HERMITE-HADAMARD TYPE INEQUALITIES FOR MAPPINGS WHOSE DERIVATIVES ARE $s$–CONVEX IN THE SECOND SENSE VIA FRACTIONAL INTEGRALS

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Abstract. In this paper we establish Hermite-Hadamard type inequalities for mappings whose derivatives are $s$–convex in the second sense and concave.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

(1.1)

is known that the Hermite-Hadamard inequality for convex function. Both inequalities hold in the reserved direction if $f$ is concave. We note that Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example see ([1]–[21]).

Definition 1.1. ([18]) A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be $s$–convex in the second sense if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$
for all $x, y \in [0, \infty), \lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. This class of $s$–convex functions is usually denoted by $K_2^s$.

In ([15]) Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for $s$–convex functions in the second sense:

**Theorem 1.2.** Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s$–convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty), a < b$. If $f' \in L^1([a, b])$, then the following inequalities hold:

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}$$

(1.2)

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2)

The following results are proved by M.I.Bhatti et al. (see [8]).

**Theorem 1.3.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$ such that $|f''|$ is convex function on $I$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:

$$\left|\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J^\alpha_a f(b) + J^\alpha_b f(a)]\right| \leq \frac{\alpha(b-a)^2}{2(\alpha + 1)(\alpha + 2)} \left[\frac{|f''(a)|}{2} + \frac{|f''(b)|}{2}\right]$$

$$\leq \frac{(b-a)^2}{\alpha + 1} \beta(2, \alpha + 1) \left[\frac{|f''(a)|}{2} + \frac{|f''(b)|}{2}\right]$$

(1.3)

where $\beta$ is Euler Beta function.

**Theorem 1.4.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$. Assume that $p \in \mathbb{R}, p > 1$ such that $|f''|^{\frac{1}{p-1}}$ is convex function on $I$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:

$$\left|\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J^\alpha_a f(b) + J^\alpha_b f(a)]\right| \leq \frac{(b-a)^2}{\alpha + 1} \beta^{\frac{1}{p}} (p + 1, \alpha p + 1) \left[\frac{|f''(a)|}{2} + \frac{|f''(b)|}{2}\right]^{\frac{1}{p}}$$

(1.4)

where $\beta$ is Euler Beta function.

**Theorem 1.5.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$. Assume that $q \geq 1$ such that $|f''|^q$ is convex function on $I$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:

$$\left|\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J^\alpha_a f(b) + J^\alpha_b f(a)]\right| \leq \frac{\alpha(b-a)^2}{4(\alpha + 1)(\alpha + 2)} \left[\left(\frac{2\alpha+4}{3\alpha+9} |f''(a)|^q + \frac{\alpha+5}{3\alpha+9} |f''(b)|^q\right)^{\frac{1}{q}} + \left(\frac{\alpha+5}{3\alpha+9} |f''(a)|^q + \frac{2\alpha+4}{3\alpha+9} |f''(b)|^q\right)^{\frac{1}{q}}\right].$$

(1.5)
Theorem 1.6. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$. Assume that $p \in \mathbb{R}, p > 1$ with $q = \frac{p}{p-1}$ such that $|f''|^q$ is concave function on $I$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma (\alpha + 1)}{2 (b - a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \leq \frac{(b - a)^2}{\alpha + 1} \beta (p, \alpha + 1) \left| f'' \left( \frac{a + b}{2} \right) \right|$$  \hspace{1cm} (1.6)

where $\beta$ is Euler Beta function.

We will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further this paper.

Definition 1.7. Let $f \in L[a, b]$. The Reimann-Liouville integrals $J_{a^+}^\alpha f(x)$ and $J_{b^-}^\alpha f(x)$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma (\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt, \quad x > a$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma (\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) dt, \quad x < b$$

respectively, where $\Gamma (\alpha) = \int_0^\infty e^{-u} u^{\alpha - 1} du$ is the Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities, see [3]-[25].

In this paper, we establish fractional integral inequalities of Hermite-Hadamard type for mappings whose derivatives are $s-$ convex and concave.

2. Main results

In order to prove our main theorems we need the following lemma (see [8]).

Lemma 2.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$, the interior of $I$. Assume that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following identity for fractional integral with $\alpha > 0$ holds:

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma (\alpha + 1)}{2 (b - a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] = \frac{(b - a)^2}{2 (\alpha + 1)} \int_0^1 t (1 - t^\alpha) \left[ f'' (ta + (1 - t) b) + f'' ((1 - t) a + tb) \right] dt$$  \hspace{1cm} (2.1)

where $\Gamma (\alpha) = \int_0^\infty e^{-u} u^{\alpha - 1} du$. 

Theorem 2.2. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a twice differentiable function on \( I^c \) and let \( a, b \in I^c \) with \( a < b \) and \( f'' \in L[a, b] \). If \(|f''|\) is \( s\)-convex in the second sense on \( I \) for some fixed \( s \in (0, 1) \), then the following inequality for fractional integrals with \( \alpha > 0 \) holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\beta - a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \leq \frac{(b - a)^2}{2(\alpha + 1)} \left[ \int_0^1 t(1 - t^\alpha) |t^s f''(a) + (1 - t)^s f''(b)| dt \right]
\]

where \( \beta \) is Euler Beta function.

Proof. From Lemma 2.1 since \(|f''|\) is \( s\)-convex in the second sense on \( I \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\beta - a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \leq \frac{(b - a)^2}{2(\alpha + 1)} \int_0^1 t(1 - t^\alpha) \left[ \|f''(ta + (1 - t)b)\| + |f''((1 - t)a + tb)| \right] dt
\]

\[
\leq \frac{(b - a)^2}{2(\alpha + 1)} \left[ \int_0^1 t(1 - t^\alpha) \left| t^s f''(a) + (1 - t)^s f''(b) \right| dt \right]
\]

\[
+ \int_0^1 t(1 - t^\alpha) \left| (1 - t)^s f''(a) + t^s f''(b) \right| dt
\]

\[
= \frac{(b - a)^2}{2(\alpha + 1)} \left[ \int_0^1 t^{s+1} (1 - t^\alpha) dt + \int_0^1 t(1 - t^\alpha)(1 - t)^s dt \right] \left[ \|f''(a)\| + |f''(b)| \right]
\]

\[
= \frac{(b - a)^2}{2(\alpha + 1)} \left[ \frac{\alpha}{(s + 2)(\alpha + s + 2)} + \beta(2, s + 1) - \beta(\alpha + 2, s + 1) \right]
\]

\[
\times \left[ |f''(a)| + |f''(b)| \right]
\]

where we used the fact that

\[
\int_0^1 t^{s+1} (1 - t^\alpha) \, dt = \frac{\alpha}{(s + 2)(\alpha + s + 2)}
\]

and

\[
\int_0^1 t(1 - t^\alpha)(1 - t)^s \, dt = \beta(2, s + 1) - \beta(\alpha + 2, s + 1)
\]

which completes the proof.

\(\square\)

Remark 2.3. In Theorem 2.2 if we choose \( s = 1 \) then (2.2) reduces the inequality (1.3) of Theorem 1.3.

Theorem 2.4. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a twice differentiable function on \( I^c \). Suppose that \( a, b \in I^c \) with \( a < b \) and \( f'' \in L[a, b] \). If \(|f''|^q\) is \( s\)-convex in the second sense on \( I \) for some fixed \( s \in (0, 1) \), \( p, q > 1 \), then the following inequality for fractional integrals with \( \alpha \in (0, 1) \) holds:
Proof. From Lemma 2.1, using the well known Hölder inequality and $|f''|^{q}$ is $s-$convex in the second sense on $I$, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left[ J_{a^{+}}^{\alpha} f(b) + J_{b^{-}}^{\alpha} f(a) \right] \right|
\leq \frac{(b - a)^{2}}{2(\alpha + 1)} \int_{0}^{1} |t (1 - t)^{\alpha}| \left[ |f''(ta + (1 - t)b)| + |f''((1 - t)a + tb)| \right] dt
\leq \frac{(b - a)^{2}}{2(\alpha + 1)} \left( \int_{0}^{1} t^{p} (1 - t)^{\alpha} dt \right)^{\frac{1}{p}}
\times \left[ \left( \int_{0}^{1} |f''(ta + (1 - t)b)|^{q} dt \right)^{\frac{1}{q}} + \left( \int_{0}^{1} |f''((1 - t)a + tb)|^{q} dt \right)^{\frac{1}{q}} \right]
\leq \frac{(b - a)^{2}}{2(\alpha + 1)} \left( \int_{0}^{1} t^{p} (1 - t)^{\alpha} dt \right)^{\frac{1}{p}}
\times \left[ \left( \int_{0}^{1} t^{s} |f''(a)|^{q} + (1 - t)^{s} |f''(b)|^{q} dt \right)^{\frac{1}{q}} + \left( \int_{0}^{1} ((1 - t)^{s} |f''(a)|^{q} + t^{s} |f''(b)|^{q} dt \right)^{\frac{1}{q}} \right]
\leq \frac{(b - a)^{2}}{2(\alpha + 1)} \left( \int_{0}^{1} t^{p} (1 - t)^{\alpha} dt \right)^{\frac{1}{p}} \left[ \left( \int_{0}^{1} t^{s} \left( \frac{|f''(a)|^{q}}{s + 1} \right) dt \right)^{\frac{1}{q}} + \left( \int_{0}^{1} t^{s} \left( \frac{|f''(b)|^{q}}{s + 1} \right) dt \right)^{\frac{1}{q}} \right]
\leq \frac{(b - a)^{2}}{2(\alpha + 1)} \frac{1}{\beta \alpha^\frac{1}{p}} (p + 1, \alpha p + 1) \left[ \frac{|f''(a)|^{q} + |f''(b)|^{q}}{s + 1} \right]^{\frac{1}{q}}
\]

where we used the fact that

\[
\int_{0}^{1} t^{s} dt = \int_{0}^{1} (1 - t)^{s} dt = \frac{1}{s + 1}
\]

and

\[
\int_{0}^{1} t^{p} (1 - t)^{\alpha} dt \leq \int_{0}^{1} t^{p} (1 - t)^{\alpha p} dt = \beta (p + 1, \alpha p + 1)
\]

which completes the proof.

Remark 2.5. In Theorem 2.4 if we choose $s = 1$ then (2.3) reduces the inequality (1.4) of Theorem 1.4.
Theorem 2.6. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a twice differentiable function on \( I \). Suppose that \( a, b \in I^0 \) with \( a < b \) and \( f'' \in L[a, b] \). If \( |f''|^q \) is \( s \)-convex in the second sense on \( I \) for some fixed \( s \in (0, 1] \) and \( q \geq 1 \) then the following inequality for fractional integrals with \( \alpha > 0 \) holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)\alpha} \left[ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \leq \frac{(b - a)^2}{4(\alpha + 1)(\alpha + 2)} \left[ \left( \left| f''(a) \right|^q \frac{2\alpha + 4}{2 + (s + 2)(\alpha + s + 2)} + \left| f''(b) \right|^q \frac{\beta(2s + 1) - \beta(\alpha + 2s + 1)(2\alpha + 4)}{\alpha} \right)^{\frac{1}{q}} + \left( \left| f''(a) \right|^q \frac{\beta(2s + 1) - \beta(\alpha + 2s + 1)(2\alpha + 4)}{\alpha} + \left| f''(b) \right|^q \frac{2\alpha + 4}{(s + 2)(\alpha + s + 2)} \right)^{\frac{1}{q}} \right]
\]

Proof. From Lemma 2.1, using power mean inequality and \( |f''|^q \) is \( s \)-convex in the second sense on \( I \) we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)\alpha} \left[ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \leq \frac{(b - a)^2}{2(\alpha + 1)} \int_0^1 |t(1 - t^\alpha)| \left[ \left| f''(ta + (1 - t)b) \right| + \left| f''((1 - t)a + tb) \right| \right] dt
\]

\[
\leq \frac{(b - a)^2}{2(\alpha + 1)} \left( \int_0^1 t(1 - t^\alpha) dt \right)^{1 - \frac{1}{q}} \left[ \left( \int_0^1 t(1 - t^\alpha) \left| f''(ta + (1 - t)b) \right|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 t(1 - t^\alpha) \left| f''((1 - t)a + tb) \right|^q dt \right)^{\frac{1}{q}} \right]
\]

\[
\leq \frac{(b - a)^2}{2(\alpha + 1)} \left( \int_0^1 t(1 - t^\alpha) dt \right)^{1 - \frac{1}{q}} \times \left[ \left( \int_0^1 t(s + 1)(1 - t^\alpha) \left| f''(a) \right|^q + t(1 - t^\alpha)(1 - t^s) \left| f''(b) \right|^q \right) dt \right]^{\frac{1}{q}}
\]

\[
+ \left( \int_0^1 t(1 - t^\alpha)(1 - t^s) \left| f''(a) \right|^q + t(s + 1)(1 - t^\alpha) \left| f''(b) \right|^q \right) dt \right]^{\frac{1}{q}}
\]

\[
= \frac{(b - a)^2}{2(\alpha + 1)} \left( \frac{\alpha}{2(\alpha + 2)} \right)^{1 - \frac{1}{q}} \times \left[ \left( \left| f''(a) \right|^q \frac{\alpha}{s + 2(\alpha + s + 2)} + \left| f''(b) \right|^q \frac{\beta(2s + 1) - \beta(\alpha + 2s + 1)(2\alpha + 4)}{\alpha} \right)^{\frac{1}{q}} + \left( \left| f''(a) \right|^q \frac{\beta(2s + 1) - \beta(\alpha + 2s + 1)(2\alpha + 4)}{2(\alpha + s + 2)} \right)^{\frac{1}{q}} \right]
\]
where we used the fact that
\[
\int_0^1 t^{s+1} (1 - t^\alpha) \, dt = \frac{\alpha}{(s+2)(\alpha+s+2)}
\]
and
\[
\int_0^1 t (1 - t^\alpha) (1 - t)^s \, dt = \beta(2, s+1) - \beta(\alpha+2, s+1)
\]
which completes the proof. □

Remark 2.7. In Theorem 2.6 if we choose \( s = 1 \) then (2.4) reduces the inequality (1.5) of Theorem 1.5.

The following result holds for \( s \)-concavity.

**Theorem 2.8.** Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a twice differentiable function on \( I^\circ \). Suppose that \( a, b \in I^\circ \) with \( a < b \) and \( f'' \in L[a, b] \). If \( |f''|^q \) is \( s \)-concave in the second sense on \( I \) for some fixed \( s \in (0, 1] \) and \( p, q > 1 \), then the following inequality for fractional integrals with \( \alpha \in (0, 1] \) holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \leq \frac{(b-a)^2}{\alpha+1} \beta^{\frac{1}{p}} (p+1, \alpha p+1) 2^{-\frac{s-1}{q}} \left| f'' \left( \frac{a+b}{2} \right) \right|
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \beta \) is Euler Beta function.

**Proof.** From Lemma 2.1 and using the Hölder inequality we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \leq \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t (1 - t^\alpha)| \left[ |f''(ta + (1-t)b)| + |f''((1-t)a + tb)| \right] \, dt
\]

\[
\leq \frac{(b-a)^2}{2(\alpha+1)} \left( \int_0^1 t^p (1 - t^\alpha)^p \, dt \right)^{\frac{1}{p}} \times \left[ \left( \int_0^1 |f''(ta + (1-t)b)|^q \, dt \right)^{\frac{1}{q}} + \left( \int_0^1 |f''((1-t)a + tb)|^q \, dt \right)^{\frac{1}{q}} \right]
\]
Since $|f''|^q$ is $s$–concave using inequality (1.2) we get (see [2])

$$
\int_0^1 |f''(ta + (1-t)b)|^q \, dt \leq 2^{s-1} \left| f'' \left( \frac{a+b}{2} \right) \right|^q \tag{2.7}
$$

and

$$
\int_0^1 |f''((1-t)a + tb)|^q \, dt \leq 2^{s-1} \left| f'' \left( \frac{b+a}{2} \right) \right|^q \tag{2.8}
$$

Using (2.7) and (2.8) in (2.6), we have

$$
\left| f(a) + f(b) - \frac{\Gamma (\alpha + 1)}{2(b-a)^{\alpha}} \left[ J_a^\alpha f(b) + J_b^\alpha f(a) \right] \right|
\leq \frac{(b-a)^2}{\alpha + 1} \frac{\beta p}{p + 1} \left( 2^{s-1} \left| f'' \left( \frac{a+b}{2} \right) \right|^q \right)
$$

which completes the proof. $\square$

**Remark 2.9.** In Theorem 2.8 if we choose $s = 1$ then (2.5) reduces inequality (1.6) of Theorem 1.6.

**References**


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