Homomorphisms between Lie JC*-Algebras and Cauchy–Rassias Stability of Lie JC*-Algebra Derivations

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Abstract. It is shown that every almost linear mapping $h: A \rightarrow B$ of a unital Lie JC*-algebra $A$ to a unital Lie JC*-algebra $B$ is a Lie JC*-algebra homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$, $h(3^n u \circ y) = h(3^n u) \circ h(y)$ or $h(q^n u \circ y) = h(q^n u) \circ h(y)$ for all $y \in A$, all unitary elements $u \in A$ and $n = 0, 1, 2, \cdots$, and that every almost linear almost multiplicative mapping $h: A \rightarrow B$ is a Lie JC*-algebra homomorphism when $h(2x) = 2h(x)$, $h(3x) = 3h(x)$ or $h(qx)qh(x)$ for all $x \in A$. Here the numbers 2, 3, $q$ depend on the functional equations given in the almost linear mappings or in the almost linear almost multiplicative mappings. Moreover, we prove the Cauchy–Rassias stability of Lie JC*-algebra homomorphisms in Lie JC*-algebras, and of Lie JC*-algebra derivations in Lie JC*-algebras.

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1. Introduction

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [18]). Let $L(H)$ be the real vector space of all bounded self-adjoint linear operators on $H$, interpreted as the (bounded) observables of the system. In 1932, Jordan observed that $L(H)$ is a (nonassociative) algebra via the anticommutator product $x \circ y := \frac{xy + yx}{2}$. A commutative algebra $X$ with product $x \circ y$ is called a Jordan algebra. A unital Jordan $C^*$-subalgebra of a $C^*$-algebra, endowed with the anticommutator product, is called a JC*-algebra.

A unital $C^*$-algebra $C$, endowed with the Lie product $[x, y] = \frac{xy - yx}{2}$ on $C$, is called a Lie $C^*$-algebra. A unital $C^*$-algebra $C$, endowed with the Lie product $[,]$ and the anticommutator product $\circ$, is called a Lie JC*-algebra if $(C, \circ)$ is a JC*-algebra and $(C, [,])$ is a Lie $C^*$-algebra (see [5], [6]).

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Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \theta(|x|^p + |y|^p)
\]
for all $x, y \in X$. Rassias [11] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that
\[
\| f(x) - T(x) \| \leq \frac{2\theta}{2 - 2p}|x|^p
\]
for all $x \in X$. Găvruţa [1] generalized the Rassias’ result: Let $G$ be an abelian group and $Y$ a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that
\[
\tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty
\]
for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying
\[
\| f(x + y) - f(x) - f(y) \| \leq \varphi(x, y)
\]
for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that
\[
\| f(x) - T(x) \| \leq \frac{1}{2} \tilde{\varphi}(x, x)
\]
for all $x \in G$. C. Park [7] applied the Găvruţa’s result to linear functional equations in Banach modules over a $C^*$-algebra.

Jun and Lee [2] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0, \infty)$ a function such that
\[
\tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y) < \infty
\]
for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying
\[
\| 2f\left(\frac{x + y}{2}\right) - f(x) - f(y) \| \leq \varphi(x, y)
\]
for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that
\[
\| f(x) - f(0) - T(x) \| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x))
\]
for all $x \in X \setminus \{0\}$. C. Park and W. Park [9] applied the Jun and Lee’s result to the Jensen’s equation in Banach modules over a $C^*$-algebra.

Recently, Trif [17] proved the following: Let $q := \frac{(d-1)}{d-1}$ and $r := -\frac{1}{d-1}$. Denote by $\varphi : X^d \to [0, \infty)$ a function such that
\[
\tilde{\varphi}(x_1, \ldots, x_d) = \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_d) < \infty
\]
for all $x_1, \ldots, x_d \in X$. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$
\|d_{d-2}C_{l-2}f(\frac{x_1 + \cdots + x_d}{d}) + d_{-2}C_{l-1}\sum_{j=1}^{d} f(x_j) - l \sum_{1 \leq j_1 < \cdots < j_l \leq d} f(\frac{x_{j_1} + \cdots + x_{j_l}}{l})\| \leq \varphi(x_1, \ldots, x_d)
$$

for all $x_1, \ldots, x_d \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$
\|f(x) - f(0) - T(x)\| \leq \frac{1}{l \cdot d_{-1}C_{l-1}} \tilde{\varphi}(qx, rx, \ldots, rx)
$$

for all $x \in X$. And C. Park [8] applied the Trif’s result to the Trif functional equation in Banach modules over a $C^*$-algebra. Several authors have investigated functional equations (see [10]–[16]).

Throughout this paper, let $q = \frac{l(d-1)}{d-l}$ and $r = -\frac{l}{d-l}$ for positive integers $l, d$ with $2 \leq l \leq d - 1$. Let $A$ be a unital Lie $JC^*$-algebra with norm $\| \cdot \|$, unit $e$ and unitary group $U(A) = \{ u \in A \mid uu^* = u^*u = e \}$, and $B$ a unital Lie $JC^*$-algebra with norm $\| \cdot \|$ and unit $e'$.

Using the stability methods of linear functional equations, we prove that every almost linear mapping $h : A \rightarrow B$ is a Lie $JC^*$-algebra homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$, $h(3^n u \circ y) = h(3^n u) \circ h(y)$ or $h(q^n u \circ y) = h(q^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \ldots$, and that every almost linear almost multiplicative mapping $h : A \rightarrow B$ is a Lie $JC^*$-algebra homomorphism when $h(2x) = 2h(x)$, $h(3x) = 3h(x)$ or $h(qx) = qh(x)$ for all $x \in A$. We moreover prove the Cauchy–Rassias stability of Lie $JC^*$-algebra homomorphisms in unital Lie $JC^*$-algebras, and of Lie $JC^*$-algebra derivations in unital Lie $JC^*$-algebras.

2. Homomorphisms between Lie $JC^*$-algebras

**Definition 2.1.** A $\mathbb{C}$-linear mapping $H : A \rightarrow B$ is called a Lie $JC^*$-algebra homomorphism if $H : A \rightarrow B$ satisfies

$$
H(x \circ y) = H(x) \circ H(y),
H([x, y]) = [H(x), H(y)],
H(x^*) = H(x)^*
$$

for all $x, y \in A$.

**Remark 2.1.** A $\mathbb{C}$-linear mapping $H : A \rightarrow B$ is a $C^*$-algebra homomorphism if and only if the mapping $H : A \rightarrow B$ is a Lie $JC^*$-algebra homomorphism. Assume that $H$ is a Lie $JC^*$-algebra homomorphism. Then

$$
H(xy) = H([x, y] + x \circ y) = H([x, y]) + H(x \circ y) = [H(x), H(y)] + H(x) \circ H(y) = H(x)H(y)
$$
for all \( x, y \in A \). So \( H \) is a \( C^* \)-algebra homomorphism.

Assume that \( H \) is a \( C^* \)-algebra homomorphism. Then

\[
H([x, y]) = H(\frac{xy - yx}{2}) = \frac{H(x)H(y) - H(y)H(x)}{2} = [H(x), H(y)],
\]

\[
H(x \circ y) = H(\frac{xy + yx}{2}) = \frac{H(x)H(y) + H(y)H(x)}{2} = H(x) \circ H(y)
\]

for all \( x, y \in A \). So \( H \) is a Lie \( JC^* \)-algebra homomorphism.

We are going to investigate Lie \( JC^* \)-algebra homomorphisms between Lie \( JC^* \)-algebras associated with the Cauchy functional equation.

Theorem 2.1. Let \( h : A \to B \) be a mapping satisfying \( h(0) = 0 \) and \( h(2^n u \circ y) = h(2^n u) \circ h(y) \) for all \( y \in A \), all \( u \in U(A) \) and \( n = 0, 1, 2, \cdots \), for which there exists a function \( \varphi : A^4 \to [0, \infty) \) such that

\[
\tilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty,
\]

\[
\|h(\mu x + \mu y + [z, w]) - \mu h(x) - \mu h(y) - [h(z), h(w)]\|
\leq \varphi(x, y, z, w),
\]

\[
\|h(2^n u^*) - h(2^n u^*)\| \leq \varphi(2^n u, 2^n u, 0, 0)
\]

for all \( \mu \in \mathbb{T} := \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} \), all \( u \in U(A) \), all \( x, y, z, w \in A \) and \( n = 0, 1, 2, \cdots \). Assume that \( (2.iv) \lim_{n \to \infty} h(2^n e) = e' \). Then the mapping \( h : A \to B \) is a Lie \( JC^* \)-algebra homomorphism.

Proof. Put \( z = w = 0 \) and \( \mu = 1 \in \mathbb{T} \) in (2.ii). It follows from Găvruta’s Theorem [1] that there exists a unique additive mapping \( H : A \to B \) such that

\[
\|h(x) - H(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x, 0, 0)
\]

for all \( x \in A \). The additive mapping \( H : A \to B \) is given by

\[
H(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x)
\]

for all \( x \in A \).

By the assumption, for each \( \mu \in \mathbb{T} \),

\[
\|h(2^n \mu x) - 2\mu h(2^n x)\| \leq \varphi(2^{n-1} x, 2^{n-1} x, 0, 0)
\]

for all \( x \in A \). And one can show that

\[
\|\mu h(2^n x) - 2\mu h(2^n x)\| \leq |\mu| \cdot \|h(2^n x) - 2h(2^n x)\| \leq \varphi(2^{n-1} x, 2^{n-1} x, 0, 0)
\]

for all \( \mu \in \mathbb{T} \) and all \( x \in A \). So

\[
\|h(2^n \mu x) - \mu h(2^n x)\| \leq \|h(2^n \mu x) - 2\mu h(2^n x)\| + \|2\mu h(2^{n-1} x) - \mu h(2^n x)\|
\]

\[
\leq \varphi(2^{n-1} x, 2^{n-1} x, 0, 0) + \varphi(2^{n-1} x, 2^{n-1} x, 0, 0)
\]
for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus $2^{-n}\|h(2^n \mu x) - \mu h(2^n x)\| \to 0$ as $n \to \infty$
for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence

$$H(\mu x) = \lim_{n \to \infty} \frac{h(2^n \mu x)}{2^n} = \lim_{n \to \infty} \frac{\mu h(2^n x)}{2^n} = \mu H(x)$$  \hspace{1cm} (2.2)

for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and $M$ an integer greater than $4|\lambda|$. Then

$$|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$ By [3, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $H(x) = H(3 \cdot \frac{1}{3} x) = 3H(\frac{1}{3} x)$ for all $x \in A$. So $H(\frac{1}{3} x) = \frac{1}{3} H(x)$ for all $x \in A$. Thus by (2.2)

$$H(\lambda x) = H\left(\frac{M}{3} \cdot 3 \frac{\lambda}{M} x\right) = M \cdot H\left(\frac{1}{3} \cdot 3 \frac{\lambda}{M} x\right) = \frac{M}{3} H(3 \frac{\lambda}{M} x)$$

$$= \frac{M}{3} H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3} (H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x))$$

$$= \frac{M}{3} (\mu_1 + \mu_2 + \mu_3) H(x) = \frac{M}{3} \cdot 3 \frac{\lambda}{M} H(x)$$

$$= \lambda H(x)$$

for all $x \in A$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C} \setminus \{0\}$ and all $x, y \in A$. And $H(0x) = 0 = 0H(x)$ for all $x \in A$. So the unique additive mapping $H : A \to B$ is a $\mathbb{C}$-linear mapping.

Since $h(2^n u \circ y) = h(2^n u) \circ h(\eta y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$,

$$H(u \circ y) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n u \circ y) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n u) \circ h(\eta y) = H(u) \circ h(\eta y)$$  \hspace{1cm} (2.3)

for all $y \in A$ and all $u \in U(A)$. By the additivity of $H$ and (2.3),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ h(2^n y)$$

for all $y \in A$ and all $u \in U(A)$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ h(2^n y) = H(u) \circ \frac{1}{2^n} h(2^n y)$$  \hspace{1cm} (2.4)

for all $y \in A$ and all $u \in U(A)$. Taking the limit in (2.4) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y)$$  \hspace{1cm} (2.5)

for all $y \in A$ and all $u \in U(A)$. Since $H$ is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [4, Theorem 4.1.7]), i.e.,

$$x = \sum_{j=1}^{m} \lambda_j u_j \; (\lambda_j \in \mathbb{C}, u_j \in U(A)),$$

$$H(x \circ y) = H\left(\sum_{j=1}^{m} \lambda_j u_j \circ y\right) = \sum_{j=1}^{m} \lambda_j H(u_j \circ y) = \sum_{j=1}^{m} \lambda_j H(u_j) \circ H(y)$$

$$= H\left(\sum_{j=1}^{m} \lambda_j u_j\right) \circ H(y) = H(x) \circ H(y)$$
for all \( x, y \in A \).

By (2.iv), (2.3) and (2.5),
\[
H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)
\]
for all \( y \in A \). So
\[
H(y) = h(y)
\]
for all \( y \in A \).

It follows from (2.1) that
\[
H(x) = \lim_{n \to \infty} \frac{h(2^n x)}{2^n}
\]
(2.6)
for all \( x \in A \). Let \( x = y = 0 \) in (2.ii). Then we get
\[
\|h([z, w]) - [h(z), h(w)]\| \leq \varphi(0, 0, z, w)
\]
for all \( z, w \in A \). So
\[
\frac{1}{2^n} \|h([2^n z, 2^n w]) - [h(2^n z), h(2^n w)]\| \leq \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w)
\]
\[
\leq \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w)
\]
(2.7)
for all \( z, w \in A \). By (2.i), (2.6), and (2.7),
\[
H([z, w]) = \lim_{n \to \infty} \frac{h(2^n [z, w])}{2^n} = \lim_{n \to \infty} \frac{h([2^n z, 2^n w])}{2^n}
\]
\[
= \lim_{n \to \infty} \frac{1}{2^n} [h(2^n z), h(2^n w)] = \lim_{n \to \infty} \frac{h(2^n z)}{2^n}, \frac{h(2^n w)}{2^n}
\]
\[
= [H(z), H(w)]
\]
for all \( z, w \in A \).

By (2.i) and (2.iii), we get
\[
H(u^*) = \lim_{n \to \infty} \frac{h(2^n u^*)}{2^n} = \lim_{n \to \infty} \frac{h(2^n u)^*}{2^n} = (\lim_{n \to \infty} \frac{h(2^n u)}{2^n})^*
\]
\[
= H(u)^*
\]
for all \( u \in U(A) \). Since \( H : A \to B \) is \( \mathbb{C} \)-linear and each \( x \in A \) is a finite linear combination of unitary elements, i.e., \( x = \sum_{j=1}^{m} \lambda_j u_j \) (\( \lambda_j \in \mathbb{C}, u_j \in U(A) \)),
\[
H(x^*) = H(\sum_{j=1}^{m} \overline{\lambda_j} u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j)^*
\]
\[
= (\sum_{j=1}^{m} \lambda_j H(u_j))^* = H(\sum_{j=1}^{m} \lambda_j u_j)^* = H(x)^*
\]
for all \( x \in A \).

Therefore, the mapping \( h : A \to B \) is a Lie \( JC^* \)-algebra homomorphism, as desired.
Corollary 2.2. Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that
\[
\|h(\mu x + \mu y + [z, w]) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \\
\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p),
\]
\[
\|h(2^n u^*) - h(2^n u)^*\| \leq 2 \cdot 2^n \theta
\]
for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, all $x, y, z, w \in A$ and $n = 0, 1, 2, \cdots$. Assume that $\lim_{n \to \infty} h(2^n, e) = e'$. Then the mapping $h : A \to B$ is a Lie $JC^*$-algebra homomorphism.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$, and apply Theorem 2.1.

Theorem 2.3. Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $y \in A$, all $u \in U(A)$ and $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : A^4 \to [0, \infty)$ satisfying (2.i), (2.iii) and (2.iv) such that
\[
\|h(\mu x + \mu y + [z, w]) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \leq \varphi(x, y, z, w) \tag{2.v}
\]
for $\mu = 1, i$, and all $x, y, z, w \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $h : A \to B$ is a Lie $JC^*$-algebra homomorphism.

Proof. Put $z = w = 0$ and $\mu = 1$ in (2.v). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $H : A \to B$ satisfying (2.0). The additive mapping $H : A \to B$ is given by
\[
H(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x)
\]
for all $x \in A$. By the same reasoning as in the proof of [11, Theorem], the additive mapping $H : A \to B$ is $\mathbb{R}$-linear.

Put $y = z = w = 0$ and $\mu = i$ in (2.v). By the same method as in the proof of Theorem 2.1, one can obtain that
\[
H(ix) = \lim_{n \to \infty} \frac{h(2^n ix)}{2^n} = \lim_{n \to \infty} \frac{ih(2^n x)}{2^n} = iH(x)
\]
for all $x \in A$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So
\[
H(\lambda x) = H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x)
\]
for all $\lambda \in \mathbb{C}$ and all $x \in A$. So
\[
H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)
\]
for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in A$. Hence the additive mapping $H : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1.
Theorem 2.4. Let \( h : A \to B \) be a mapping satisfying \( h(2x) = 2h(x) \) for all \( x \in A \) for which there exists a function \( \varphi : A^4 \to [0, \infty) \) satisfying (2.i), (2.ii), (2.iii) and (2.iv) such that

\[
\|h(2^n u \circ y) - h(2^n u) \circ h(y)\| \leq \varphi(u, y, 0, 0) \tag{2.vi}
\]

for all \( y \in A \), all \( u \in U(A) \) and \( n = 0, 1, 2, \cdots \). Then the mapping \( h : A \to B \) is a Lie \( JC^* \)-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique \( \mathbb{C} \)-linear mapping \( H : A \to B \) satisfying (2.0).

By (2.vi) and the assumption that \( h(2x) = 2h(x) \) for all \( x \in A \),

\[
\|h(2^n u \circ y) - h(2^n u) \circ h(y)\| = \frac{1}{4^m}\|h(2^m 2^n u \circ 2^m y) - h(2^m 2^n u) \circ h(2^m y)\|
\]

\[
\leq \frac{1}{4^m}\varphi(2^m u, 2^m y, 0, 0) \leq \frac{1}{2^m}\varphi(2^m u, 2^m y, 0, 0),
\]

which tends to zero as \( m \to \infty \) by (2.i). So

\[
h(2^n u \circ y) = h(2^n u) \circ h(y)
\]

for all \( y \in A \), all \( u \in U(A) \) and \( n = 0, 1, 2, \cdots \). But by (2.1),

\[
H(x) = \lim_{n \to \infty} \frac{1}{2^n}h(2^n x) = h(x)
\]

for all \( x \in A \).

The rest of the proof is the same as in the proof of Theorem 2.1. \( \square \)

Now we are going to investigate Lie \( JC^* \)-algebra homomorphisms between Lie \( JC^* \)-algebras associated with the Jensen functional equation.

Theorem 2.5. Let \( h : A \to B \) be a mapping satisfying \( h(0) = 0 \) and \( h(3^n u \circ y) = h(3^n u) \circ h(y) \) for all \( y \in A \), all \( u \in U(A) \) and \( n = 0, 1, 2, \cdots \), for which there exists a function \( \varphi : (A \setminus \{0\})^4 \to [0, \infty) \) such that

\[
\bar{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y, 3^j z, 3^j w) < \infty, \tag{2.viii}
\]

\[
\|2h(\mu x + \mu y + [z, w]) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \leq \varphi(x, y, z, w), \tag{2.viii}
\]

\[
\|h(3^n u^*) - h(3^n u)^*\| \leq \varphi(3^n u, 3^n u, 0, 0) \tag{2.ix}
\]

for all \( \mu \in \mathbb{T}^1 \), all \( u \in U(A) \), all \( x, y, z, w \in A \) and \( n = 0, 1, 2, \cdots \). Assume that \( \lim_{n \to \infty} \frac{h(3^n u)}{3^n} = e' \). Then the mapping \( h : A \to B \) is a Lie \( JC^* \)-algebra homomorphism.
Put \( z = w = 0 \) and \( \mu = 1 \in \mathbb{T}^1 \) in (2.viii). It follows from Jun and Lee’s Theorem [2, Theorem 1] that there exists a unique additive mapping \( H : A \to B \) such that

\[
\|h(x) - H(x)\| \leq \frac{1}{3}(\tilde{\varphi}(x, -x, 0, 0) + \tilde{\varphi}(-x, 3x, 0, 0))
\]

for all \( x \in A \setminus \{0\} \). The additive mapping \( H : A \to B \) is given by

\[
H(x) = \lim_{n \to \infty} \frac{1}{3^n} h(3^n x)
\]

for all \( x \in A \).

By the assumption, for each \( \mu \in \mathbb{T}^1 \),

\[
\|2h(3^n \mu x) - \mu h(2 \cdot 3^{n-1} x) - \mu h(4 \cdot 3^{n-1} x)\| \leq \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0)
\]

for all \( x \in A \setminus \{0\} \). And one can show that

\[
\begin{align*}
\|\mu h(2 \cdot 3^{n-1} x) + \mu h(4 \cdot 3^{n-1} x) - 2\mu h(3^n x)\| & \\
& \leq |\mu| \cdot \|h(2 \cdot 3^{n-1} x) + h(4 \cdot 3^{n-1} x) - 2h(3^n x)\| \\
& \leq \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0)
\end{align*}
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \setminus \{0\} \). So

\[
\begin{align*}
\|h(3^n \mu x) - \mu h(3^n x)\| & = \|h(3^n \mu x) - \frac{1}{2} \mu h(2 \cdot 3^{n-1} x) - \frac{1}{2} \mu h(4 \cdot 3^{n-1} x) \\
& + \frac{1}{2} \mu h(2 \cdot 3^{n-1} x) + \frac{1}{2} \mu h(4 \cdot 3^{n-1} x) - \mu h(3^n x)\| \\
& \leq \frac{1}{2} \|2h(3^n \mu x) - \mu h(2 \cdot 3^{n-1} x) - \mu h(4 \cdot 3^{n-1} x)\| \\
& + \frac{1}{2} \|\mu h(2 \cdot 3^{n-1} x) + \mu h(4 \cdot 3^{n-1} x) - 2\mu h(3^n x)\| \\
& \leq \frac{2}{2} \varphi(2 \cdot 3^{n-1} x, 4 \cdot 3^{n-1} x, 0, 0)
\end{align*}
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \setminus \{0\} \). Thus \( 3^{-n} \|h(3^n \mu x) - \mu h(3^n x)\| \to 0 \) as \( n \to \infty \) for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \setminus \{0\} \). Hence

\[
H(\mu x) = \lim_{n \to \infty} \frac{h(3^n \mu x)}{3^n} = \lim_{n \to \infty} \frac{\mu h(3^n x)}{3^n} = \mu H(x)
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \setminus \{0\} \).

By the same reasoning as in the proof of Theorem 2.1, the unique additive mapping \( H : A \to B \) is a \( \mathbb{C} \)-linear mapping.

By a similar method to the proof of Theorem 2.1, one can show that the mapping \( h : A \to B \) is a Lie \( JC^* \)-algebra homomorphism.
Corollary 2.6. Let \( h : A \to B \) be a mapping satisfying \( h(0) = 0 \) and \( h(3^n u \circ y) = h(3^n u) \circ h(y) \) for all \( y \in A \), all \( u \in U(A) \) and \( n = 0, 1, 2, \ldots \), for which there exist constants \( \theta \geq 0 \) and \( p \in [0, 1) \) such that 

\[
\|2h\left(\frac{\mu x + \mu y + [z, w]}{2}\right) - \mu h(x) - \mu h(y) - [h(z), h(w)]\| \\
\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p),
\]

\[
\|h(3^n u^*) - h(3^n u)^*\| \leq 2 \cdot 3^n \|h\| \theta
\]

for all \( \mu \in T^1 \), all \( u \in U(A) \), all \( x, y, z, w \in A \setminus \{0\} \) and \( n = 0, 1, 2, \ldots \). Assume that \( \lim_{n \to \infty} \frac{h(3^n u)}{3^n} = e' \). Then the mapping \( h : A \to B \) is a Lie JC* -algebra homomorphism.

Proof. Define \( \varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \), and apply Theorem 2.5.

One can obtain similar results to Theorems 2.3 and 2.4 for the Jensen functional equation.

Finally, we are going to investigate Lie JC* -algebra homomorphisms between Lie JC* -algebras associated with the Trif functional equation.

Theorem 2.7. Let \( h : A \to B \) be a mapping satisfying \( h(0) = 0 \) and \( h(q^n u \circ y) = h(q^n u) \circ h(y) \) for all \( y \in A \), all \( u \in U(A) \) and \( n = 0, 1, 2, \ldots \), for which there exists a function \( \varphi : A^{d+2} \to [0, \infty) \) such that 

\[
\bar{\varphi}(x_1, \ldots, x_d, z, w) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_d, q^j z, q^j w) < \infty, \quad (2.x)
\]

\[
\|d \cdot d^{-2} C_{l-2} h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d}\right) + \frac{[z, w]}{d \cdot d^{-2} C_{l-2}} + d^{-2} C_{l-1} \sum_{j=1}^{d} \mu h(x_j)
\]

\[-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(x_{j_1} + \cdots + x_{j_l}) - [h(z), h(w)]\|

\leq \varphi(x_1, \ldots, x_d, z, w), \quad (2.xi)

\[
\|h(q^n u^*) - h(q^n u)^*\| \leq \varphi(q^n u^*, \ldots, q^n u, 0, 0, 0) \quad (2.xii)
\]

for all \( \mu \in T^1 \), all \( u \in U(A) \), all \( x_1, \ldots, x_d, z, w \in A \) and \( n = 0, 1, 2, \ldots \). Assume that \( \lim_{n \to \infty} \frac{h(q^n u)}{q^n} = e' \). Then the mapping \( h : A \to B \) is a Lie JC* -algebra homomorphism.

Proof. Put \( z = w = 0 \) and \( \mu = 1 \in T^1 \) in (2.xi). It follows from Trif’s Theorem [17, Theorem 3.1] that there exists a unique additive mapping \( H : A \to B \) such that 

\[
\|h(x) - H(x)\| \leq \frac{1}{l \cdot d^{-1} C_{l-1}} \bar{\varphi}(q^l x, r x, \cdots, r x, 0, 0) \quad \text{d times}
\]

for all \( x \in A \). The additive mapping \( H : A \to B \) is given by 

\[
H(x) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n x)
\]

for all \( x \in A \).
Theorem 2.7.

\[ \lim_{n \to \infty} \mu_{x, \ldots, x, 0, 0} \]

for all \( n \in \mathbb{N} \). So

\[ q^{-n} \| d_{-2} C_{l-2} (h(\mu q^n x) - \mu h(q^n x)) \| \leq q^{-n} \varphi(q^n x, \ldots, q^n x, 0, 0) \]

for all \( x \in A \). By (2.x),

\[ q^{-n} \| d_{-2} C_{l-2} (h(\mu q^n x) - \mu h(q^n x)) \| \to 0 \]

as \( n \to \infty \) for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \). Thus

\[ q^{-n} \| h(\mu q^n x) - \mu h(q^n x) \| \to 0 \]

as \( n \to \infty \) for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \). Hence

\[ H(\mu x) = \lim_{n \to \infty} \frac{h(q^n \mu x)}{q^n} = \lim_{n \to \infty} \frac{\mu h(q^n x)}{q^n} = \mu H(x) \]

for all \( \mu \in \mathbb{T}^1 \) and all \( x \in A \).

By the same reasoning as in the proof of Theorem 2.1, the unique additive mapping \( H : A \to B \) is a \( \mathbb{C} \)-linear mapping.

By a similar method to the proof of Theorem 2.1, one can show that the mapping \( h : A \to B \) is a Lie \( JC^* \)-algebra homomorphism.

Corollary 2.8. Let \( h : A \to B \) be a mapping satisfying \( h(0) = 0 \) and \( h(q^n u \circ y) = h(q^n u) \circ h(y) \) for all \( y \in A \), all \( u \in U(A) \) and \( n = 0, 1, 2, \ldots \), for which there exist constants \( \theta \geq 0 \) and \( p \in [0, 1) \) such that

\[ \| d_{-2} C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z, w]}{d_{-2} C_{l-2}}) + d_{-2} C_{l-1} \sum_{j=1}^d \mu h(x_j) \] \[ - \sum_{1 \leq j_1 < \ldots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - [h(z), h(w)] \| \] \[ \leq \theta(\sum_{j=1}^d \| x_j \|^p + \| z \|^p + \| w \|^p) \]

\[ \| h(q^n u^*) - h(q^n u)^* \| \leq dq^{np} \theta \]

for all \( \mu \in \mathbb{T}^1 \), all \( u \in U(A) \), all \( x_1, \ldots, x_d, z, w \in A \) and \( n = 0, 1, 2, \ldots \). Assume that \( \lim_{n \to \infty} \frac{h(q^n x)}{q^n} = e' \). Then the mapping \( h : A \to B \) is a Lie \( JC^* \)-algebra homomorphism.

Proof. Define \( \varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^d \| x_j \|^p + \| z \|^p + \| w \|^p) \), and apply Theorem 2.7.

One can obtain similar results to Theorems 2.3 and 2.4 for the Trif functional equation.
3. Stability of Lie $JC^*$-algebra homomorphisms in Lie $JC^*$-algebras

We are going to show the Cauchy–Rassias stability of Lie $JC^*$-algebra homomorphisms in Lie $JC^*$-algebras associated with the Cauchy functional equation.

**Theorem 3.1.** Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^6 \to [0, \infty)$ such that

$$
\bar{\varphi}(x, y, z, w, a, b) := \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y, 2^j z, 2^j w, 2^j a, 2^j b) < \infty, \quad (3.i)
$$

$$
\|h(\mu x + \mu y + [z, w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| \\
\leq \varphi(x, y, z, w, a, b), \quad (3.ii)
$$

$$
\|h(2^n u^*) - h(2^n u)^*\| \leq \varphi(2^n u, 2^n u, 0, 0, 0) \quad (3.iii)
$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A$. Then there exists a unique Lie $JC^*$-algebra homomorphism $H : A \to B$ such that

$$
\|h(x) - H(x)\| \leq \frac{1}{2} \bar{\varphi}(x, 0, 0, 0, 0) \quad (3.iv)
$$

for all $x \in A$.

**Proof.** Put $z = w = a = b = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (3.ii). It follows from Gavruta’s Theorem [1] that there exists a unique additive mapping $H : A \to B$ satisfying (3.iv). The additive mapping $H : A \to B$ is given by

$$
H(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x)
$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. $\blacksquare$

**Corollary 3.2.** Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$
\|h(\mu x + \mu y + [z, w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| \\
\leq \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p),
$$

$$
\|h(2^n u^*) - h(2^n u)^*\| \leq 2 \cdot 2^p \theta
$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A$. Then there exists a unique Lie $JC^*$-algebra homomorphism $H : A \to B$ such that

$$
\|h(x) - H(x)\| \leq \frac{2\theta}{2 - 2^p} ||x||^p
$$

for all $x \in A$.

**Proof.** Define $\varphi(x, y, z, w, a, b) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p)$, and apply Theorem 3.1. $\blacksquare$
Theorem 3.3. Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^6 \to [0, \infty)$ satisfying (3.i) and (3.iii) such that

$$
\|h(\mu x + \mu y + [z, w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| 
\leq \varphi(x, y, z, w, a, b)
$$

for $\mu = 1, i$, and all $x, y, z, w, a, b \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique Lie JC$^*$-algebra homomorphism $H : A \to B$ satisfying (3.iv).

Proof. The proof is similar to the proof of Theorem 2.3. ■

We are going to show the Cauchy–Rassias stability of Lie JC$^*$-algebra homomorphisms in Lie JC$^*$-algebras associated with the Jensen functional equation.

Theorem 3.4. Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : (A \setminus \{0\})^6 \to [0, \infty)$ such that

$$
\tilde{\varphi}(x, y, z, w, a, b) = \sum_{j=0}^{\infty} 3^{-j}\varphi(3^jx, 3^jy, 3^jz, 3^jw, 3^ja, 3^jb) < \infty, (3.v)
$$

$$
\|2h(\frac{\mu x + \mu y + [z, w] + a \circ b}{2}) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| 
\leq \varphi(x, y, z, w, a, b), \quad (3.vi)
$$

$$
\|h(3^n u^*) - h(3^n u)^*\| \leq \varphi(3^n u, 3^n u, 0, 0, 0), \quad (3.vii)
$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, 2, \cdots$, and all $x, y, z, w, a, b \in A \setminus \{0\}$. Then there exists a unique Lie JC$^*$-algebra homomorphism $H : A \to B$ such that

$$
\|h(x) - H(x)\| \leq \frac{1}{3}(\tilde{\varphi}(x, -x, 0, 0, 0, 0) + \tilde{\varphi}(-x, 3x, 0, 0, 0, 0)) \quad (3.viii)
$$

for all $x \in A \setminus \{0\}$.

Proof. Put $z = w = a = b = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (3.vi). It follows from Jun and Lee’s Theorem [2, Theorem 1] that there exists a unique additive mapping $H : A \to B$ satisfying (3.viii). The additive mapping $H : A \to B$ is given by

$$
H(x) = \lim_{n \to \infty} \frac{1}{3^n} h(3^n x)
$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.5. ■

Corollary 3.5. Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$
\|2h(\frac{\mu x + \mu y + [z, w] + a \circ b}{2}) - \mu h(x) - \mu h(y) - [h(z), h(w)] - h(a) \circ h(b)\| 
\leq \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p),
$$

$$
\|h(3^n u^*) - h(3^n u)^*\| \leq 2 \cdot 3^{np}\theta
$$
for all \( \mu \in \mathbb{T}^1 \), all \( u \in U(A) \), \( n = 0, 1, 2, \cdots \), and all \( x, y, z, w, a, b \in A \setminus \{0\} \).

Then there exists a unique Lie \( JC^* \)-algebra homomorphism \( H : A \to B \) such that

\[
\|h(x) - H(x)\| \leq \frac{3 + 3p}{3 - 3p} \theta \|x\|^p
\]

for all \( x \in A \setminus \{0\} \).

**Proof.** Define \( \varphi(x, y, z, w, a, b) = \theta(||x||^p + ||y||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p) \), and apply Theorem 3.4.

One can obtain a similar result to Theorem 3.3 for the Jensen functional equation.

Now we are going to show the Cauchy–Rassias stability of Lie \( JC^* \)-algebra homomorphisms in Lie \( JC^* \)-algebras associated with the Trif functional equation.

**Theorem 3.6.** Let \( h : A \to B \) be a mapping with \( h(0) = 0 \) for which there exists a function \( \varphi : A^{d+4} \to [0, \infty) \) such that

\[
\bar{\varphi}(x_1, \cdots, x_d, z, w, a, b) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \cdots, q^j x_d, q^j z, q^j w, q^j a, q^j b)
\]

\[
< \infty,
\]

\[
\|d_{d-2}C_{l-2}h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[z, w] + a \circ b}{d_{d-2}C_{l-2}}\right) + d_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)
\]

\[
-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - [h(z), h(w)] - h(a) \circ h(b)
\]

\[
\leq \varphi(x_1, \cdots, x_d, z, w, a, b),
\]

\[
\|h(q^n u^*) - h(q^n u^*)\| \leq \varphi(q^n u, \cdots, q^n u, 0, 0, 0, 0)
\]

\[
(3.x)
\]

for all \( \mu \in \mathbb{T}^1 \), all \( u \in U(A) \), \( n = 0, 1, 2, \cdots \), and all \( x_1, \cdots, x_d, z, w, a, b \in A \).

Then there exists a unique Lie \( JC^* \)-algebra homomorphism \( H : A \to B \) such that

\[
\|h(x) - H(x)\| \leq \frac{1}{l \cdot d_{d-1}C_{l-1}} \bar{\varphi}(q^x, r^x, \cdots, r^x, 0, 0, 0, 0)
\]

\[
(3.xii)
\]

for all \( x \in A \).

**Proof.** Put \( z = w = a = b = 0 \) and \( \mu = 1 \in \mathbb{T}^1 \) in (3.x). It follows from Trif’s Theorem [17, Theorem 3.1] that there exists a unique additive mapping \( H : A \to B \) satisfying (3.xii). The additive mapping \( H : A \to B \) is given by

\[
H(x) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n x)
\]

for all \( x \in A \).

The rest of the proof is similar to the proof of Theorem 2.7.  \( \blacksquare \)
Corollary 3.7. Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

\[
\|d_{d-2}C_{l-2}h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d}\right) + \frac{[z, w] + a \circ b}{d_{d-2}C_{l-2}} + d_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)
- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(x_{j_1} + \cdots + x_{j_l}) - [h(z), h(w)] - h(a) \circ h(b)\| \\
\leq \theta \left(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p\right),
\]

\[
\|h(q^n u^\ast) - h(q^n u)\|^p \leq dq^n p \theta
\]

for all $\mu \in T^1$, all $u \in U(A)$, $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d, z, w, a, b \in A$. Then there exists a unique Lie $JC^\ast$-algebra homomorphism $H : A \to B$ such that

\[
\|h(x) - H(x)\| \leq \frac{q^{1-p}(q^p + (d-1)p^p) \theta}{l d_{d-1}C_{l-1}(q^{1-p} - 1)} ||x||^p
\]

for all $x \in A$.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w, a, b) = \theta \left(\sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p + ||a||^p + ||b||^p\right)$, and apply Theorem 3.6.

One can obtain a similar result to Theorem 3.3 for the Trif functional equation.

4. Stability of Lie $JC^\ast$-algebra derivations in Lie $JC^\ast$-algebras

Definition 4.1. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a Lie $JC^\ast$-algebra derivation if $D : A \to A$ satisfies

\[
D(x \circ y) = (Dx) \circ y + x \circ (Dy),
\]

\[
D([x, y]) = [Dx, y] + [x, Dy],
\]

\[
D(x^\ast) = D(x)^\ast
\]

for all $x, y \in A$.

Remark 4.1. A $\mathbb{C}$-linear mapping $D : A \to A$ is a $C^\ast$-algebra derivation if and only if the mapping $D : A \to A$ is a Lie $JC^\ast$-algebra derivation.

Assume that $D$ is a Lie $JC^\ast$-algebra derivation. Then

\[
D(xy) = D([x, y] + x \circ y) = D([x, y]) + D(x \circ y)
= [Dx, y] + [x, Dy] + (Dx) \circ y + x \circ (Dy) = (Dx)y + x(Dy)
\]

for all $x, y \in A$. So $D$ is a $C^\ast$-algebra derivation.
Assume that $D$ is a $C^*$-algebra derivation. Then

$$D([x, y]) = D\left(\frac{xy - yx}{2}\right) = \frac{(Dx)y + x(Dy) - (Dy)x - y(Dx)}{2}$$

$$= [Dx, y] + [x, Dy],$$

$$D(x \circ y) = D\left(\frac{xy + yx}{2}\right) = \frac{(Dx)y + x(Dy) + (Dy)x + y(Dx)}{2}$$

$$= (Dx) \circ y + x \circ (Dy)$$

for all $x, y \in A$. So $H$ is a Lie $JC^*$-algebra derivation.

We are going to show the Cauchy–Rassias stability of Lie $JC^*$-algebra derivations in Lie $JC^*$-algebras associated with the Cauchy functional equation.

**Theorem 4.1.** Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^6 \to [0, \infty)$ satisfying (3.1) and (3.3) such that

$$\|h(\mu x + \mu y + [z, w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)] - h(a) \circ b - a \circ h(b)\| \leq \varphi(x, y, z, w, a, b) \quad (4.i)$$

for all $\mu \in T^1$ and all $x, y, z, w, a, b \in A$. Then there exists a unique Lie $JC^*$-algebra derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{1}{2}\tilde{\varphi}(x, x, 0, 0, 0, 0) \quad (4.ii)$$

for all $x \in A$.

**Proof.** Put $z = w = a = b = 0$ in (4.i). By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (4.ii). The $\mathbb{C}$-linear mapping $D : A \to A$ is given by

$$D(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x) \quad (4.1)$$

for all $x \in A$.

It follows from (4.1) that

$$D(x) = \lim_{n \to \infty} \frac{h(2^n x)}{2^{2n}} \quad (4.2)$$

for all $x \in A$. Let $x = y = a = b = 0$ in (4.i). Then we get

$$\|h([z, w]) - [h(z), w] - [z, h(w)]\| \leq \varphi(0, 0, z, w, 0, 0)$$

for all $z, w \in A$. Since

$$\frac{1}{2^{2n}} \varphi(0, 0, 2^n z, 2^n w, 0, 0) \leq \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w, 0, 0),$$
there exists a unique Lie \(JC\) for all \(\mu\) for all \(z, w\) for all \(\theta\) derivation satisfying (4.ii), as desired.

Let \(\Phi\) for all \(\mu x\) and apply Theorem 4.1.

Theorem 4.3. Let \(h : A \to A\) be a mapping with \(h(0) = 0\) for which there exists a function \(\varphi : A^6 \to [0, \infty)\) satisfying (3.i) and (3.iii) such that

\[
\|h(\mu x + \mu y + [z, w] + a \circ b) - \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)]
- h(a) \circ b - a \circ h(b)\| 
\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p),
\]

for all \(\mu \in \mathbb{T}^1\), all \(u \in U(A)\), \(n = 0, 1, 2, \ldots\), and all \(x, y, z, w, a, b \in A\). Then there exists a unique Lie \(JC^*\)-algebra derivation \(D : A \to A\) such that

\[
\|h(x) - D(x)\| \leq \frac{2\theta}{2 - 2\theta}\|x\|^p
\]

for all \(x \in A\).

Proof. Define \(\varphi(x, y, z, w, a, b) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p)\), and apply Theorem 4.1.
for \( \mu = 1, i \), and all \( x, y, z, w, a, b \in A \). If \( h(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then there exists a unique Lie \( JC^* \)-algebra derivation \( D : A \rightarrow A \) satisfying (4.ii).

**Proof.** By the same reasoning as in the proof of Theorem 2.3, there exists a unique \( \mathbb{C} \)-linear mapping \( D : A \rightarrow A \) satisfying (4.ii).

The rest of the proof is the same as in the proofs of Theorems 2.1, 3.1 and 4.1.

We are going to show the Cauchy–Rassias stability of Lie \( JC^* \)-algebra derivations in Lie \( JC^* \)-algebras associated with the Jensen functional equation.

**Theorem 4.4.** Let \( h : A \rightarrow A \) be a mapping with \( h(0) = 0 \) for which there exists a function \( \varphi : (A \setminus \{0\})^6 \rightarrow [0, \infty) \) satisfying (3.v) and (3.vii) such that

\[
\|2h\left(\frac{\mu x + \mu y + [z, w] + a \circ b}{2}\right) - \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)] - h(a) \circ b - a \circ h(b)\| \leq \varphi(x, y, z, w, a, b)
\]

for all \( \mu \in T^1 \) and all \( x, y, z, w, a, b \in A \setminus \{0\} \). Then there exists a unique Lie \( JC^* \)-algebra derivation \( D : A \rightarrow A \) such that

\[
\|h(x) - D(x)\| \leq \frac{1}{3}\left(\tilde{\varphi}(x, -x, 0, 0, 0, 0) + \tilde{\varphi}(-x, 3x, 0, 0, 0, 0)\right)
\]

for all \( x \in A \setminus \{0\} \).

**Proof.** Put \( z = w = a = b = 0 \) in (4.iii). By the same reasoning as in the proof of Theorem 2.5, there exists a unique \( \mathbb{C} \)-linear involutive mapping \( D : A \rightarrow A \) satisfying (4.iv). The \( \mathbb{C} \)-linear mapping \( D : A \rightarrow A \) is given by

\[
D(x) = \lim_{n \to \infty} \frac{1}{3^n} h(3^n x)
\]

for all \( x \in A \).

It follows from (4.4) that

\[
D(x) = \lim_{n \to \infty} \frac{h(3^n x)}{3^n}
\]

for all \( x \in A \). Let \( x = y = a = b = 0 \) in (4.iii). Then we get

\[
\|2h\left(\frac{[z, w]}{2}\right) - [h(z), w] - [z, h(w)]\| \leq \varphi(0, 0, z, w, 0, 0)
\]

for all \( z, w \in A \). Since

\[
\frac{1}{3^{2n}} \varphi(0, 0, 3^n z, 3^n w, 0, 0) \leq \frac{1}{3^n} \varphi(0, 0, 3^n z, 3^n w, 0, 0),
\]

\[
\frac{1}{3^{2n}} \|2h\left(\frac{1}{2}[3^n z, 3^n w]\right) - [h(3^n z), 3^n w] - [3^n z, h(3^n w)]\| \leq \frac{1}{3^{2n}} \varphi(0, 0, 3^n z, 3^n w, 0, 0)
\]

\[
\leq \frac{1}{3^n} \varphi(0, 0, 3^n z, 3^n w, 0, 0)
\]

(4.6)
for all \( z, w \in A \). By (3.v), (4.5), and (4.6),
\[
2D\left(\frac{[z, w]}{2}\right) = \lim_{n \to \infty} \frac{2h\left(\frac{3^{2n}}{2} [z, w]\right)}{3^{2n}} = \lim_{n \to \infty} \frac{2h\left(\frac{1}{2} 3^n z, 3^n w\right)}{3^{2n}}
\]
\[
= \lim_{n \to \infty} \left( \frac{h(3^n z)}{3^n}, \frac{h(3^n w)}{3^n} \right)
\]
\[
= [D(z), w] + [z, D(w)]
\]
for all \( z, w \in A \). But since \( D \) is \( \mathbb{C} \)-linear,
\[
D([z, w]) = 2D\left(\frac{[z, w]}{2}\right) = [D(z), w] + [z, D(w)]
\]
for all \( z, w \in A \).

Similarly, one can obtain that
\[
2D\left(\frac{a \circ b}{2}\right) = \lim_{n \to \infty} \frac{2h\left(\frac{3^{2n}}{2} a \circ b\right)}{3^{2n}} = \lim_{n \to \infty} \frac{2h\left(\frac{1}{2} 3^n a \circ (3^n b)\right)}{3^{2n}}
\]
\[
= \lim_{n \to \infty} \left( \frac{h(3^n a)}{3^n} \circ (\frac{3^n b}{3^n}) + \frac{3^n a}{3^n} \circ \left( \frac{h(3^n b)}{3^n}\right) \right)
\]
\[
= (Da) \circ b + a \circ (Db)
\]
for all \( a, b \in A \). So
\[
D(a \circ b) = 2D\left(\frac{a \circ b}{2}\right) = (Da) \circ b + a \circ (Db)
\]
for all \( a, b \in A \). Hence the \( \mathbb{C} \)-linear mapping \( D : A \to A \) is a Lie \( JC^* \)-algebra derivation satisfying (4.iv), as desired.

**Corollary 4.5.** Let \( h : A \to A \) be a mapping with \( h(0) = 0 \) for which there exist constants \( \theta \geq 0 \) and \( p \in [0, 1) \) such that
\[
\|2h\left(\frac{\mu x + \mu y + [z, w] + a \circ b}{2}\right) - \mu h(x) - \mu h(y) - [h(z), w] - [z, h(w)]
\]
\[
- h(a) \circ b - a \circ h(b) \| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p),
\]
\[
\|h(3^n u^*) - h(3^n u)^*\| \leq 2 \cdot 3^{np} \theta
\]
for all \( \mu \in \mathbb{T}^1 \), all \( u \in U(A) \), \( n = 0, 1, 2, \ldots \), and all \( x, y, z, w, a, b \in A \setminus \{0\} \). Then there exists a unique Lie \( JC^* \)-algebra derivation \( D : A \to A \) such that
\[
\|h(x) - D(x)\| \leq \frac{3 + 3^p}{3 - 3^p} \theta \|x\|^p
\]
for all \( x \in A \setminus \{0\} \).

**Proof.** Define \( \varphi(x, y, z, w, a, b) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p), \)
and apply Theorem 4.4.

One can obtain a similar result to Theorem 4.3 for the Jensen functional equation.

Finally, we are going to show the Cauchy–Rassias stability of Lie \( JC^* \)-algebra derivations in Lie \( JC^* \)-algebras associated with the Trif functional equation.
**Theorem 4.6.** Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^{d+4} \to [0, \infty)$ satisfying (3.ix) and (3.xi) such that

$$\|d_{d-2}C_{l-2}h\left(\sum_{j=1}^{d} \left[\frac{\mu x_1 + \cdots + \mu x_d}{d} \right] + \frac{[z, w] + a \circ b}{d_{d-2}C_{l-2}}\right) + d_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)$$

$$-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\left(\sum_{j=1}^{d} x_{j_i}ight) - [h(z), w] - [z, h(w)]$$

(4.v)

for all $\mu \in T^1$ and all $x_1, \ldots, x_d, z, w, a, b \in A$. Then there exists a unique Lie $JC^*$-algebra derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{1}{l \cdot d_{d-2}C_{l-1}} \varphi(qx, rx, \cdots, rx, 0, 0, 0, 0)$$

(4.vi)

for all $x \in A$.

**Proof.** Put $z = w = a = b = 0$ in (4.v). By the same reasoning as in the proof of Theorem 2.7, there exists a unique $\mathbb{C}$-linear involutive mapping $D : A \to A$ satisfying (4.vi). The $\mathbb{C}$-linear mapping $D : A \to A$ is given by

$$D(x) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n x)$$

(4.7)

for all $x \in A$.

It follows from (4.7) that

$$D(x) = \lim_{n \to \infty} \frac{h(q^{2n} x)}{q^{2n}}$$

(4.8)

for all $x \in A$. Let $x_1 = \cdots = x_d = a = b = 0$ in (4.v). Then we get

$$\|d_{d-2}C_{l-2}h\left(\frac{[z, w]}{d_{d-2}C_{l-2}}\right) - [h(z), w] - [z, h(w)]\| \leq \varphi(0, \cdots, 0, z, w, 0, 0)$$

for all $z, w \in A$. Since

$$\frac{1}{q^{2n}} \varphi(0, \cdots, 0, q^n z, q^n w, 0, 0) \leq \frac{1}{q^n} \varphi(0, \cdots, 0, q^n z, q^n w, 0, 0),$$

$$\frac{1}{q^{2n}} \|d_{d-2}C_{l-2}h\left(\frac{1}{d_{d-2}C_{l-2}} [q^n z, q^n w]\right) - [h(q^n z), q^n w] - [q^n z, h(q^n w)]\|$$

$$\leq \frac{1}{q^{2n}} \varphi(0, \cdots, 0, q^n z, q^n w, 0, 0) \leq \frac{1}{q^n} \varphi(0, \cdots, 0, q^n z, q^n w, 0, 0)$$

(4.9)
for all $z, w \in A$. By (3.1x), (4.8), and (4.9),

$$d_{d-2}C_{l-2}D\left(\frac{[z, w]}{d_{d-2}C_{l-2}}\right) = \lim_{n \to \infty} \frac{d_{d-2}C_{l-2}h\left(\frac{q^n z, q^n w}{q^{2n}}\right)}{q^{2n}}$$

$$= \lim_{n \to \infty} \frac{d_{d-2}C_{l-2}h\left(\frac{1}{d_{d-2}C_{l-2}}[q^n z, q^n w]\right)}{q^{2n}} = \lim_{n \to \infty} \left(\left[\frac{h(q^n z)}{q^n}, \frac{q^n w}{q^n}\right] + \left[\frac{q^n z}{q^n}, h(q^n w)\right]\right) = [D(z), w] + [z, D(w)]$$

for all $z, w \in A$.

But since $D$ is $\mathbb{C}$-linear, vglue-8pt

$$D([z, w]) = d_{d-2}C_{l-2}D([z, w]$$

Similarly, one can obtain that $D(a \circ b) = (Da) \circ b + a \circ (Db)$ for all $a, b \in A$. Hence the $\mathbb{C}$-linear mapping $D: A \to A$ is a Lie $JC^n$-algebra derivation satisfying (4.6), as desired.

**Corollary 4.7.** Let $h: A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|d_{d-2}C_{l-2}h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d}\right) + [z, w] + a \circ b \|_{d_{d-2}C_{l-2}} + d_{d-2}C_{l-1} \sum_{j=1}^d \mu h(x_j)$$

$$-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - [h(z), w] - [z, h(w)] - h(a) \circ b$$

$$-a \circ h(b) \| \leq \theta \left(\sum_{j=1}^d \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p\right),$$

$$\|h(q^n u) - h(q^n u)^*\| \leq dq^n \theta$$

for all $\mu \in T^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x_1, \cdots, x_d, z, w, a, b \in A$. Then there exists a unique Lie $JC^n$-algebra derivation $D: A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{q^{1-p}(q^p + (d-1)p\theta)}{l_{d-1}C_{l-1}(q^{l-1-p} - 1)} \|x\|^p$$

for all $x \in A$.

**Proof.** Define $\varphi(x_1, \cdots, x_d, z, w, a, b) = \theta(\sum_{j=1}^d \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p), and apply Theorem 4.6.

One can obtain a similar result to Theorem 4.3 for the Trif functional equation.

**References**


