On Prime $\mathbb{Z}$-graded Lie algebras of growth one

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Abstract. We will give the structure of $\mathbb{Z}$-graded prime nondegenerate algebras $L = \sum_{i \in \mathbb{Z}} L_i$ containing the Virasoro algebra and having the dimensions of the homogeneous components, $\dim L_i$, uniformly bounded.

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1. Introduction

Throughout the paper we consider algebras over an algebraically closed field $F$ of zero characteristic.

By a $\mathbb{Z}$-graded algebra we mean an algebra $L = \sum_{i \in \mathbb{Z}} L_i$, $L_i L_j \subseteq L_{i+j}$, having all homogeneous components $L_i$ finite dimensional. In [Ma1], [Ma2] (see also the earlier work [K1]) O. Mathieu classified all graded simple Lie algebras with polynomial growth of dimensions $\dim L_i$. He proved that every such algebra is a (twisted) loop algebra or an algebra of Cartan type or the Virasoro algebra $\text{Vir}$.

The problem of classification of $\mathbb{Z}$-graded Lie superalgebras with all $\dim L_i$ uniformly bounded is still open. Of particular interest is the case when the even part of $L$ contains Vir, that is, when $L$ is a superconformal algebra (see [KvL]). In this paper we modify O. Mathieu’s result [Ma1] to make it applicable to the study of the even part of a superconformal algebra (see [MZ1], [KMZ]).

Recall that an algebra $L$ is called prime if for any two nonzero ideals $I$, $J \subset L$ we have $IJ \neq (0)$. A Lie algebra $L$ is nondegenerate if $a \in L$, $[[L, a], a] = (0)$ implies $a = 0$. Following [Z2] we say that $L$ is a Lie algebra with finite grading if $L = \sum_{i \in \mathbb{Z}} L_{(i)}$, $[L_{(i)}, L_{(j)}] \subseteq L_{(i+j)}$, the subspaces $L_{(i)}$ can be infinite dimensional, but $\{i \mid L_{(i)} \neq (0)\}$ is finite. The grading is not trivial if $\sum_{i \neq 0} L_{(i)} \neq (0)$. All Jordan algebras and their generalizations can be interpreted as Lie algebras with finite gradings (see [Z2]).

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Let \( L = \sum_{i \in \mathbb{Z}} L_i \) be a graded Lie algebra, all dimensions \( \dim L_i \) are uniformly bounded and \( L_0 \) is not solvable. Then \( L_0 \) contains a copy of \( \mathfrak{sl}_2(F) = Fe + Fh + Ff, \ [e, f] = h, \ [h, e] = 2e, \ [h, f] = -2f \). The adjoint operator \( \text{ad}(h) : L \to L \) has only finitely many eigenvalues and the decomposition of \( L \) into a direct sum of eigenspaces is a finite grading on \( L \), which is compatible with the initial \( \mathbb{Z} \)-grading.

For a finite dimensional simple algebra \( G \) let \( \mathcal{L}(G) = G \otimes F[t^{-1}, t] \) be its loop algebra. Every finite grading on \( G \) extends to a finite grading on \( \mathcal{L}(G) \) which is compatible with the \( \mathbb{Z} \)-grading. If \( G \) is graded by a finite cyclic group \( \mathbb{Z}/l\mathbb{Z} \), \( G = G_0 + \cdots + G_{l-1} \), then we will refer to \( \sum_{i=j \mod l} G_i \otimes t^j \) as a twisted loop algebra.

The Virasoro algebra naturally acts on \( \mathcal{L}(G) \) and the semidirect sum \( L = \mathcal{L}(G) \rtimes \text{Vir} \) for some finite dimensional simple Lie algebra \( G \).

We prove also the following theorem on Jordan pairs (see [L]) which generalizes [MZ1] and determines the structure of \( \mathbb{Z} \)-graded prime nondegenerated Jordan pairs having the dimensions of the homogeneous components uniformly bounded.

**Theorem 2.** Let \( V = (V^-, V^+) = \sum_{i \in \mathbb{Z}} V_i \) be a prime nondegenerate \( \mathbb{Z} \)-graded Jordan pair having all \( \dim V_i \) uniformly bounded. Then either \( V \) is isomorphic to a (twisted) loop pair \( \mathcal{L}(W) \), where \( W \) is a finite dimensional simple Jordan pair or \( V \) is embeddable in \( \mathcal{L}(W) \) and \( \sum_{i \geq k} \mathcal{L}(W)_i \subseteq V \subseteq \mathcal{L}(W) \) or \( \sum_{i \geq k} \mathcal{L}(W)_{-i} \subseteq V \subseteq \mathcal{L}(W) \).

### 2. The strongly PI case

Let \( f(x_1, \ldots, x_n) \) be a nonzero element of the free associative algebra. We say that an associative algebra \( A \) satisfies the polynomial identity \( f(x_1, \ldots, x_n) = 0 \) if \( f(a_1, \ldots, a_n) = 0 \) for arbitrary elements \( a_1, \ldots, a_n \in A \). An algebra satisfying some polynomial identity is said to be a PI-algebra.

For an arbitrary algebra \( A \) the multiplication algebra \( M(A) \) of \( A \) is the subalgebra of \( \text{End}_F(A) \) generated by all right and left multiplications \( R(a) : x \mapsto xa, \ L(a) : x \mapsto ax, \ a \in A \).

An algebra \( A \) is **strongly PI** if its multiplications algebra \( M(A) \) is PI.

An element \( a \) in a Lie algebra \( L \) over a field \( F \) is said to have rank 1 if \( [[L, a], a] \subseteq Fa \).

**Lemma 2.1.** ([Z1]) There exists a function \( R(n) \) such that an arbitrary Lie algebra generated by \( n \)-elements of rank 1 has dimension \( \leq R(n) \).

An ideal of the free Lie (resp. associative) algebra is said to be a \( T \)-ideal if it is invariant under all substitutions. For an arbitrary algebra \( L \) the ideal of all identities satisfied by \( L \) is a \( T \)-ideal.
Lemma 2.2. Let $L$ be a Lie algebra over a field $F$, $ch F = 0$ and $a \in L$ an element of rank 1. Let’s consider $s$ elements $a_i = a a d(x_1) \cdots a d(x_{ir})$, $1 \leq i \leq s$, $x_{ij} \in L$. Let $m = 2^1 + 2^2 + \cdots + 2^s$ and let $T$ be the $T$-ideal of all identities that are satisfied by all Lie algebras of dimension $\leq R(m)$. Then the subalgebra $< a_1, \ldots, a_s >$ satisfies all identities of $T$

Proof. Let’s consider the Lie algebra $\tilde{L} = L((t^{-1}, t))$ of Laurent series over $L$. Clearly, $\tilde{L}$ is an algebra over the field of Laurent series $F((t^{-1}, t))$. The element $a$ is an element of rank 1 in $\tilde{L}$, $[[\tilde{L}, a], a] \subseteq F((t^{-1}, t))a$.

For a series $b = \sum_i b_i t^i$, $b_i \in L$, let’s denote $\min(b) = b_k$ if $b_k \neq 0$ and $b_i = 0$ for every $i < k$.

For arbitrary elements $x_{ij}$, $1 \leq i \leq s$, $1 \leq j \leq r_i$, we have $e^{2 a d(x_{ij} t)} - e^{a d(x_{ij} t)} = a d(x_{ij} t) + (\cdot \cdot \cdot ) t^2$.

Therefore,

$$aad(x_{i1}) \cdots a d(x_{ir}) = \min(a(e^{2 a d(x_{i1} t)} - e^{a d(x_{i1} t)})) \cdots (e^{2 a d(x_{ir} t)} - e^{a d(x_{ir} t)}), \quad (\ast)$$

Since $e^{a d(x_{ij} t)}$, $e^{2 a d(x_{ij} t)}$ are automorphisms of $\tilde{L}$ it follows that the elements $ae^{k_1 a d(x_{i1} t)} \cdots e^{k_{r_i} a d(x_{ir} t)}$, $1 \leq k_1, \ldots, k_{r_i} \leq 2$, are elements of rank 1 in $\tilde{L}$.

Let’s denote as $B$ the subalgebra of $\tilde{L}$ generated by $m$ elements: $ae^{k_1 a d(x_{i1} t)} \cdots e^{k_{r_i} a d(x_{ir} t)}$, where $k_1, \ldots, k_{r_i} \in \{1, 2\}$, $1 \leq i \leq s$. We have $\dim F((t^{-1}, t))B \leq R(m)$.

Taking $(\ast)$ into account, an arbitrary commutator $\sigma$ in $a_1, \ldots, a_s$ is either $0$ or $\min(b)$ where $b \in B$.

Let $f(x_1, \ldots, x_k) \in T$. Without loss of generality we will assume that $f$ is multilinear. Let us consider $s$ arbitrary commutators $\sigma_1, \ldots, \sigma_k$ in $a_1, \ldots, a_s$. If $\sigma_i = 0$ for some $i$, then $f(\sigma_1, \ldots, \sigma_k) = 0$. In the other case, there exist elements $b_1, \ldots, b_k \in B$ such that $\sigma_i = \min(b_i)$, $1 \leq i \leq s$. Hence, $f(\sigma_1, \ldots, \sigma_k) = 0$ or $f(\sigma_1, \ldots, \sigma_k) = \min f(b_1, \ldots, b_k)$. But $f(b_1, \ldots, b_k) = 0$ and so Lemma is proved.

Recall that a centroid of an algebra $A$ is the centralizer of the multiplication algebra $M(A)$ in $\text{End}_F(A)$

Lemma 2.3. Let $A = \sum_{i \in Z} A_i$ be a graded algebra whose centroid $\Gamma = \sum_{i \in Z} \Gamma_i$ contains a homogeneous invertible element $\gamma \in \Gamma_i$ of degree $i \neq 0$. Then $A \cong \mathcal{L}(G)$ is a (twisted) loop algebra.

Proof. Let $\gamma_i \in \Gamma_i$ with $\gamma_i^{-1} = \gamma_{-i} \in \Gamma_{-i}$ and let $a_{i1}^1, \ldots, a_{id}^d \in A_j$ be linearly independent elements. Then

$$\gamma_i a_{i1}^1, \ldots, \gamma_i a_{id}^d \in A_{i+j}$$

are also linearly independent. Hence $\dim A_j = \dim A_{i+j} = \dim A_{-i+j}$, for arbitrary $j \in Z$.

Taking $i$ the smallest index such that there exists an invertible $\gamma_i$, we can define a finite dimensional algebra structure in $G = A_0 + A_1 + \cdots + A_{i-1}$ by the new law:
\[ a_i \ast b_h = \begin{cases} a_i b_h & \text{if } l + h < i \\ \gamma_{i}^{-1}(a_i b_h) & \text{if } l + h \geq i \end{cases} \]

It is clear that \( A \) is isomorphic to \( \sum_{i=j \mod 1} G_i \otimes t^j \). Lemma is proved.

**Lemma 2.4.** Let \( \Lambda \) be a subset of \( Z \) closed under addition and let \( m = \gcd(\Lambda) \). Then either \( \Lambda = mZ \) or \( m\{i \in Z, i \geq k\} \subseteq \Lambda \subseteq mZ_{\geq 0} \) or \(-m\{i \in Z, i \geq k\} \subseteq \Lambda \subseteq mZ_{\leq 0} \) for some \( k \geq 1 \).

**Proof.** Suppose at first that \( \Lambda \) contains both a positive element \( i \geq 1 \) and a negative element \( -j, j \geq 1 \). Then \( \Lambda \) contains the additive subgroup \( ijZ \).

The quotient \( \Lambda/ijZ \subseteq Z/ijZ \) is a sub-semigroup of a finite group, hence \( \Lambda/ijZ \) is a group. Hence \( \Lambda \) is a subgroup of \( Z \) and therefore \( \Lambda = mZ \).

Now suppose that \( \Lambda \subseteq Z_{\geq 0} \). Then, clearly \( \Lambda \subseteq mZ_{\geq 0} \). Choose \( k \geq 1 \) such that \( km \in \Lambda \). There exist elements \( \lambda_1, \ldots, \lambda_r \in \Lambda \) and integers \( k_1, \ldots, k_r \) in \( Z \) such that \( k_1 \lambda_1 + \cdots + k_r \lambda_r = m \).

Choose a sufficiently large integer \( q \) such that \( q + ik_j \geq 0 \) for all \( j = 1, \ldots, r \) and for all \( i, 0 \leq i \leq k - 1 \). The element \( \lambda = q(\sum^r_{i=1} \lambda_i) \) is in \( \Lambda \). We claim that \( \lambda + mZ_{\geq 0} \subseteq \Lambda \).

Indeed, for \( 0 \leq i \leq k - 1 \) we have \( \lambda + mi \in \sum^r_{i=1} Z_{\geq 0} \lambda_i \subseteq \Lambda \).

Now it is easy to see that for an arbitrary element \( \lambda' \in \Lambda \), if \( \lambda', \lambda' + m, \ldots, \lambda' + (k-1)m \in \Lambda \) then \( \lambda' + km \in \Lambda \) as well and therefore the element \( \lambda'' = \lambda + m \) has the same property as \( \lambda' \). Hence \( \lambda' + mZ_{\geq 0} \subseteq \Lambda \). Lemma is proved.

**Lemma 2.5.** Let \( \Gamma = \sum \Gamma_i \) be a \( Z \)-graded (commutative and associative) domain over an algebraically closed field \( F \) such that the dimensions \( \dim_F \Gamma_i \) are uniformly bounded. Then, either \( \Gamma \simeq F[t^{-m}, t^m] \) or \( \sum_{i \geq k} F t^m \subseteq \Gamma \subseteq F[t^m] \) or \( \sum_{i \geq k} F t^{-m} \subseteq \Gamma \subseteq F[t^{-m}] \), where \( m \geq 1, k \geq 1 \).

**Proof.** Let us prove first that \( \dim_F \Gamma_i \leq 1 \) for every \( i \). Let \( d = \max\{\dim \Gamma_i | i \in Z\} \). Choose two arbitrary nonzero elements, \( a_i, b_i \in \Gamma_i \).

Since \( \dim_F \Gamma_d \leq d \), there exists a nontrivial linear dependence relation
\[ \gamma_d a_i^d + \gamma_{d-1} a_i^{d-1} b_i + \cdots + \gamma_0 b_i^d = 0. \]

The polynomial \( f(x) = \gamma_d x^d + \gamma_{d-1} x^{d-1} + \cdots + \gamma_0 \) can be decomposed as \( f(x) = \gamma_d (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d) \), with \( \gamma_d \neq 0 \), \( \alpha_1, \alpha_2, \ldots, \alpha_d \in F \).

We have \( 0 = f(\frac{a_i}{b_i}) = \gamma_d (\frac{a_i}{b_i} - \alpha_1)(\frac{a_i}{b_i} - \alpha_2) \cdots \)

Hence \( a_i = \alpha_k b_i \) for some \( k \). Now \( \Lambda = \{ i \in Z | \Gamma_i \neq (0) \} \) is a subsemigroup of \( Z \) and the result is a consequence of Lemma 2.4.

Let \( L = \sum_{i \in Z} L_i \) be a strongly PI \( Z \)-graded prime nondegenerate Lie algebra. Let \( d = \max_{i \in Z} \dim L_i \). Let \( \Gamma \) denote the centroid of \( L \), \( \Gamma_h \) is the set of homogeneous elements from \( \Gamma \).
Lemma 2.6. (1) $\Gamma \neq (0)$ is an integral domain and the ring of fractions $(\Gamma \setminus \{0\})^{-1}L$ is a simple finite dimensional Lie algebra over the field $K = (\Gamma \setminus \{0\})\Gamma$.

(2) The algebra $\tilde{L} = (\Gamma_h \setminus \{0\})^{-1}L$ is a graded simple algebra and $\dim F L_i \leq d$, for an arbitrary $i \in Z$.

(3) Either $L$ is isomorphic to a (twisted) loop algebra or there is a graded embedding $\varphi : \Gamma \to F[t^{-m}, t^m]$ such that

$$\sum_{i \geq k} F t^{im} \subseteq \varphi(\Gamma) \subseteq F[t^m] \text{ or } \sum_{i \geq k} F t^{-im} \subseteq \varphi(\Gamma) \subseteq F[t^{-m}].$$

Proof. For the assertion (1) cf. see [Ro].

(2) We only need to check that $\tilde{L}$ is graded simple. Let $I$ be a non-zero graded ideal of $L$. By (1), $(\Gamma \setminus \{0\})^{-1}I = (\Gamma \setminus \{0\})^{-1}L$.

Let $\dim_K (\Gamma \setminus \{0\})^{-1}L = r$ and $f_r(x_1, \ldots, x_q)$ is a multilinear central polynomial that corresponds to $r \times r$ matrices. Then $(\Gamma \setminus \{0\})^{-1}L$ is a faithful irreducible module over the multiplication algebra $M < (\Gamma \setminus \{0\})^{-1}L >$. Hence, $M < (\Gamma \setminus \{0\})^{-1}L \simeq M_r(K)$. Consequently, there exist operators $\omega_i = ad(a_{i1}) \cdots ad(a_{iq}), 1 \leq i \leq q$, homogeneous elements of $I$ such that $f_r(\omega_1, \ldots, \omega_q) \neq 0$. Clearly, $f_r(\omega_1, \ldots, \omega_q) \in \Gamma_h$. Now,$$L = (Lf_r(\omega_1, \ldots, \omega_q))f_r(\omega_1, \ldots, \omega_q)^{-1} \subseteq I f_r(\omega_1, \ldots, \omega_q)^{-1} \subseteq (\Gamma_h \setminus \{0\})^{-1}I.$$

This proves $(\Gamma_h \setminus \{0\})^{-1}I = (\Gamma_h \setminus \{0\})^{-1}L$ and so $\tilde{L}$ is graded simple.

In order to prove (3) we will show that $\dim \Gamma_k \leq d$ for an arbitrary $k$. Let’s take $d + 1$ arbitrary elements $\gamma_1, \ldots, \gamma_{d+1} \in \Gamma_k$ and a non zero homogeneous element $a_i \in L_i$. Since $a_i \gamma_1, a_i \gamma_2, \ldots, a_i \gamma_{d+1} \in L_{i+k}$, there exists a non trivial linear dependence relation $\sum_{j=1}^{d+1} \xi_j a_i \gamma_j = 0, \xi_j \in F$. Since non zero elements in $\Gamma$ have zero nuclei and $a_i \in K e r \sum_{j=1}^{d+1} \xi_j \gamma_j$, it follows that $\sum_{j=1}^{d+1} \xi_j \gamma_j = 0$.

We have proved that $\dim F \Gamma_k \leq d$ and so the assertion (3) follows from Lemmas 2.3 and 2.5.

Indeed, by Lemma 2.5, either $\Gamma \simeq F[t^{-m}, t^m]$ or there exists the wanted embedding. If $\Gamma \simeq F[t^{-m}, t^m]$, then $L$ is a loop algebra by Lemma 2.3.

Lemma 2.7. Let $L = \sum_{i \in Z} L_i$ be a prime, nondegenerate, strongly PI Lie algebra, $\dim L_i \leq d$, as in the previous lemma. Let’s assume that $Vir = \sum_{i \in Z} Vir_i$ can be embedded into $Der(L)$ as a graded algebra. Then $L$ is isomorphic to a (nontwisted) loop algebra.

Proof. If $L$ is not isomorphic to a (twisted) loop algebra, then by Lemma 2.6 there exists a graded embedding $\varphi : \Gamma \to F[t^{-m}, t^m], m \geq 1$, such that either $\sum_{i \geq k} F t^{im} \subseteq \varphi(\Gamma) \subseteq F[t^m]$ or $\sum_{i \geq k} F t^{-im} \subseteq \varphi(\Gamma) \subseteq F[t^{-m}]$ for some $k \geq 1$.

Let us assume that $\sum_{i \geq k} F t^{im} \subseteq \varphi(\Gamma) \subseteq F[t^m]$. This implies that $\Gamma$ is generated by a finite set of elements $\gamma_i \in \Gamma_{s_i}, i = 1, 2, \ldots, r$.

Let $s = \max_{1 \leq i \leq r} s_i$. The Virasoro algebra acts on $\Gamma$. For each generator $\gamma_i$ the subspace $\gamma_i Vir_{-(s+1)} = (0)$, since it is contained in $\Gamma$ and has negative degree.
So $\text{Vir}_{-(s+1)}$ is contained in the kernel of the action of the Virasoro algebra on the derivations of $\Gamma$. By the simplicity of the Virasoro algebra, we have that $\Gamma\text{Vir} = (0)$.

Now the Virasoro algebra acts on a finite dimensional Lie algebra $\hat{L}_K = (\Gamma \setminus \{0\})^{-1}L$ and the action is not trivial since $\text{Vir} \subseteq \text{Der}(L)$. This leads to a contradiction, since the Virasoro algebra is not strongly PI.

We showed that $L$ is isomorphic to a loop algebra. Let us show that this loop algebra is not twisted. Indeed, let $\Gamma \text{Vir} = (0)$. Now we can argue as above.

**Lemma 2.8.** Let $L$ be a prime nondegenerate Lie algebra and let $I$ be a nonzero ideal of $L$. Then $I$ is a prime nondegenerate algebra.

**Proof.** We will prove first that $I$ is nondegenerate. Indeed, let $0 \neq a \in I$ and $[[I, a], a] = (0)$. Since $L$ is nondegenerate, there exists an element $x \in L$ such that $[[x, a], a] \neq 0$. Now, $\text{Lad}([[x, a], a])^2 = \text{Lad}(a)^2 \text{ad}(x)^2 \text{ad}(a)^2 \subseteq \text{Lad}(a)^2 = (0)$, (cf. [Ko]), a contradiction.

Now we will prove that $I$ is prime. Let $I', I''$ be non-zero ideals of $I$, with $[I', I''] = 0$. Let $id_L(I'')$ the ideal of $L$ generated by $I''$. If $[id_L(I''), I'] = 0$, then the nonzero ideal of $L$, $id_L(I'')$, has a non zero centralizer, which contradicts primeness of $L$. Hence, $J = [I', id_L(I'')]$ is a non zero ideal of $I$. We have

$$\text{ad}(L)\text{ad}(I')^2 \subseteq \text{ad}(I')\text{ad}(L)\text{ad}(I') + \text{ad}(I)\text{ad}(I') \subseteq \text{ad}(I')M < L >.$$

Let’s choose an arbitrary nonzero element $a \in J$, $a = \sum a_i \text{ad}(x_{i1}) \cdots \text{ad}(x_{ir_i})$ with $a_i \in I''$, $x_{ij} \in L$, $r_i \geq 0$. So, for $r = \max_i r_i$ we have

$$\text{aad}(I')^{2r} \subseteq \sum a_i \text{ad}(I')M < L > = (0).$$

Hence, $\text{aad}(J)^{2r} = (0)$.

This proves that $J$ has a nontrivial center, what contradicts the nondegeneracy of $I$ and proves the lemma.

**Lemma 2.9.** Let $L = \sum_{i \in \mathbb{Z}} L_i$ be a $\mathbb{Z}$-graded prime nondegenerate Lie algebra containing the Virasoro algebra and having all the dimensions $\dim L_i$ uniformly bounded. Suppose that $L$ contains a nonzero graded ideal $I$ which is strongly PI. Then $L$ is isomorphic to the semidirect sum of a loop algebra $\hat{L}(G)$ (for some finite dimensional simple Lie algebra $G$) and the Virasoro algebra

**Proof.** By Lemma 2.8 $I$ is a prime nondegenerate algebra. Moreover, since $L$ is prime, the action of Vir on $I$ is faithful. Hence by Lemma 2.7 $I \simeq \hat{L}(G)$, with $\dim G < \infty$. Again, since $I$ is prime and nondegenerate it follows that the algebra $G$ is simple. For an arbitrary element $a \in L$ let $\text{ad}_I(a)$ denote the linear operator $\text{ad}_I(a) : I \to I$, $x \to [x, a]$. The mapping $a \to \text{ad}_I(a)$ is an embedding of $L$ into the Lie algebra

$$\text{Der}(\hat{L}(G)) = \hat{L}(G) \prec \text{Vir}.$$
Since the Virasoro algebra is simple and not strongly PI, it follows that \(\text{Vir} \cap I = (0)\). Now comparing the dimensions of the homogeneous components we conclude that the embedding \(L \to \text{Der}(L(G))\), \(a \to \text{ad}_I(a)\) is an isomorphism. The Lemma is proved.

3. Lie-Jordan Connections

In this section we will study connections between Lie algebras and Jordan systems.

A Jordan pair \(P = (P^-, P^+)\) is a pair of vector spaces with a pair of trilinear operations

\[
\{ , , \} : P^- \times P^+ \times P^- \to P^-, \quad \{ , , \} : P^+ \times P^- \times P^+ \to P^+
\]

that satisfies the following identities:

\begin{align*}
(P.1) \quad &\{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\}, \\
(P.2) \quad &\{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, u^\sigma\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, u^\sigma\}, \\
(P.3) \quad &\{x^\sigma, y^{-\sigma}, x^\sigma\}, x^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} = \\
&\{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}, \\
\end{align*}

for every \(x^\sigma, u^\sigma \in P^\sigma, y^{-\sigma}, z^{-\sigma} \in P^{-\sigma}, \sigma = \pm\) (see [L]).

If \(L = \sum_{i=-n}^n L(i)\) is a finite grading, then the pair \((L(-n), L(n))\) with the operations \(\{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}\), \(\sigma = \pm\) is a Jordan pair.

An element \(a \in P^\sigma\) is called an absolute zero divisor of the pair \(P\) if \(\{a, P^{-\sigma}, a\} = (0)\). A Jordan pair is said to be nondegenerate if it does not contain nonzero absolute zero divisors.

A Jordan pair is said to be prime if the product of any two nonzero ideals is not zero, where an ideal of \(P\) is a pair of subspaces \(I = (I^-, I^+)\) that satisfies the obvious condition.

The smallest ideal \(M(P)\) of the pair \(P\) whose quotient is nondegenerate is called the McCrimmon radical of \(P\).

An element \(a\) of a Lie algebra is a sandwich if \([L, a], a\) = 0. The Kostrikin radical of a Lie algebra \(L\) is the smallest ideal \(K(L)\) whose quotient is nondegenerate.

The central pair in this connection is given by the following two lemmas, that reduce our original problem in Lie algebras to a Jordan pairs problem.

**Lemma 3.1.** Let \(L\) be a Lie algebra with a finite grading \(L = \sum_{k=-n}^n L(k)\), \(L(0) = \sum_{k=1}^n [L(-k), L(k)]\) and \(L(n) \neq (0)\). If \(L\) is prime and nondegenerate, then:

1. Every nonzero ideal of \(L\) has a nonzero intersection with \(L(n)\),
2. The Jordan pair \(V = (L(-n), L(n))\) is prime and nondegenerate.
Proof. (1) Let \( (0) \neq I \leq L \) and suppose that \( I \cap L_{(n)} = (0) \). Then, \([I, L_{(n)}], L_{(n)} \subseteq I \cap L_{(n)} = (0)\). Consider the subalgebra \( L' = I + L_{(n)} \).

Clearly, \([L', L_{(n)}], L_{(n)} \) = (0). Hence, \( L_{(n)} \) is in the Koszul radical of \( L' \) and using Lemma 2.8 and Proposition 2 of [Z1] we conclude that \([I, L_{(n)}] \subseteq K(L') \cap I = K(I) = (0)\). This contradicts primeness of \( L \).

(2) The non-degeneracy of \( V \) follows from the fact that every absolute zero divisor of \( V \) is a sandwich of \( L \).

Now, let us assume that \( I \) and \( J \) are nonzero ideals of \( V \) and that \( I \cap J = (0) \). Let \( I \) and \( J \) be the ideals of \( V \) generated by \( I \) and \( J \) respectively. By (1), the nonzero ideal \( I \cap J \) has nonzero intersection with \( V \). Let \( P = (I \cap L_{(-n)} \cap J, I \cap J \cap L_{(n)} \) \) \( \subseteq V \).

Zelmanov proved in [Z1] that the quotient pairs \( I \cap V / I \) and \( J \cap V / J \) coincide with their McCrimmon radicals. We will prove that this implies that \( P \subseteq M(V) \).

Let's recall that a sequence of elements in a Jordan pair \( x_1, x_2, \ldots \in V^\sigma \), \( \sigma = \pm \), is called an m-sequence if \( x_{i+1} \in \{x_i, V^{-\sigma}, x_i\} \). In [Z3] it was proved that the McCrimmon radical consists of those elements \( x \) such that every m-sequence starting by \( x \) finishes in zero.

Let \( x \in P^\sigma \) and let \( x = x_1, x_2, \ldots \) be an m-sequence. Since \( x \in I \cap V^\sigma \), it follows that there exists \( s_1 \geq 1 \) such that \( x_i \in I \) for all \( i \geq s_1 \).

Similarly, there exists \( s_2 \geq 1 \) s.t. \( x_j \in J \) for all \( j \geq s_2 \). Hence, for every \( k \geq \max(s_1, s_2) \) we have that \( x_k \in I \cap J = (0) \). Now, \( (0) \neq P \subseteq M(V) \) contradicts the nondegeneracy of \( V \), what proves the lemma.

Lemma 3.2. Let \( L = \sum_{k=-n}^n L_{(k)} \) be a Lie algebra with a finite grading. Let us assume that the Jordan pair \( V = (L_{(-n)}, L_{(n)}) \) is prime and nondegenerate and that an arbitrary nonzero ideal of \( L \) has nonzero intersection with \( V \). Then \( L \) is prime and nondegenerate.

Proof. Clearly, the algebra \( L \) is prime, because if \( I, J \) are non zero ideals of \( L \) with \([I, J] = (0)\), then \( I' = I \cap V \), \( J' = J \cap V \) are nonzero ideals of \( V \) and \([I^\sigma, J^{-\sigma}, V^\sigma \) = \( J^{-\sigma}, I^\sigma, V^{-\sigma} \) \subseteq I \cap J = (0), \( \sigma = \pm \), what contradicts primeness of \( V \).

In [Z2] it was proved that \( K(L) \cap L_{(\pm n)} \) is contained in the McCrimmon radical of the pair \( V \), hence \( K(L) \cap L_{(\pm n)} = (0) \), what implies, under our assumptions, that \( K(L) = (0) \) and so \( L \) is nondegenerate.

4. The Jordan Case

The last two lemmas have reduced our original problem to a problem concerning Jordan pairs. So, our aim now will be to prove Theorem 2.

We will need the following lemma
Lemma 4.1. Let $G$ be a simple finite dimensional Lie algebra with a $\mathbb{Z}/l\mathbb{Z}$-grading, $G = \sum_{i \in \mathbb{Z}/l\mathbb{Z}} G_i$.

If $\dim G_0 \leq d$, then $\dim_F G \leq N(d) = \max(d(2d + 1), 248)$.

Proof. The mapping $d : G \to G$, $a_i \mapsto ia_i$ is a derivation. Since every derivation is inner, there exists an element $h \in G$ such that $d = ad(h)$. So $h$ is semisimple and is contained in some Cartan subalgebra $H$. Since $H$ is abelian, the elements of $H$ commute with $h$ and given that $[a_i, h] = d(a_i) = ia_i$, necessarily $H \subseteq G_0$. But $\dim G_0 \leq d$, which implies $\dim H \leq d$.

Now the bound follows from the classification of simple finite dimensional Lie algebras.

Proof of Theorem 2

We will divide the proof of the theorem in three cases

Case 1. We will assume first that $K(V)$ is strongly PI (where $K(V)$ denotes the Lie algebra associated to $V$ via the Tits-Kantor-Koecher construction).

Recall that the Tits-Kantor-Koecher Lie algebra $K(V)$ can be characterized in the following way: $K(V) = K(V)_{-1} + K(V)_0 + K(V)_1$ is a $\mathbb{Z}$-graded Lie algebra, $K(V)_0 = [K(V)_{-1}, K(V)_1]$, $(K(V)_{-1}, K(V)_1) = V$ and $K(V)_0$ does not contain nonzero ideals of $K(V)$.

We will see that under our assumption, the algebra $K(V)$ is prime. Let us show that every nonzero ideal of $K(V)$ has non zero intersection with $V^+$. Since the Jordan pair $V$ is prime, there are no elements $0 \neq x^- \in V^-$ with $[x^-, V^+, V^+] = (0)$. Similarly, there are no elements $0 \neq x^+ \in V^+$ with $[x^+, V^-, V^-] = (0)$.

If $I \cap V^+ \neq (0)$, then $(0) \neq [I \cap V^+, V^-, V^-] \subseteq I \cap V^-$. That is, for an arbitrary ideal $I$ of $V$, $I \cap V^+ \neq (0)$ if and only if $I \cap V^- \neq (0)$.

Let $x = x_- + x_0 + x_+ \in I$. Let us assume that $x_- \neq 0$. Then $[x, V^+, V^+] = [x_-, V^+, V^+] \neq 0$ and $[x, V^+, V^+] \subseteq I$. So $[x, V^+, V^+] \subseteq I \cap V^+$ and $I \cap V^+ \neq (0)$. Similarly, if $x_+ \neq 0$, then $I \cap V^- \neq (0)$.

Hence $I \subseteq [V^-, V^+]$, which implies $I = (0)$.

Now we can prove that $K(V)$ is prime. Indeed, let’s consider $I_1, I_2$ two non zero ideals of $K(V)$. Then $I_1 \cap V \neq (0)$, $I_2 \cap V \neq (0)$. Since $V$ is prime, $I_1 \cap I_2 \cap V \neq (0)$ and, in particular, $I_1 \cap I_2 \neq (0)$.

Since $L = K(V)$, is a prime and strongly PI Lie algebra it follows that the centroid $\Gamma$ of $L$ is nonzero and the algebra $(\Gamma \setminus \{0\})^{-1}L$ is finite dimensional over $(\Gamma \setminus \{0\})^{-1}\Gamma$.

Let us see that $\Gamma$ can be identified with the centroid of $V$, that is, $V^+ \Gamma \subseteq V^+$ and $V^- \Gamma \subseteq V^-$. Indeed, let’s consider the derivation $d : L \to L$, $d(a_i) = ia_i$, that multiplies $V^\pm$ by $\pm 1$ and annihilates $[V^-, V^+]$. The centroid $\Gamma$ decomposes into eigenspaces with respect to the action of $d : \Gamma = \Gamma_{-1} + \Gamma_{-1} + \Gamma_{0} + \Gamma_{1} + \Gamma_{2}$. Since every element of $\cup_{i \neq 0} \Gamma_i$ is nilpotent and $L$ is prime, we have that $\Gamma = \Gamma_{0}$, that is, $\Gamma$ maps $V^+$ to $V^+$ and $V^-$ to $V^-$. 
The centroid \( \Gamma \) is a graded commutative domain, \( \Gamma = \sum_{i \in \mathbb{Z}} \Gamma_i \) with \( \dim \Gamma_i \leq 1 \). If \( \Gamma = \Gamma_0 \), then \( \Gamma = F \) and \( \dim_F V < \infty \).

If there exist \( i, j \geq 1 \) with \( \Gamma_i \neq (0) \neq \Gamma_{-j} \), then \( V \) is a (twisted) loop Jordan pair.

Let’s consider finally the case when every negative component of \( \Gamma \) is zero (the case with all positive components of \( \Gamma \) equal to zero is similar).

Let \( \gamma_l \) be a homogeneous element of the centroid with degree \( l \), \( \gamma_l : V \to V \). Then \( \ker \gamma_l \subseteq V \), \( \text{Im} \gamma_l \subseteq V \) and they annihilate each other. Since \( V \) is prime, it follows that \( \gamma_l \) is injective.

From \( \gamma_l(V_i) \subseteq V_{i+l} \), it follows that \( \dim V_i = \dim V_i \gamma_l \leq \dim V_{i+l} \). For every \( i \), \( 0 \leq i \leq l-1 \), the ascending sequence: \( \cdots \dim V_i \leq \dim V_{i+l} \leq \dim V_{i+2l} \leq \cdots \) stabilizes in some \( k_i \), that is, \( \dim V_{i+k_i l} = \dim V_{i+(k_i+1) l} \).

Let \( k(\gamma_l) = \max \{ k_i \mid 0 \leq i \leq l-1 \} \). For every \( h \geq k(\gamma) \) the linear mapping \( \gamma_l : V_h \to V_{h+l} \) is bijective.

Let \( \Gamma_h \) be the set of homogeneous elements in \( \Gamma \) (so \( (\Gamma_h \setminus \{0\})^{-1} V \) is a graded Jordan pair over \( (\Gamma_h \setminus \{0\})^{-1} \Gamma \) and an arbitrary nonzero homogeneous element of \( \Gamma_h^{-1} \Gamma \) is invertible).

Let \( n = \min \{ l > 0 \mid C_l = (\Gamma_h^{-1} \Gamma)^l \neq 0 \} \). If \( 0 \neq c_n \in C_n \), then there exist \( i, j \), \( i > j \), and \( 0 \neq \gamma_i \in \Gamma_i \), \( 0 \neq \gamma_j \in \Gamma_j \) with \( c_n = \gamma_j \gamma_i \). Let \( k \) be a multiple of \( n \) such that \( k \geq \max \{ k(\gamma_i), k(\gamma_j) \} \) (let’s notice that we can write \( V_{h+j} \gamma_j^{-1} \subseteq V_h \subseteq V \) if \( h \geq k \), even if there is no \( \gamma_j^{-1} \) in \( \Gamma \)). Hence, \( V_{h+n} = V_{h+n+j} \gamma_j^{-1} = V_{h+n+j-i} \gamma_i \gamma_j^{-1} = V_h c_n \).

Let’s consider the finite-dimensional vector space \( V = V_0 + V_1 + \cdots + V_{n-1} \) with \( V_h = V_{h+k} \) for \( 0 \leq h \leq n-1 \).

If \( 0 \leq r, s \leq n-1 \), \( b^r_{k+r} \in V^r_{k+r}, b^{-r}_s \in V^{-r}_{k+s}, \sigma = \pm 1 \), then

\[ \{ b^r_{k+r}, b^{-r}_s, b^\sigma_s \} \in V^\sigma_{2k+2r+s}. \]

Let \( 2k+2r+s = ln+t, l \geq 0, 0 \leq t \leq n-1 \). Then \( V_{2k+2r+s} = V_{k+ln+t} = V_{k+t} c_n \).

Define \( \{ b^r_{k+r}, b^{-r}_s, b^\sigma_s \}^* = \{ b^r_{k+r}, b^{-r}_s, b^\sigma_s \}^\sigma c_n \in V_{k+t} = V_t \).

Then \( V \) becomes a finite-dimensional \( \mathbb{Z}/n\mathbb{Z} \)-graded Jordan pair with this new product and we get the wanted result.

**Case 2.** We will assume now that \( V \) is finitely generated

According to the classification of prime non-degenerated Jordan pairs by E. Zelmanov, we know that a finitely generated prime Jordan pair \( V \) is either special or strongly PI. Since the strongly PI case is already known, we only need to consider the special case.

In order to prove Theorem 2 in this case, we need to know the relation between the Gelfand Kirillov dimension of a special Jordan pair and the Gelfand Kirillov dimension of its associative enveloping algebra. We will use a result similar to the one used by Skosirskii ([SK1]) for algebras.
Lemma 4.2. Let \((P^-, P^+)\) be a special Jordan pair finitely generated by \(a_1, a_2, \ldots, a_n\). Then every word in the associative enveloping pair can be expressed as a linear combination of elements of the form \(\omega'\omega\), where \(\omega\) is a Jordan word and the lengths of \(\omega'\) and \(\omega\) are not greater than \(2n\).

Proof. There exists an associative algebra \(A\) (that can be assumed finitely generated by \(a_1, \ldots, a_n\)) such that \((P^-, P^+) \subseteq (A^-, A^+)\) and \(A = A^- + (A^-A^+ + A^+A^-) + A^+\).

Let \(\omega = v_1^\sigma v_2^\sigma v_3^\sigma \cdots\) be a product of Jordan words \(v_i\) and the total degree of \(\omega\) in \(a_1, \ldots, a_n\) is \(N\).

We will use an inverse induction on the length of \(v_\sigma\), maximal among the lengths of elements \(v_i^\sigma\). If the length is \(N\), then \(v = v^\sigma\). Let us assume that some \(v_i^\sigma\) placed to the right (similarly to the left) of the element \(v_\sigma\) has length \(\geq 3\). Using that \(v_k^-v_j^++v_i^- = v_k^-v_j^+v_i^-\), we can assume, without loss of generality, that this element and \(v_\sigma\) are adjacent.

But

\[
v^\sigma a^{-\sigma}b^\sigma a^{-\sigma} = (v^\sigma a^{-\sigma}b^\sigma + b^\sigma a^{-\sigma}v^\sigma)a^{-\sigma} - b^\sigma(a^{-\sigma}v^\sigma a^{-\sigma})
\]

where elements in brackets are Jordan words of length strictly greater than the length of \(v^\sigma\).

Rewrite every Jordan word \(v_i^\sigma\) except \(v^\sigma\) as an expression in the generators \(a_j^\pm, \sigma = \sum \cdots v^\sigma a_j^{-\sigma}a_j^\sigma a_j^{-\sigma} \cdots\).

A double occurrence of a generator \(a_j^{-\sigma}\) to the right of \(v^\sigma\) gives rise to \(a_j^{-\sigma}a_k^\sigma a_j^{-\sigma}\), the case which has been considered above.

Finally, we get that \(\omega\) is of the form:

\[
\omega = (\cdots)v^\sigma a_{i1}^{-\sigma}a_{i2}^\sigma a_{i3}^{-\sigma} \cdots
\]

where all the generators \(a_{i1}^{-\sigma}, a_{i2}^{-\sigma}, \ldots\) are distinct.

Hence the length to the right of \(v^\sigma\) (and similarly to the left) is \(\leq 2n\), where \(n\) is the number of generators.

Lemma 4.3. If \(P\) is a finitely generated special Jordan pair and \(A\) is an associative algebra as in Lemma 4.2 with \((P^-, P^+) \subseteq (A^-, A^+)\), then \(GK - \dim(P) = GK - \dim(A)\).

Proof. Let \(U\) be a finite dimensional vector space that generates \(P\) and \(A\).

Then

\[
GK - \dim(A) = \limsup_{n \to \infty} \frac{\ln \dim U^n}{\ln n}
\]

But \(U^n \subseteq U'WmU''\), where \(U'\) and \(U''\) are subspaces of bounded dimensions (not more than \(C\)) and \(Wm\) is spanned by Jordan words in elements of \(U\) of length \(\geq m = n - 4r\) where \(r\) is the dimension of the vector space \(U\). So \(\dim U^n \leq C^2 \dim Wm\).

Hence,
\[GK - \dim(A) = \limsup_{n \to \infty} \frac{\ln \dim U^n}{\ln n} \leq \limsup_{n \to \infty} \frac{\ln(C^2, \dim W^m)}{\ln n} = \]

\[\limsup_{n \to \infty} \frac{\ln C^2 + \ln(\dim W^m)}{\ln(m + 4r)} = \limsup_{m \to \infty} \frac{\ln \dim W^m}{\ln m} = GK - \dim P\]

Now we can conclude the proof of Theorem 2 in the finitely generated case.

If the considered Jordan pair \(P\) is finitely generated and special, its associative enveloping algebra \(A\) is finitely generated and \(GK - \dim(A) = 1\). By the result by Small, Stafford and Warfield Jr. [SSW] we know that \(A\) is PI. Hence \(P\) is strongly PI and the result follows from Case 1.

**Case 3. The General Case**

**Lemma 4.4.** Let \(V = \sum_{i \in \mathbb{Z}} V_i\) be a \(\mathbb{Z}\)-graded Jordan pair having all dimensions \(\dim V_i\) uniformly bounded. Then the locally nilpotent radical \(\text{Loc}(V)\) is equal to the McCrimmon radical \(M(V)\).

**Proof.** It is known that \(M(V) \subseteq \text{Loc}(V)\) (see [Z4]).

Choose an arbitrary homogeneous element \(v_k^\sigma \in V_k^\sigma\) and consider the homotope Jordan algebra \(J = V^{-\sigma}; x \star y = \{x, v_k^\sigma, y\}\). Assign a new degree to homogeneous elements of \(J\), \(\deg(V_i^{-\sigma}) = i + k\). With this degree \(J\) becomes a graded Jordan algebra having all dimensions \(\dim J_i\) uniformly bounded. In [MZ1] it was proved that \(\text{Loc}(J) = M(J)\). Since \(\text{Loc}(V)^{-\sigma} \subseteq \text{Loc}(J)\) and \(\{v_k^\sigma, M(J), v_k^\sigma\} \subseteq M(V)\) (see [Z4]), we conclude that \(\{v_k^\sigma, \text{Loc}(V), v_k^\sigma\} \subseteq M(V)\).

In particular, an arbitrary homogeneous element of \(\text{Loc}(V)\) lies in \(M(\text{Loc}(V)) \subseteq M(V)\). This implies that \(\text{Loc}(V) \subseteq M(V)\). The Lemma is proved.

Let \(V\) be a Jordan pair satisfying the assumptions of Theorem 2 and let \(\tilde{V}\) be a finitely generated graded subpair of \(V\). The nondegenerate pair \(\tilde{V}/M(\tilde{V})\) can be approximated by finitely generated prime nondegenerate Jordan pairs. By the Case 2 each of these pairs is either \(L(U)\) or can be embedded into a loop pair \(L(U)\), where \(U\) is a simple finite dimensional pair. By Lemma 4.1, \(\dim \tilde{V} \leq N(d)\), where \(d = \max \dim V_i\).

Let \(T\) be the ideal of the free Jordan pair consisting of those elements which are identically zero in all Jordan pairs of dimension \(\leq N(d)\).

We proved that for an arbitrary finitely generated subpair \(\tilde{V}\) of \(V\), the set of values \(T(\tilde{V})\) lies in the locally nilpotent radical \(\text{Loc}(\tilde{V})\). This implies that \(T(V) \subseteq \text{Loc}(V)\). By Lemma 4.4 \(\text{Loc}(V) = M(V) = (0)\), which implies \(T(V) = (0)\). Hence the pair \(V\) is strongly PI, which is the Case 1. Theorem 2 is proved.

In the next section we will need the following lemma about loop Jordan pairs.

Let \(W\) be a simple finite dimensional Jordan pair graded by \(\mathbb{Z}/l\mathbb{Z}\), \(W = \sum_{i=0}^{l-1} W_i\), and let \(L(W) = \sum_{i=q \mod l} W_i \otimes t^q\) be a (twisted) loop pair.
Lemma 4.5. For any $k \geq 1$ we have

1) The subpair $\sum_{i \geq k} L(W)_i$ is finitely generated,

2) Every subpair $P \subseteq L(W)$ containing $\sum_{i \geq k} L(W)_i$ is prime and nondegenerate.

Proof. 1) We will prove that $\sum_{i \geq k} L(W)_i$ is generated by $\sum_{i=k}^{3k+2l} L(W)_i$.

Let $q > 3k + 2l$, $a \in W_j^\sigma$, $0 \leq j \leq l - 1$, $j \equiv q \mod l$ and $a \otimes t^q \in L(W)_q$.

We have that $W^\sigma = \{W^\sigma, W^{-\sigma}, W^\sigma\}$ (by simplicity of $W$), so $a = \sum_i \{a_i^\sigma, b_i^\sigma, c_i^\sigma\}$, with $a_i^\sigma \in W_{\pi(i)}$, $b_i^\sigma \in W_{\mu(i)}$, and $c_i^\sigma \in W_{\rho(i)}$, $0 \leq \pi(i) < \mu(i), \rho(i) < l - 1$.

Choose integers $k \leq q_1(i), q_2(i) \leq k + l - 1$ such that $q_1(i) \equiv \pi(i) \mod l$, $q_2(i) = \rho(i) \mod l$ and $q_3(i) = q - q_1(i) - q_2(i)$.

From $q > 3q + 2l$, it follows that $q_3(i) > k$. Now,

$$a \otimes t^q = \sum_i \{a_i^\sigma \otimes t^{q_1(i)}, b_i^\sigma \otimes t^{q_2(i)}, c_i^\sigma \otimes t^{q_3(i)}\},$$

that is,

$$L(W)_q \subseteq \sum \{L(W)_{q_1}, L(W)_{q_3}, L(W)_{q_2}\},$$

where $k \leq q_1, q_2, q_3 \leq q$.

2) Note that if $\Omega$ is a homogeneous operator in the multiplication algebra of $L(W)$ and $(\sum_{i=k}^{k+l-1} L(W)_i)\Omega = (0)$, then $\Omega = 0$.

Let $P$ be a subpair of $L(W)$ with $P \supseteq \sum_{i=k}^\infty L(W)_i$. If $a^\sigma \in P^\sigma$ is an absolute zero divisor of the pair $P$, then $(\sum_{i=k}^{k+l-1} L(W)_i)U(a) = (0)$. This implies that $L(W)U(a) = (0)$. Since $L(W)$ is nondegenerate, it follows that $a = 0$. We have proved that $P$ is nondegenerate.

Let $I, J$ be non zero graded ideals of $P$ with $I \cap J = (0)$.

Take $0 \neq a^\sigma \otimes t^p \in I$, $0 \neq b^\sigma \otimes t^q \in J$ and $c(x_1, \ldots, x_n, \ldots)$ an arbitrary multilinear expression in the free Jordan pair. Then

$$c(a^\sigma \otimes t^p, b^\sigma \otimes t^q, \sum_{i \geq k} L(W)_i, \sum_{i \geq k} L(W)_i, \ldots) = (0).$$

This implies that $c(a^\sigma, b^\sigma, W, W, \ldots) = (0)$, what contradicts primeness of $W$. This proves the lemma.

5. The Lie Case

Lemma 5.1. Let $A$ be a simple $\mathbb{Z}/l\mathbb{Z}$-graded finite dimensional algebra and let $a$ be a homogeneous element of degree $d(a)$. Consider the loop algebra $\sum_{i=1}^{\infty} A_i \otimes t^i$ and its subalgebra $\sum_{j \geq m} A_i \otimes t^j$. Choose an integer $n \geq m$ such that $n = d(a) \mod l$ and let $I$ be the ideal generated by $a \otimes t^n$ in $\sum_{j \geq m} A_i \otimes t^j$. Then $I \supseteq \sum_{j \geq p} A_i \otimes t^j$ for some $p \geq m$. 

Proof.

Let $a_1, \ldots, a_s$ be homogeneous elements of $A$ and $b = aP(a_1) \cdots P(a_s)$, where $P = R$ or $L$. We choose integers $j_1, \ldots, j_s \geq m$ such that $j_k \equiv d(a_k) \pmod{l}$, $k = 1, \ldots, s$. Then $(a \otimes t^m)P(a_1 \otimes t^{j_1}) \cdots P(a_s \otimes t^{j_s}) = b \otimes t^m \in I$ and for an arbitrary $k \in Z_{\geq 0}$ we have that

$$b \otimes t^{n+k} = (a \otimes t^m)P(a_1 \otimes t^{j_1+k}) \cdots P(a_s \otimes t^{j_s}) \in I.$$ 

Let’s take a basis $e_1, \ldots, e_r$ of $A$ that consists of elements of the type $e_i = aR(a_{i_1}) \cdots R(a_{i_r})$, where the elements $a_{ij}$ are homogeneous. According to what we have mentioned above, there exist integers $q_1, \ldots, q_r \geq m$ such that $e_i \otimes t^{q_i+lZ_{\geq 0}} \in I$. It suffices to take $p = \max_{1 \leq i \leq r} q_i$.

Remark. The assertion of the Lemma 5.1 is true also for $Z/lZ$-graded simple finite dimensional Jordan pairs.

We can already prove the main result giving the structure of prime $Z$-graded Lie algebras.

Proof of Theorem 1

Let $L = \sum_{i \in Z} L_i = \sum_{k=-n}^n L_{(k)}$ be a Lie algebra that satisfies the assumptions of Theorem 1. By Lemma 3.1 and Theorem 2, we know that $V = (L_{(-n)}, L_{(n)})$ can be embedded into a loop pair $\mathcal{L}(W)$, $V \hookrightarrow \mathcal{L}(W)$, where $W$ is a simple finite-dimensional Jordan pair and either $\sum_{i \geq k} \mathcal{L}(W)_i \subseteq V$ or $\sum_{i \leq k} \mathcal{L}(W)_{-i} \subseteq V$, for some $k \geq 1$. Let’s assume that $\sum_{i \geq k} \mathcal{L}(W)_i \subseteq V$.

For an arbitrary scalar $\alpha \in F$ we define a homomorphism

$$\varphi_\alpha : W \otimes_F F[t^{-1}, t] \rightarrow W$$

via $t \rightarrow \alpha$. Since $\varphi_\alpha(\sum_{i \geq k} \mathcal{L}(W)_i) = \varphi_\alpha(\sum_{i \geq k} \mathcal{L}(W)_{-i}) = W$, it follows that $\varphi_\alpha(V) = W$.

Let’s denote $I_\alpha = Ker \varphi_\alpha \cap V$ and $\tilde{I}_\alpha$ the ideal in the Lie algebra generated by $I_\alpha$. Using Lemma 14 in [Z1] we have that $\tilde{I}_\alpha \cap V = I_\alpha$.

Let $\mathcal{G}$ be the Tits-Kantor-Koecher construction associated to the Jordan pair $W$. A $Z/lZ$-gradation of $W$ induces a $Z/lZ$-gradation of $\mathcal{G}$ and so $\mathcal{G}$ is $Z \times Z/lZ$-graded. The 0 component of this $Z \times Z/lZ$-gradation contains a Cartan subalgebra $H$.

Every $Z \times Z/lZ$-homogeneous component of $\mathcal{G}$ decomposes as a sum of eigenspaces with respect to the action of $H$. All the eigenspaces have dimension 1 and there exists a nonzero eigenvector $x$ such that $[[\mathcal{G}, x], x] = Fx$. Hence, every homogeneous component $W_\sigma$ contains a non zero element $a'$ such that $\{a', W^{-\sigma}, a'\} = Fa'$.

Choose an integer $q \geq k$, $q = p \pmod{l}$ and let $a' \otimes t^q = a \in \sum_{i \geq k} \mathcal{L}(W)_i \subseteq V$.

By Lemma 5.1 the ideal $id_V(a)$ of the Jordan pair (generated by the element $a$) contains a $\sum_{i \geq m} \mathcal{L}(W)_i$, for some $m \geq k$. 


By Lemma 4.4(1), the subpair $\sum_{i \geq m} L(W)_i$ is finitely generated. Choose, inside of the ideal $id_L(a)$ generated by $a$ in the algebra $L$, a finite set of elements $a_i = aad(x_{i1}) \cdots ad(x_{is})$, $1 \leq i \leq s$, $x_{ij} \in L$ that are $0Z \times 0Z/1Z$-homogeneous and include generators of $\sum_{i \geq m} L(W)_i$.

Consider $L' = \langle a_1, \ldots, a_s \rangle$ the subalgebra generated by the elements $a_1, \ldots, a_s$, $m = 2^r_1 + \cdots + 2^r_s$ (as in Lemma 2.1) and $T$ the $T$-ideal generated by all identities satisfied by all Lie algebras of dimension $\leq R(m)$.

For an arbitrary scalar, $0 \neq \alpha \in F$, we have $\varphi_{\alpha}(a) = \alpha^s a'$

Hence $[[\varphi_{\alpha}(L), \varphi_{\alpha}(a)], \varphi_{\alpha}(a)] \subseteq \{a', W^{-\sigma}, a'\} = F a' = F \varphi_{\alpha}(a)$.

By Lemma 2.1, the Lie algebra $\varphi_{\alpha}(L')$ satisfies all the identities of $T$. Since $\cap_{0 \neq \alpha \in F} I_\alpha = (0)$ (notice that $(\cap_{0 \neq \alpha \in F} I_\alpha) \cap V = \emptyset$), it follows that $T(L') = (0)$.

Let $J(L')$ a $Z \times Z/|Z|$-graded maximal ideal of $L'$ such that $J(L') \cap L'_{(n)} = J(L') \cap L'_{(-n)} = (0)$ (it exists by Zorn Lemma). The Jordan pair $(L'_{(-n)}, L'_{(n)})$ is prime and nondegenerate by Lemma 4.4(1).

An arbitrary non-zero graded ideal of $L'/J(L')$ has nonzero intersection with the pair $(L'_{(-n)}, L'_{(n)})$. By Lemma 3.2, the algebra $L'/J(L')$ is prime and nondegenerate. Furthermore, $T(L'/J(L')) = (0)$, so $L'/J(L')$ is strongly PI. Using Lemma 2.6(2) and Mathieu's theorem [see [Ma2]], $(T_h (L'/J(L')) \setminus \{0\})^{-1}(L'/J(L'))$ is isomorphic to a loop algebra $L(G)$. By Lemma 4.1, $\dim_F (G) \leq m = \max(d(2d+1),248)$. Let $T_m$ be the ideal of the free Lie that consists of all the identities that are satisfied identically in all Lie algebras of dimension $\leq m$. Then $T_m(L') \subseteq J(L')$ and so $T_m(L') \cap L'_{(n)} = (0)$.

Since $L'$ is an arbitrary finitely generated subalgebra of $id_L(a)$ containing a given (finite) subset and such subalgebras cover the ideal $id_L(a)$, we conclude that $T_m(id_L(a)) \cap L'_{(n)} = (0)$.

But the ideal $T_m(id_L(a))$ of $id_L(a)$ is invariant with respect to all the derivations of $id_L(a)$. Hence $T_m(id_L(a))$ is an ideal of $L$. By Lemma 3.1(1), $T_m(id_L(a)) \cap L'_{(n)} = (0)$ implies $T_m(id_L(a)) = (0)$. So the algebra $id_L(a)$ is strongly PI. Finally it suffices to apply Lemma 2.9 to finish the proof of Theorem 1.

6. References


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