X,
\]

\[1 \setminus \lambda(x) = 1 - \lambda(x), \quad \forall x \in X.
\]

A fuzzy topology [2] on X is a family \( \tau \) of fuzzy sets in X which satisfies the following conditions:

(a) \( 0,1 \in \tau \),
(b) \( \lambda, \mu \in \tau \Rightarrow \lambda \cap \mu \in \tau \),
(c) for each \( \alpha \in \Lambda \), \( \lambda_\alpha \in \tau \Rightarrow \bigcup_{\alpha \in \Lambda} \lambda_\alpha \in \tau \).

The pair \( (X, \tau) \) is called a fuzzy topological space and the members of \( \tau \) are called fuzzy open sets. The complements of the fuzzy open sets are called fuzzy closed sets. The closure denoted by \( \text{cl}(\lambda) \) (interior, denoted by \( \text{int}(\lambda) \)) of a fuzzy set \( \lambda \) of X is the
intersection (union) of all fuzzy closed supersets (fuzzy open subsets, respectively) of $\lambda$ [2]. For a fuzzy set $\lambda$ of a fuzzy topological space $X$, $1 - \text{int}(\lambda) = \text{cl}(1 - \lambda)$ and $1 - \text{cl}(\lambda) = \text{int}(1 - \lambda)$. A fuzzy set $\lambda$ in $X$ is said to be quasi-coincident [8] with a fuzzy set $\mu$ in $X$ denoted by $\lambda \approx \mu$ if there exists a point $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If $\lambda$ and $\mu$ are two fuzzy sets of $X$, then $\lambda \approx \mu$ if and only if $\lambda$ and $1 - \mu$ are not quasi-coincident. A fuzzy topological space $(X, \tau)$ is said to be fuzzy connected [3] if there is no proper fuzzy set in $X$ which is both fuzzy open and fuzzy closed. A fuzzy topological space $(X, \tau)$ is said to be fuzzy connected [6] between its subsets $\lambda$ and $\mu$ if and only if there is no fuzzy closed fuzzy open set $\delta$ in $X$ such that $\lambda \approx \delta$ and $\neg(\delta \approx \mu)$.

A system $(X, \tau_1, \tau_2)$ consisting of a set $X$ with two topologies $\tau_1$ and $\tau_2$ on $X$ is called a fuzzy bitopological space [5]. A fuzzy bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise fuzzy connected [8] if it has no proper fuzzy set which is both $\tau_1$-fuzzy open and $\tau_2$-fuzzy closed, $i, j = 1, 2$, $i \neq j$. The purpose of this paper is to introduce and study the concept of pairwise fuzzy connectedness between fuzzy sets in fuzzy bitopological spaces.

Throughout this paper $i, j = 1, 2$ where $i \neq j$. If $P$ is any fuzzy topological property then $\tau_1$-$P$ and $\tau_2$-$P$ denote the property $P$ with respect to the fuzzy topology $\tau_1$ and $\tau_2$, respectively and $\chi_A$ denotes the characteristic function of a subset $A$ of $X$.

2. Pairwise fuzzy connectedness between fuzzy sets

**Definition 2.1.** A fuzzy bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise fuzzy connected between fuzzy sets $\lambda$ and $\mu$ if there is no $(i, j)$-fuzzy clopen ($\tau_i$-fuzzy closed and $\tau_j$-fuzzy open) set $\delta$ in $X$ such that $\lambda \approx \delta$ and $\neg(\delta \approx \mu)$.

**Remark 2.1.** Pairwise fuzzy connectedness between fuzzy sets $\lambda$ and $\mu$ is not equal to the fuzzy connectedness of $(X, \tau_1)$ and $(X, \tau_2)$ between $\lambda$ and $\mu$.

**Example 2.1.** Let $X = \{a, b\}$ and let $\lambda$, $\mu$, $\nu_1$ and $\nu_2$ be fuzzy sets on $X$ defined as follows:

$$\begin{align*}
\lambda(a) &= 0.2, & \lambda(b) &= 0.3, \\
\mu(a) &= 0.5, & \mu(b) &= 0.4, \\
\nu_1(a) &= 0.3, & \nu_1(b) &= 0.4, \\
\nu_2(a) &= 0.7, & \nu_2(b) &= 0.6.
\end{align*}$$

Let $\tau_1 = \{0, \nu_1, 1\}$ and $\tau_2 = \{0, \nu_2, 1\}$ be fuzzy topologies on $X$. Then $(X, \tau_1)$ and $(X, \tau_2)$ are fuzzy connected between the fuzzy sets $\lambda$ and $\mu$ but $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected between $\lambda$ and $\mu$.

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Example 2.2. Let $X = \{a, b\}$. Let fuzzy sets $\nu_1, \nu_2, \delta_1, \delta_2, \lambda$ and $\mu$ be defined as follows:

$$
\begin{align*}
\nu_1(a) &= 0.5, \quad \nu_1(b) = 0.6, \\
\nu_2(a) &= 0.5, \quad \nu_2(b) = 0.7, \\
\delta_1(a) &= 0.5, \quad \delta_1(b) = 0.4, \\
\delta_2(a) &= 0.5, \quad \delta_2(b) = 0.3, \\
\lambda(a) &= 0.5, \quad \lambda(b) = 0.3, \\
\mu(a) &= 0.5, \quad \mu(b) = 0.2.
\end{align*}
$$

Let $\tau_1 = \{0, \nu_1, \delta_1, 1\}$ and $\tau_2 = \{0, \nu_2, \delta_2, 1\}$ be fuzzy topologies on $X$. Then the fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between $\lambda$ and $\mu$, but neither $(X, \tau_1)$ nor $(X, \tau_2)$ are fuzzy connected between $\lambda$ and $\mu$.

**Theorem 2.1.** A fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between fuzzy sets $\lambda$ and $\mu$ if and only if there is no $(i, j)$-fuzzy clopen set $\delta$ in $X$ such that $\lambda \leq \delta \leq 1 - \mu$.

**Proof.** Obvious. \qed

**Theorem 2.2.** If a fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between fuzzy sets $\lambda$ and $\mu$ then $\lambda$ and $\mu$ are non-empty.

**Proof.** Evident. \qed

**Theorem 2.3.** If a fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between fuzzy sets $\lambda$ and $\mu$ and if $\lambda \leq \lambda_1$ and $\mu \leq \mu_1$ then $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between $\lambda_1$ and $\mu_1$.

**Proof.** Suppose the fuzzy bitopological space $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected between the fuzzy sets $\lambda_1$ and $\mu_1$. Then there is an $(i, j)$-fuzzy clopen set $\delta$ in $X$ such that $\lambda_1 \leq \delta$ and $\neg(\delta \cap \mu_1)$. Clearly $\lambda \leq \delta$. Now we claim that $\neg(\delta \cap \mu)$. If $(\delta \cap \mu)$ then there exists a point $x \in X$ such that $\delta(x) + \mu(x) > 1$. Therefore $\delta(x) + \mu_1(x) > \delta(x) + \mu(x) > 1$ and $\delta \cap \mu_1$, a contradiction. Consequently, $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected between $\lambda$ and $\mu$. \qed

**Theorem 2.4.** A fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between $\lambda$ and $\mu$ if and only if it is pairwise fuzzy connected between $\tau_i$-$\text{cl}(\lambda)$ and $\tau_j$-$\text{cl}(\mu)$.

**Proof.** Necessity: It follows by using Theorem (2.3).
Sufficiency: Suppose the fuzzy bitopological space \((X, \tau_1, \tau_2)\) is not pairwise fuzzy connected between \(\lambda\) and \(\mu\). Then there is an \((i,j)\)-fuzzy clopen set \(\delta\) in \(X\) such that \(\lambda \leq \delta\) and \(\neg(\delta, q, \mu)\). Since \(\lambda \leq \delta\), \(\tau_i\)-cl(\(\lambda\)) \(\leq \tau_i\)-cl(\(\delta\)) \(< \delta\) because \(\delta\) is \(\tau_i\)-fuzzy closed. Now,

\[
\neg(\delta, q, \mu) \Rightarrow \delta \leq 1 - \mu \\
\Rightarrow \delta \leq \tau_j\text{-int}(1 - \mu) \\
\Rightarrow \delta \leq 1 - \tau_j\text{-cl}(\mu) \\
\Rightarrow \neg(\delta, q, \tau_j\text{-cl}(\mu)).
\]

Hence \(X\) is not pairwise fuzzy connected between \(\tau_i\text{-cl}(\lambda)\) and \(\tau_j\text{-cl}(\mu)\), a contradiction. \(\square\)

**Theorem 2.5.** Let \((X, \tau_1, \tau_2)\) be a fuzzy bitopological space and let \(\lambda\) and \(\mu\) be two fuzzy sets in \(X\). If \(\lambda, q, \mu\) then \((X, \tau_1, \tau_2)\) is pairwise fuzzy connected between \(\lambda\) and \(\mu\).

**Proof.** If \(\delta\) is any \((i,j)\)-fuzzy clopen set in \(X\) such that \(\lambda \leq \delta\) then \(\lambda, q, \mu \Rightarrow \delta, q, \mu\). \(\square\)

**Remark 2.2.** The converse of Theorem (2.5) may not be true as is shown by the next example.

**Example 2.3.** Let \(X = \{a, b\}\) and let the fuzzy sets \(\lambda, \mu, \delta_1\) and \(\delta_2\) be defined as follows:

\[
\lambda(a) = 0.5, \quad \lambda(b) = 0.4, \\
\mu(a) = 0.3, \quad \mu(b) = 0.5, \\
\delta_1(a) = 0.2, \quad \delta_1(b) = 0.9, \\
\delta_2(a) = 0.8, \quad \delta_2(b) = 0.1.
\]

Let \(\tau_1 = \{0, \delta_1, 1\}\) and \(\tau_2 = \{0, \delta_2, 1\}\) be fuzzy topologies on \(X\). Then the fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise fuzzy connected between \(\lambda\) and \(\mu\) but \(\neg(\lambda, q, \mu)\).

**Theorem 2.6.** If a fuzzy bitopological space \((X, \tau_1, \tau_2)\) is pairwise fuzzy connected neither between \(\lambda\) and \(\mu\), nor between \(\lambda\) and \(\mu_0\), then it is not pairwise fuzzy connected between \(\lambda\) and \(\mu_0 \cup \mu_1\).

**Proof.** Since \(X\) is pairwise fuzzy connected neither between \(\lambda\) and \(\mu_0\) nor between \(\lambda\) and \(\mu_1\), there exists \((i,j)\)-fuzzy clopen fuzzy sets \(\delta_0\) and \(\delta_1\) in \((X, \tau_1, \tau_2)\) such that \(\lambda \leq \delta_0\), \(\neg(\delta_0, q, \mu_0)\) and \(\lambda \leq \delta_1\), \(\neg(\delta_1, q, \mu_1)\). Put \(\delta = \delta_0 \cap \delta_1\). Then \(\delta\) is
(i, j)-fuzzy clopen and $\lambda \leq \delta$. Now we claim that $-\left(\delta \ q \ (\mu_0 \cup \mu_1)\right)$. If $\delta \ q \ (\mu_0 \cup \mu_1)$ then there exists a point $x \in X$ such that $\delta(x) + (\mu_0 \cup \mu_1)(x) > 1$. This implies that $\delta \ q \mu_0$ or $\delta \ q \mu_1$, a contradiction. Hence $X$ is not pairwise fuzzy connected between $\lambda$ and $\mu_0 \cup \mu_1$. \hfill \qed

**Theorem 2.7.** A fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected if and only if it is pairwise fuzzy connected between every pair of its non-empty fuzzy subsets.

**Proof.** Necessity: Let $\lambda$ and $\mu$ be any pair of non-empty fuzzy subsets of $X$. Suppose $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected between $\lambda$ and $\mu$. Then there is an $(i, j)$-fuzzy clopen set $\delta$ in $X$ such that $\lambda \leq \delta$ and $-(\delta \ q \mu)$. Since $\lambda$ and $\mu$ are non-empty, it follows that $\delta$ is a non-empty proper $(i, j)$-fuzzy clopen subset of $X$. Hence $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected.

Sufficiency: Suppose $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected. Then there exists a non-empty proper $(i, j)$-fuzzy clopen subset $\delta$ of $X$. Consequently, $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected between $\delta$ and $1 - \delta$, a contradiction. \hfill \qed

**Remark 2.3.** If fuzzy bitopological space $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between a pair of its subsets then it need not necessarily hold that $(X, \tau_1, \tau_2)$ is pairwise fuzzy between every pair of its subsets and so it is not necessarily pairwise fuzzy connected as is shown by the next example.

**Example 2.4.** Let $X = \{a, b\}$ and let $\delta_1, \delta_2, \lambda_1, \lambda_2, \mu_1$ and $\mu_2$ be defined as follows.

\[
\begin{align*}
\delta_1(a) &= 0.4, & \delta_1(b) &= 0.6, \\
\delta_2(a) &= 0.6, & \delta_2(b) &= 0.4, \\
\lambda_1(a) &= 0.7, & \lambda_1(b) &= 0.8, \\
\lambda_2(a) &= 0.3, & \lambda_2(b) &= 0.2, \\
\mu_1(a) &= 0.8, & \mu_1(b) &= 0.7, \\
\mu_2(a) &= 0.2, & \mu_2(b) &= 0.3.
\end{align*}
\]

Let $\tau_1 = \{0, \delta_1, 1\}$ and $\tau_2 = \{0, \delta_2, 1\}$ be two fuzzy topologies on $X$. Then $(X, \tau_1, \tau_2)$ is pairwise fuzzy connected between $\lambda$, and $\mu$, but it is not pairwise fuzzy connected between $\lambda_2$ and $\mu_2$. Also $(X, \tau_1, \tau_2)$ is not pairwise fuzzy connected.

**Theorem 2.8.** Let $(Y, (\tau_1)_Y, (\tau_2)_Y)$ be a subspace of a fuzzy bitopological space $(X, \tau_1, \tau_2)$ and let $\lambda$, $\mu$ be fuzzy sets of $Y$. If $(Y, (\tau_1)_Y, (\tau_2)_Y)$ is pairwise fuzzy connected between $\lambda$ and $\mu$ then $(X, \tau_1, \tau_2)$ is also pairwise fuzzy connected between $\lambda$ and $\mu$.

**Proof.** Evident. \hfill \qed

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Theorem 2.9. Let \((Y, (\tau_1)_Y, (\tau_2)_Y)\) be a subspace of a fuzzy bitopological space \((X, \tau_1, \tau_2)\) and let \(\lambda, \mu\) be fuzzy sets of \(Y\). If \((X, \tau_1, \tau_2)\) is pairwise fuzzy connected between \(\lambda\) and \(\mu\) and \(\chi_Y\) is bifuzzy clopen in \((X, \tau_1, \tau_2)\) then \((Y, (\tau_1)_Y, (\tau_2)_Y)\) is pairwise fuzzy connected between \(\lambda\) and \(\mu\).

\textbf{Proof.} Suppose \((Y, (\tau_1)_Y, (\tau_2)_Y)\) is not pairwise fuzzy connected between \(\lambda\) and \(\mu\) then there exists an \((i, j)\)-bifuzzy clopen set \(\delta\) in \(X\) such that \(\lambda \leq \delta\) and \(\neg(\lambda \cap \delta)\). Since \(\chi_Y\) is bifuzzy open and bifuzzy closed in \((X, \tau_1, \tau_2), \delta\) is an \((i, j)\)-bifuzzy clopen in \((X, \tau_1, \tau_2)\). Therefore \((X, \tau_1, \tau_2)\) is not pairwise fuzzy connected between \(\lambda\) and \(\mu\), which is a contradiction. \(\square\)

\textbf{References}


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