PERIODIC SOLUTIONS FOR THIRD-ORDER NONLINEAR
DELAY DIFFERENTIAL EQUATIONS WITH VARIABLE
COEFFICIENTS

ABDELOUAHEB ARDJOUNI\textsuperscript{1,*}, FARID NOUIOUA\textsuperscript{1} AND AHCENE DJOUDI\textsuperscript{2}

Communicated by F.H. Ghane

Abstract. In this paper, the following third-order nonlinear delay differential equation with periodic coefficients
\begin{align*}
x''''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) &= f(t,x(t),x(t-\tau(t))) + \frac{d}{dt}g(t,x(t-\tau(t))),
\end{align*}
is considered. By employing Green’s function, Krasnoselskii’s fixed point theorem and the contraction mapping principle, we state and prove the existence and uniqueness of periodic solutions to the third-order nonlinear delay differential equation.

1. Introduction

Third order differential equations arise from in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on \cite{19,23}.

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the monographs \cite{8,20} and the papers \cite{1–18,21–23,25–28} and the references therein.

\textsuperscript{*}Corresponding author.
The second order nonlinear delay differential equation with periodic coefficients
\[ x''(t) + p(t)x'(t) + q(t)x(t) = f(t, x(t), x(t - \tau(t))) + \frac{d}{dt}g(t, x(t - \tau(t))), \]
has been investigated in [5]. By using Krasnoselskii’s fixed point theorem and
the contraction mapping principle, Ardjouni and Djoudi obtained existence and
uniqueness of periodic solutions.

In [23], Ren, Siegmund and Chen discussed the existence of positive periodic
solutions for the third-order differential equation
\[ x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = g(t, x(t)). \]
By employing the fixed point index, the authors obtained existence results for
positive periodic solutions.

Inspired and motivated by the works mentioned above and the papers [1–
18], [21–23], [25–28] and the references therein, we concentrate on the existence
of periodic solutions for the third-order nonlinear delay differential equation
\[ x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = f(t, x(t), x(t - \tau(t))) + \frac{d}{dt}g(t, x(t - \tau(t))), \tag{1.1} \]
where \( p, q, r, \tau \) are continuous real-valued functions. The functions \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \)
and \( f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous in their respective arguments. To show
the existence of periodic solutions, we transform (1.1) into an integral equation
and then use Krasnoselskii’s fixed point theorem. The obtained integral equation
splits in the sum of two mappings, one is a contraction and the other is compact.
We also obtain the existence of a unique periodic solution of (1.1) by employing
the contraction mapping principle as the basic mathematical tool.

The organization of this paper is as follows. In Section 2, we introduce some
notations and lemmas, and state some preliminary results needed in later section.
Then we give the Green’s function of (1.1) which plays an important role in this
paper. In Section 3, we present our main results on existence and uniqueness.

We state Krasnoselskii’s fixed point theorem which enables us to prove the
existence of periodic solutions to (1.1). For its proof we refer the reader to [24].

**Theorem 1.1** (Krasnoselskii). Let \( \mathcal{M} \) be a closed convex nonempty subset of a
Banach space \((\mathcal{B}, ||.||))\). Suppose that \( H_1 \) and \( H_2 \) map \( \mathcal{M} \) into \( \mathcal{B} \) such that
(i) \( x, y \in \mathcal{M} \), implies \( H_1x + H_2y \in \mathcal{M} \),
(ii) \( H_1 \) is compact and continuous,
(iii) \( H_2 \) is a contraction mapping.
Then there exists \( z \in \mathcal{M} \) with \( z = H_1z + H_2z \).

In this paper, we give the assumptions as follows that will be used in the main
results.

(h1) There exist differentiable positive \( T \)-periodic functions \( a_1 \) and \( a_2 \) and a
positive real constant \( \rho \) such that
\[
\begin{align*}
\frac{d}{dt}a_1(t) + \rho a_1(t) &= p(t), \\
\frac{d}{dt}a_2(t) + \rho a_1(t) &= q(t), \\
\frac{d}{dt}a_2(t) + \rho a_2(t) &= r(t).
\end{align*}
\]
(h2) $p, q, r, \tau \in C(\mathbb{R}, \mathbb{R}^+)$ are $T$-periodic functions with $\tau(t) \geq \tau^* > 0$ and
\[
\int_0^T p(s)ds > \rho T, \quad \int_0^T q(s)ds > 0.
\]

(h3) The functions $g(t, x)$ and $f(t, x, y)$ are continuous $T$-periodic in $t$ and globally Lipschitz continuous in $x$ and in $x$ and $y$, respectively. That is
\[
g(t + T, x) = g(t, x), \quad f(t + T, x, y) = f(t, x, y),
\]
and there are positive constants $k_1, k_2$ and $k_3$ such that
\[
|g(t, x) - g(t, y)| \leq k_1 |x - y|,
\]
and
\[
|f(t, x, y) - f(t, z, w)| \leq k_1 |x - z| + k_2 |y - w|.
\]

2. **Green’s function of third-order differential equation**

For $T > 0$, let $P_T$ be the set of all continuous scalar functions $x$, periodic in $t$ of period $T$. Then $(P_T, \|\|)$ is a Banach space with the supremum norm
\[
\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0, T]} |x(t)|.
\]

We consider
\[
x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t), \quad (2.1)
\]
where $h$ is a continuous $T$-periodic function. Obviously, by the condition $(h1)$, (2.1) is transformed into
\[
\begin{align*}
\{ y'(t) + \rho y(t) &= h(t), \\
x''(t) + a_1(t)x'(t) + a_2(t)x(t) &= y(t).
\end{align*}
\]

**Lemma 2.1** ([3]). If $y, h \in P_T$, then $y$ is a solution of equation
\[
y'(t) + \rho y(t) = h(t),
\]
if and only if
\[
y(t) = \int_t^{t+T} G_1(t, s)h(s)ds, \quad (2.2)
\]
where
\[
G_1(t, s) = \frac{\exp(\rho (s-t))}{\exp(\rho T) - 1}. \quad (2.3)
\]

**Corollary 2.2.** Green function $G_1$ satisfies the following properties
\[
\begin{align*}
G_1(t + T, s + T) &= G_1(t, s), \quad G_1(t, t + T) = G_1(t, t) \exp(\rho T), \\
G_1(t + T, s) &= G_1(t, s) \exp(-\rho T), \quad G_1(t, s + T) = G_1(t, s) \exp(\rho T),
\end{align*}
\]
\[
\begin{align*}
\frac{\partial}{\partial t} G_1(t, s) &= -\rho G_1(t, s), \\
\frac{\partial}{\partial s} G_1(t, s) &= \rho G_1(t, s),
\end{align*}
\]
and

\[ m_1 \leq G_1(t, s) \leq M_1, \]

where

\[ m_1 = \frac{1}{\exp(\rho T) - 1}, \quad M_1 = \frac{\exp(\rho T)}{\exp(\rho T) - 1}. \]

Lemma 2.3 ([22]). Suppose that (h1) and (h2) hold and

\[ R_1 \left[ \exp \left( \int_0^T a_1(v) dv \right) - 1 \right] \geq 1, \tag{2.4} \]

where

\[ R_1 = \max_{t \in [0, T]} \left| \int_t^{t+T} \exp \left( \int_0^T a_1(v) dv \right) \frac{a_2(s)}{\exp(\int_0^T a(v) dv) - 1} ds \right|, \]

\[ Q_1 = \left( 1 + \exp \left( \int_0^T a_1(v) dv \right) \right)^2 R_1^2. \]

Then there are continuous \( T \)-periodic functions \( a \) and \( b \) such that

\[ b(t) > 0, \quad \int_0^T a(v) dv > 0, \]

and

\[ a(t) + b(t) = a_1(t), \quad b'(t) + a(t)b(t) = a_2(t), \quad \text{for } t \in \mathbb{R}. \]

Lemma 2.4 ([26]). Suppose the conditions of Lemma 2.3 hold and \( y \in P_T \). Then the equation

\[ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t), \]

has a \( T \)-periodic solution. Moreover, the periodic solution can be expressed by

\[ x(t) = \int_t^{t+T} G_2(t, s)y(s)ds, \tag{2.5} \]

where

\[ G_2(t, s) = \frac{\int_t^s \exp \left[ \int_0^\tau b(u) du + \int_\tau^s a(u) du \right] dv + \int_s^{t+T} \exp \left[ \int_t^\tau b(u) du + \int_\tau^{s+T} a(u) du \right] dv}{\exp \left( \int_0^T a(v) dv \right) - 1} \frac{1}{\exp \left( \int_0^T b(v) dv \right) - 1}. \tag{2.6} \]
Corollary 2.5. Green’s function \( G_2 \) satisfies the following properties

\[
G_2(t + T, s + T) = G_2(t, s), \quad G_2(t, t + T) = G_2(t, t),
\]

\[
G_2(t + T, s) = \exp \left( - \int_0^T b(v) dv \right) \left[ G_2(t, s) + \int_t^{t+T} E(t, u) F(u, s) du \right],
\]

\[
\frac{\partial}{\partial t} G_2(t, s) = -b(t) G_2(t, s) + F(t, s),
\]

\[
\frac{\partial}{\partial s} G_2(t, s) = a(t) G_2(t, s) - E(t, s),
\]

where

\[
E(t, s) = \frac{\exp \left( \int_t^s b(v) dv \right)}{\exp \left( \int_0^T b(v) dv \right) - 1}, \quad F(t, s) = \frac{\exp \left( \int_t^s a(v) dv \right)}{\exp \left( \int_0^T a(v) dv \right) - 1}.
\]

Lemma 2.6 ([22]). Let \( A = \int_0^T a_1(v) dv \) and \( B = T^2 \exp \left( \frac{1}{T} \int_0^T \ln (a_2(v)) dv \right) \). If

\[
A^2 \geq 4B,
\]

then

\[
\min \left\{ \int_0^T a(v) dv, \int_0^T b(v) dv \right\} \geq \frac{1}{2} \left( A - \sqrt{A^2 - 4B} \right) = l,
\]

\[
\max \left\{ \int_0^T a(v) dv, \int_0^T b(v) dv \right\} \leq \frac{1}{2} \left( A + \sqrt{A^2 - 4B} \right) = L.
\]

Corollary 2.7. Functions \( G_2, E \) and \( F \) satisfy

\[
m_2 \leq G_2(t, s) \leq M_2,
\]

\[
E(t, s) \leq \frac{e^L}{e^l - 1}, \quad F(t, s) \leq e^L,
\]

where

\[
m_2 = \frac{T}{(\exp (L) - 1)^2}, \quad M_2 = \frac{T \exp \left( \int_0^T a_1(v) dv \right)}{(\exp (l) - 1)^2}.
\]

Lemma 2.8 ([11]). Suppose the conditions of Lemma 2.3 hold and \( h \in P_T \). Then the equation

\[
x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = h(t),
\]

has a \( T \)-periodic solution. Moreover, the periodic solution can be expressed by

\[
x(t) = \int_t^{t+T} G(t, s) h(s) ds,
\]

where

\[
G(t, s) = \int_t^{t+T} G_2(t, \sigma) G_1(\sigma, s) d\sigma.
\]
Corollary 2.9. Green’s function $G$ satisfies the following properties

\[
G(t+T, s+T) = G(t, s), \quad G(t, t+T) = G(t, t) \exp(\rho T),
\]

\[
\frac{\partial}{\partial t} G(t, s) = (\exp(-\rho T) - 1) G_1(t, t) G_2(t, s) - b(t) G(t, s)
\]

\[
+ \int_t^{t+T} F(t, \sigma) G_1(\sigma, s) d\sigma,
\]

\[
\frac{\partial}{\partial s} G(t, s) = \rho G(t, s),
\]

and

\[
m \leq G(t, s) \leq M,
\]

where

\[
m = \frac{T^2}{(\exp(l) - 1)^2(\exp(\rho T) - 1)}, \quad M = \frac{T^2 \exp\left(\rho T + \int_0^T a(v) dv\right)}{(\exp(l) - 1)^2(\exp(\rho T) - 1)}.
\]

3. Main Results

In this section, we will study the existence and uniqueness of periodic solutions of (1.1).

Lemma 3.1. Suppose (h1) – (h3) and (2.4) hold. The function $x \in P_T$ is a solution of (1.1) if and only if

\[
x(t) = (\exp(\rho T) - 1) G(t, t) g(t, x(t - \tau(t)))
\]

\[
+ \int_t^{t+T} G(t, s) \left\{-\rho g(s, x(s - \tau(s))) + f(s, x(s), x(s - \tau(s)))\right\} ds. \quad (3.1)
\]

Proof. Let $x \in P_T$ be a solution of (1.1). From Lemma 2.8, we have

\[
x(t) = \int_t^{t+T} G(t, s) \left[f(s, x(s), x(s - \tau(s))) + \frac{\partial}{\partial s} g(s, x(s - \tau(s)))\right] ds
\]

\[
= \int_t^{t+T} G(t, s) f(s, x(s), x(s - \tau(s))) ds
\]

\[
+ \int_t^{t+T} G(t, s) \frac{\partial}{\partial s} g(s, x(s - \tau(s))) ds. \quad (3.2)
\]

Performing an integration by parts, we get

\[
\int_t^{t+T} G(t, s) \frac{\partial}{\partial s} g(s, x(s - \tau(s))) ds
\]

\[
= G(t, s) g(s, x(s - \tau(s))) |_t^{t+T} - \int_t^{t+T} \left[ \frac{\partial}{\partial s} G(t, s) \right] g(s, x(s - \tau(s))) ds
\]

\[
= (\exp(\rho T) - 1) G(t, t) g(t, x(t - \tau(t))) - \rho \int_t^{t+T} G(t, s) g(s, x(s - \tau(s))) ds.
\]

(3.3)
We obtain (3.1) by substituting (3.3) in (3.2). Since each step is reversible, the converse follows easily. This completes the proof.

Define the mapping $H : P_T \to P_T$ by

$$(H \varphi)(t) = \int_t^{t+T} G(t,s) \{-\rho g(s, \varphi (s - \tau (s))) + f(s, \varphi (s), \varphi (s - \tau (s)))\} \, ds + (\exp (\rho T) - 1) G(t,t) g(t, \varphi(t - \tau(t))) .$$

(3.4)

Note that to apply Krasnoselskii’s fixed point theorem we need to construct two mappings, one is a contraction and the other is compact. Therefore, we express (3.4) as

$$(H \varphi)(t) = (H_1 \varphi)(t) + (H_2 \varphi)(t),$$

where $H_1, H_2 : P_T \to P_T$ are given by

$$(H_1 \varphi)(t) = \int_t^{t+T} G(t,s) \{-\rho g(s, \varphi (s - \tau (s))) + f(s, \varphi (s), \varphi (s - \tau (s)))\} \, ds ,$$

and

$$(H_2 \varphi)(t) = (\exp (\rho T) - 1) G(t,t) g(t, \varphi(t - \tau(t))) .$$

(3.5)

(3.6)

To simplify notation, we introduce the constants

$$\beta = \max_{t \in [0,T]} \{b(t)\} , \ \delta = \frac{\exp (L)}{\exp (l) - 1} .$$

(3.7)

**Lemma 3.2.** Suppose $(h1) - (h3)$, (2.4) and (2.7) hold. Then $H_1 : P_T \to P_T$ is compact.

**Proof.** Let $H_1$ be defined by (3.5). Obviously, $H_1 \varphi$ is continuous and it is easy to show that $(H_1 \varphi)(t+T) = (H_1 \varphi)(t)$. To see that $H_1$ is continuous, we let $\varphi, \psi \in P_T$. Given $\varepsilon > 0$, take $\theta = \varepsilon / N$ with $N = MT (\rho k_1 + k_2 + k_3)$ where $k_1, k_2$ and $k_3$ are given by (h3). Now, for $\|\varphi - \psi\| < \theta$, we obtain

$$\|H_1 \varphi - H_1 \psi\| \leq M \int_t^{t+T} [\rho k_1 \|\varphi - \psi\| + (k_2 + k_3) \|\varphi - \psi\|] \, ds \leq N \|\varphi - \psi\| < \varepsilon .$$

This proves that $H_1$ is continuous. To show that the image of $H_1$ is contained in a compact set, we consider $\mathcal{D} = \{\varphi \in P_T : \|\varphi\| \leq \mathcal{L}\}$, where $\mathcal{L}$ is a fixed positive constant. Let $\varphi_n \in \mathcal{D}$, where $n$ is a positive integer. Observe that in view of (h3) we have

$$|g(t,x)| = |g(t, x) - g(t,0) + g(t,0)| \leq |g(t, x) - g(t,0)| + |g(t,0)| \leq k_1 \|x\| + \mu_1 ,$$

and

$$|f(t,x,y)| = |f(t, x,y) - f(t,0,0) + f(t,0,0)| \leq |f(t, x,y) - f(t,0,0)| + |f(t,0,0)| \leq k_2 \|x\| + k_3 \|y\| + \mu_2 ,$$
where \( \mu_1 = \max_{t \in [0,T]} |g(t, 0)| \) and \( \mu_2 = \max_{t \in [0,T]} |f(t, 0, 0)| \). Hence if \( H_1 \) is given by (3.5) we obtain

\[ \|H_1 \varphi_n\| \leq D, \]

for some positive \( D \). Next we calculate \( \frac{d}{dt} \left( H_1 \varphi_n \right)(t) \) and show that it is uniformly bounded. By making use of \( (h1), (h2) \) and \( (h3) \) we obtain by taking the derivative in (3.5) that

\[
\frac{d}{dt} \left( H_1 \varphi_n \right)(t) = \int_{t}^{t+T} \left[ (\exp(-\rho T) - 1) G_1(t, t) G_2(t, s) - b(t) G(t, s) \right. \\
+ \int_{t}^{t+T} F(t, \sigma) G_1(\sigma, s) d\sigma \left] \left[ -\rho g(s, \varphi(s - \tau(s))) + f(s, \varphi(s), \varphi(s - \tau(s))) \right] ds. \]

Consequently, by invoking \( (h3) \) and (3.7), we obtain

\[
\left| \frac{d}{dt} \left( H_1 \varphi_n \right)(t) \right| \leq \left[ (1 - \exp(-\rho T)) M_1 M_2 + M_\beta + M_1 \delta T \right] \left( \rho (k_1 \mathcal{L} + \mu_1) + (k_2 + k_3) \mathcal{L} + \mu_2 \right) T \leq K,
\]

for some positive \( K \). Hence the sequence \( \left( H_1 \varphi_n \right) \) is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence \( \left( H_1 \varphi_{n_k} \right) \) of \( \left( H_1 \varphi_n \right) \) converges uniformly to continuous \( T \)-periodic function. Thus \( H_1 \) is continuous and \( H_1(\mathbb{D}) \) is contained in a compact subset of \( P_T \). \( \square \)

**Lemma 3.3.** If \( H_2 \) is given by (3.6) with

\[ k_1 \left( \exp(\rho T) - 1 \right) M < 1, \]  

then \( H_2 : P_T \to P_T \) is a contraction.

**Proof.** Let \( H_2 \) be defined by (3.6). It is easy to show that \( (H_2 \varphi)(t + T) = (H_2 \varphi)(t) \). To see that \( H_2 \) is a contraction, let \( \varphi, \psi \in P_T \). Then we have

\[
\|H_2 \varphi - H_2 \psi\| = \sup_{t \in [0,T]} \left| (H_2 \varphi)(t) - (H_2 \psi)(t) \right| \leq k_1 \left( \exp(\rho T) - 1 \right) M \|\varphi - \psi\|. 
\]

Hence \( H_2 : P_T \to P_T \) is a contraction. \( \square \)

**Theorem 3.4.** Suppose that conditions \( (h1) - (h3), (2.4), (2.7) \) and (3.8) hold. Suppose there exists a positive constant \( J \) satisfying the inequality

\[ \left( \exp(\rho T) - 1 \right) M (k_1 J + \mu_1) + (\rho (k_1 J + \mu_1) + (k_2 + k_3) J + \mu_2) T \leq J. \]

Then (1.1) has a solution \( x \in P_T \) such that \( \|x\| \leq J \).

**Proof.** Define \( \mathbb{M} = \{ \varphi \in P_T : \|\varphi\| \leq J \} \). By Lemma 3.2, the operator \( H_1 : \mathbb{M} \to P_T \) is compact and continuous. Also, from Lemma 3.3, the operator \( H_2 : \mathbb{M} \to P_T \) is a contraction. Therefore, by the Schauder fixed point theorem, there exists a solution \( x \in P_T \) to (1.1) such that \( \|x\| \leq J \). \( \square \)
is a contraction. Conditions (ii) and (iii) of Krasnoselskii theorem are satisfied. We need to show that condition (i) is fulfilled. To this end, let \( \varphi, \psi \in \mathbb{M} \). Then
\[
\left| (H_1 \varphi)(t) + (H_2 \psi)(t) \right|
\leq M \int_{t}^{t+T} \left[ \rho (k_1 \|\varphi\| + \mu_1) + (k_2 + k_3) \|\varphi\| + \mu_2 \right] ds
+ (\exp (\rho T) - 1) M (k_1 \|\psi\| + \mu_1)
\leq (\exp (\rho T) - 1) M (k_1 J + \mu_1) + (\rho (k_1 J + \mu_1) + (k_2 + k_3) J + \mu_2) T \leq J.
\]
Thus \( \|H_1 \varphi + H_2 \psi\| \leq J \) and so \( H_1 \varphi + H_2 \psi \in \mathbb{M} \). All the conditions of Krasnoselskii theorem are satisfied and consequently the operator \( H \) defined in (3.4) has a fixed point in \( \mathbb{M} \). By Lemma 3.1 this fixed point is a solution of (1.1) and the proof is complete.

**Theorem 3.5.** Suppose that conditions \((h1) - (h3), (2.4)\) and \((2.7)\) hold. If
\[(\exp (\rho T) - 1) M k_1 + (\rho k_1 + k_2 + k_3) T < 1,
\]
then (1.1) has a unique \( T \)-periodic solution.

**Proof.** Let the mapping \( H \) be given by (3.4). For \( \varphi, \psi \in P_T \), we have
\[
\left| (H \varphi)(t) - (H \psi)(t) \right|
\leq M \int_{t}^{t+T} \left[ \rho k_1 \|\varphi - \psi\| + (k_2 + k_3) \|\varphi - \psi\| \right] ds + (\exp (\rho T) - 1) M k_1 \|\varphi - \psi\|.
\]
Hence
\[
\|H \varphi - H \psi\| \leq [(\exp (\rho T) - 1) M k_1 + (\rho k_1 + k_2 + k_3) T] \|\varphi - \psi\|.
\]
By the contraction mapping principle, \( H \) has a fixed point in \( P_T \) and by Lemma 3.1, this fixed point is a solution of (1.1). The proof is complete.

**References**


1 Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria.
E-mail address: abd_ardjouni@yahoo.fr, fnouioua@gmail.com

2 Department of Mathematics, University of Annaba, P.O. Box 12, Annaba, 23000, Algeria.
E-mail address: adjoudi@yahoo.com