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Poisson Algebras of Spinor Functions

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Abstract. Poisson algebras of spinor-valued functions arise as we extend the classical Hamiltonian formalism to vector-valued symplectic forms.

1. Introduction

The classical Hamiltonian formalism involves a non-degenerate skew-symmetric bilinear form ψ on a finite dimensional real vector space $\mathfrak{v}$; a space of functions, $C^\infty(\mathfrak{v})$, or just the polynomial functions $S(\mathfrak{v})$; a Lie algebra of first order differential operators, $\mathrm{Vect}(\mathfrak{v})$, or $\mathrm{Vect}_{pol}(\mathfrak{v})$, acting on the function space; and a map from the former to the latter, $f \mapsto H_f$. These objects satisfy a number of relations coded into the fact that the Poisson bracket

$$\{f, g\} = \psi(H_f, H_g)$$

defines a Lie algebra structure on the function space.

Furthermore, as H. Weyl observed, both the Heisenberg Lie algebra $\mathfrak{n}_\psi = \mathfrak{v} \times \mathbb{R}$ with bracket

$$[(v,t),(v',t')] = (0, \psi(v,v'))$$

and the symplectic Lie algebra $\mathfrak{sp}(\psi)$ are naturally subalgebras of the Poisson Lie algebra — namely those constituted by the polynomials of degree ≤ 1 and of degree 2, respectively. These identifications are compatible with the inclusion $\mathfrak{sp}(\psi) \hookrightarrow \mathrm{Der}(\mathfrak{n}_\psi)$ and, moreover, $\mathrm{Der}(\mathfrak{n}_\psi) = \mathfrak{sp}(\psi) \oplus \mathfrak{v} \oplus \mathbb{R}\delta$ with $\mathfrak{v}$ acting by inner derivations and $\delta \cdot (v,t) = (v, 2t)$.

In this article we generalize this formalism to vector-valued skew-symmetric forms

$$\Phi : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$$

which are *symplectic*, in the sense that there exist inner products in $\mathfrak{v}$ and in $\mathfrak{z}$ such that the transformations $J_z \in \mathrm{End}(\mathfrak{v})$ defined by

$$\langle J_z u, v\rangle_{\mathfrak{v}} = \langle z, \Phi(u,v)\rangle_{\mathfrak{z}} \tag{1.1}$$

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satisfy $J_z^2 = -|z|^2 I$ or, polarizing the latter,

$$J_z J_w + J_w J_z = -2\langle z, w\rangle I.$$

The datum of Φ and the two compatible inner products is therefore equivalent to a structure of $\mathrm{C}(\mathfrak{z})$-unitary module on $\mathfrak{v}$.

The inner product in $\mathfrak{z}$ is determined by Φ up to a positive multiple, as we explain below. We will fix one and use it as freely as Φ itself. On the other hand, the existence of one compatible inner product on $\mathfrak{v}$ implies that of infinitely many. Of course, the "Weyl Calculus" should be purely symplectic and not depend on any particular choice of metric in $\mathfrak{v}$, as will indeed be the case.

The Poisson Lie algebra attached to a symplectic Φ will be modeled on the vector space

$$\tilde{\mathcal{F}} \;=\; C^\infty(\mathfrak{v}\times\mathfrak{z}) \oplus \Lambda^2\mathfrak{z}^*.$$

The Hamiltonian vector fields will be ordinary vector fields on $\mathfrak{v}\times\mathfrak{z}$ acting on $\tilde{\mathcal{F}}$ as linear differential operators. The role of the Heisenberg Lie algebra will be played by $\mathfrak{n} = \mathfrak{v}\times\mathfrak{z} \cong \mathfrak{v}\oplus\mathfrak{z}$ endowed with the bracket

$$[(v, z), (v', z')] = (0, \Phi(v, v'));$$

$\mathfrak{n}$ is a two-step nilpotent Lie algebra with center $\mathfrak{z}$, often called *of Heisenberg type* [4]. One has

$$\mathrm{Der}(\mathfrak{n}) \cong \mathfrak{sp}(\Phi) \oplus \mathrm{Hom}(\mathfrak{v}, \mathfrak{z}) \oplus \mathbb{R}\delta,$$

where

$$\mathfrak{sp}(\Phi) = \{(A, B) \in \mathfrak{sl}(\mathfrak{v}) \times \mathfrak{sl}(\mathfrak{z}) : \Phi(Au, v) + \Phi(u, Av) = B\Phi(u, v)\}.$$

Moreover, letting

$$\mathfrak{sp}_o(\Phi) = \{A \in \mathfrak{gl}(\mathfrak{v}) : \Phi(Au, v) + \Phi(u, Av) = 0\}$$

one has

$$\mathfrak{sp}(\Phi) \cong \mathfrak{sp}_o(\Phi) \oplus \mathfrak{so}(\mathfrak{z}),$$

with $\mathfrak{so}(\mathfrak{z})$ acting on $\mathfrak{v}$ by a direct sum of spin representations [9]. All these Lie algebras will be realized as subalgebras of the Poisson algebra, defined by algebraic and differential conditions along $\mathfrak{z}$.

A similar calculus is obtained if we replace functions and vector fields on $\mathfrak{v}\times\mathfrak{z}$ by objects defined on $\mathfrak{v}\times S(\mathfrak{z})$, with $S(\mathfrak{z})$ the unit sphere in $\mathfrak{z}$ and we replace the covariant derivative D by the induced one on the sphere. While that alternative setup is more natural in some ways, the present one simplifies the calculations considerably and induces the alternative one upon restriction.

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2. Preliminaries

Fix a symplectic $\Phi : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ and a corresponding inner product $\langle z, w\rangle$ in $\mathfrak{z}$. This determines the linear family of ordinary skew-forms on $\mathfrak{v}$

$$\phi_z(u,v) = \langle z, \Phi(u,v)\rangle.$$

which are non-degenerate for all $z \neq 0$. It should be emphasized that this non-degeneracy is strictly weaker than the condition for Φ to be symplectic [7], but it does not seem to lead to an analogous generalization of the Weyl Calculus.

For $z, w \in \mathfrak{z}$, $z \neq 0$, define $K_{z,w}$, $A_{z,w} \in \mathrm{End}(\mathfrak{v})$ by

$$\phi_z(K_{z,w}u, v) = -|z|^2\phi_w(u,v) \tag{2.1}$$

$$A_{z,w} = \frac{1}{2}\big(K_{z,w} + \langle z,w\rangle I\big).$$

Both operators depend linearly on z and w. Since $A_{z,w} = -A_{w,z}$, one has a linear map

$$A : \Lambda^2\mathfrak{z} \to \mathrm{End}(\mathfrak{v}).$$

Explicitly,

$$A : \sum_{i<j} c_{ij} z_i \wedge z_j \mapsto \sum_{i<j} c_{ij} A_{z_i,z_j}.$$

Identifying $\mathfrak{z}$ with $\mathfrak{z}^*$ via the given inner product, ones gets a linear map $\alpha \mapsto A_\alpha$ from $\Lambda^2\mathfrak{z}^*$ into $\mathrm{End}(\mathfrak{v})$.

Note that $K_{z,w}$ and $A_{z,w}$ are defined by the linear family of forms ϕ_z, independent of any metric in $\mathfrak{v}$. If, however, a compatible metric is chosen so that the J_z are defined by (1.1), then

$$K_{z,w} = J_zJ_w.$$

Identify $\Lambda^2\mathfrak{z}^*$ with the orthogonal Lie algebra $\mathfrak{so}(\mathfrak{z})$ in the usual way: $B \in \mathfrak{so}(\mathfrak{z})$ is identified with the 2-form $\alpha_B(z,w) = \langle Bz, w\rangle$, or, equivalently, $\alpha \in \Lambda^2\mathfrak{z}^*$ is identified with the element $B_\alpha \in \mathfrak{so}(\mathfrak{z})$ such that $\langle B_\alpha z, w\rangle = \alpha(z,w)$.

Recall that the map $z \mapsto J_z$ extends to a representation of the Clifford algebra $\mathrm{C}(\mathfrak{z})$ by endomorphisms of $\mathfrak{v}$. The multiplicative subgroup of $\mathrm{C}(\mathfrak{z})$ generated by the double products zz', with $|z| = |z'| = 1$, is $\mathrm{Spin}(\mathfrak{z})$ and the corresponding representation on $\mathfrak{v}$ is a direct sum of spin representations.

Proposition 2.1. *With the identification $\Lambda^2\mathfrak{z}^* \cong \mathfrak{so}(\mathfrak{z})$,*

(a) $\alpha \mapsto A_\alpha$ is the spin representation of $\mathfrak{so}(\mathfrak{z})$,

(b) $(A_\alpha, B_\alpha) \in \mathfrak{sp}(\Phi)$.

Proof. For (a), just note that the spin representation satisfies

$$4J_{z\wedge w}v = J_zJ_wv - J_wJ_zv$$

(see, e.g., Corollary I.6.3 in [6]) and that, in C($\mathfrak{z}$), $zw + wz = -2\langle z, w\rangle$ for all $z, w \in \mathfrak{z}$. For (b), take $s, z, w \in \mathfrak{z}$ and $u, v \in \mathfrak{v}$ and compute

$$\begin{aligned}\langle s, \Phi(J_z J_w u, v)\rangle &= \langle J_s J_z J_w u, v\rangle \\ &= -\langle J_z J_s J_w u, v\rangle - 2\langle z, s\rangle\langle J_w u, v\rangle \\ &= \langle J_z J_w J_s u, v\rangle + 2\langle s, w\rangle\langle J_z u, v\rangle - 2\langle z, s\rangle\langle J_w u, v\rangle \\ &= \langle J_s u, J_w J_z v\rangle + 2\langle s, w\rangle\langle J_z u, v\rangle - 2\langle z, s\rangle\langle J_w u, v\rangle.\end{aligned}$$

Since $J_w J_z = -J_z J_w - 2\langle z, w\rangle I$,

$$\Phi(J_z J_w u, v) + \Phi(u, J_z J_w v) = -2\langle z, w\rangle\Phi(u, v) + 2\langle z, \Phi(u, v)\rangle w - 2\langle w, \Phi(u, v)\rangle z.$$

Let $B_{z,w}$ be the infinitesimal rotation in $\mathfrak{z}$ defined by

$$B_{z,w}(z') = \langle z, z'\rangle w - \langle w, z'\rangle z$$

and recall that $A_{z,w} = \frac{1}{2}(K_{z,w} + \langle z, w\rangle I)$. One obtains

$$\Phi(A_{z,w} u, v) + \Phi(u, A_{z,w} v) = B_{z,w}\Phi(u, v).$$ ■

Recall that $\mathfrak{n} = \mathfrak{v} \times \mathfrak{z} \cong \mathfrak{v} \oplus \mathfrak{z}$ endowed with the bracket

$$[(v, z), (v', z')] = (0, \Phi(v, v'))$$

is a two-step nilpotent Lie algebra, with center $\mathfrak{z}$. Its algebra of derivations has the following structure [9]. If $(A, B) \in \mathfrak{sp}(\Phi)$, then $(v, z) \mapsto (Av, Bz)$ is a derivation of $\mathfrak{n}$, yielding an inclusion $\mathfrak{sp}(\Phi) \hookrightarrow \mathrm{Der}(\mathfrak{n})$. $\mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$ is also contained in $\mathrm{Der}(\mathfrak{n})$, as the abelian subalgebra consisting of the maps $(v, z) \mapsto (0, T(v))$, $T \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$. Furthermore, one has the semidirect sum decomposition

(2.2) $$\mathrm{Der}(\mathfrak{n}) = \mathfrak{sp}(\Phi) \oplus \mathrm{Hom}(\mathfrak{v}, \mathfrak{z}) \oplus \mathbb{R}\delta$$

and the direct sum decomposition

(2.3). $$\mathfrak{sp}(\Phi) = \mathfrak{sp}_o(\Phi) \oplus \mathfrak{so}(\mathfrak{z})$$

Both $\mathfrak{sp}(\Phi)$ and $\mathfrak{sp}_o(\Phi)$ are real reductive Lie algebras, the latter acts trivially on the center, $\mathfrak{so}(\mathfrak{z})$ acts by rotations on $\mathfrak{z}$ and by the spin representation on $\mathfrak{v}$ and $\delta(v, z) = (v, 2z)$. In matrix form, if $\mathfrak{z} \cong \mathbb{R}^m$ and $\mathfrak{v} \cong \mathbb{R}^n$, then

$$\mathrm{Der}(\mathfrak{n}) \cong \{\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : \Phi(Au, v) + \Phi(u, Av) = B\Phi(u, v)\}$$

where A, B, C, are $n \times n$, $m \times m$ and $m \times n$ real matrices, respectively,

$$\mathfrak{sp}(\Phi) \cong \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \Phi(Au, v) + \Phi(u, Av) = B\Phi(u, v), \quad tr(A) = tr(B) = 0\}$$

$$\mathfrak{sp}_o(\Phi) \cong \{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : \Phi(Au, v) + \Phi(u, Av) = 0, \},$$

$$\mathrm{Hom}(\mathfrak{v},\mathfrak{z}) \cong \{\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\}, \qquad \delta \cong \{\begin{pmatrix} 2I_n & 0 \\ 0 & I_m \end{pmatrix}\}.$$

The presence of the summand $\mathfrak{so}(\mathfrak{z})$ shows that the inner product in $\mathfrak{z}$ is determined by Φ up to a positive scalar. Indeed, $\mathfrak{sp}(\Phi)$ and $\mathfrak{sp}_o(\Phi)$ are defined by Φ and are both real reductive, hence the sum of an abelian and a semisimple subalgebra. Therefore $\mathfrak{so}(\mathfrak{z})$ is the semisimple part of the centralizer of $\mathfrak{sp}_o(\Phi)$ in $\mathfrak{sp}(\Phi)$ and, as such, it is canonically attached to Φ. Since it acts in the standard irreducible manner on $\mathfrak{z}$ and preserves the inner product, the latter is unique up to homotheties. Furthermore,

$$\mathfrak{so}(\mathfrak{z}) \cong \{\begin{pmatrix} A_B & 0 \\ 0 & B \end{pmatrix} : B \in \mathfrak{so}(m)\}.$$

where $B \mapsto A_B$ is a direct sum of spin representations of $\mathfrak{so}(m)$.

Let

$$\mathcal{F} = C^\infty(\mathfrak{n}) = C^\infty(\mathfrak{v} \times \mathfrak{z}).$$

Those functions $f(v,z)$ which are polynomial in both v and z constitute a bi-graded subspace of $\mathcal{F}$

$$\mathcal{F}^{(\cdot,\cdot)} = \bigoplus_{p,q\geq 0} \mathcal{F}^{(p,q)},$$

where $\mathcal{F}^{(p,q)}$ are the polynomials which are homogeneous of degree p in v and of degree q in z.

For any subspace $\mathcal{P} \subset \mathcal{F}$ we let

$$\tilde{\mathcal{P}} = \mathcal{P} \oplus \Lambda^2\mathfrak{z}^*.$$

An element of $\tilde{\mathcal{F}}$ can be viewed as a function $F : \mathfrak{v} \times \mathfrak{z}^3 \to \mathbb{R}$ of the form

$$F(v, z_1, z_2, z_3) = f(v, z_1) + \alpha(z_2, z_3),$$

with f smooth and α bilinear and skew-symmetric.

We will consistently identify a vector space $\mathfrak{u}$ with its tangent space at each point and denote by

$$D_u f = uf = u \cdot f$$

($u \in \mathfrak{u}$) the derivative of the function f in the direction u. If X is a vector field on $\mathfrak{u}$, $D_u X$ will denote the canonical covariant derivative of X in the direction u. Constant vector fields will be identified with the corresponding elements of $\mathfrak{u}$. If X is a vector field on $\mathfrak{n} = \mathfrak{v} \times \mathfrak{z}$, we will sometimes write $X = X' + X''$ with X', X'', tangent to $\mathfrak{v}$ and $\mathfrak{z}$, respectively. If X, Y, are vector fields on $\mathfrak{n}$ which are tangential to $\mathfrak{v}$, we will denote by $\phi(X,Y)$ the function on $\mathfrak{n}$

$$\phi(X,Y)(v,z) = \phi_z(X_{(v,z)}, Y_{(v,z)}) = \langle z, \Phi(X_{(v,z)}, Y_{(v,z)})\rangle$$

3. The main result

The differential equation

$$D_z D_x f(v,s) + |s|^{-2} D_{K_{s,z}(x)} f(v,s) = 0 \tag{3.1}$$

where $s, z \in \mathfrak{z}$ and $v, x \in \mathfrak{v}$, is linear and homogeneous of degree -1 in each of the variables v and s. Therefore, the set $\mathcal{E} \subset \mathcal{F}$ of its solutions is a linear subspace, containing

$$\mathcal{E}^{(\cdot,\cdot)} = \bigoplus_{p,q \geq 0} \mathcal{E}^{(p,q)}, \qquad \mathcal{E}^{(p,q)} := \mathcal{E} \cap \mathcal{F}^{(p,q)},$$

as a dense subspace. For emphasis: $\mathcal{E}^{(p,q)}$ consists of the polynomial functions on $\mathfrak{v} \times \mathfrak{z}$ of bidegree (p,q) which satisfy the equation (3.1).

Theorem 3.1. *There exist an extension of the natural action of* $\mathrm{Vect}(\mathfrak{n})$ *on* $\mathcal{F} = C^\infty(\mathfrak{n})$ *to a bilinear map* $\mathrm{Vect}(\mathfrak{n}) \times \tilde{\mathcal{F}} \to \mathcal{F}$*,*

$$(X, F) \mapsto X \cdot F,$$

a linear map $\tilde{\mathcal{F}} \to \mathrm{Vect}(\mathfrak{n})$

$$F \mapsto H_F$$

and a bilinear operation $\tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$

$$(F, G) \mapsto \{F, G\},$$

satisfying the following properties. Let $X, Z,$ *be vector fields on* $\mathfrak{n}$ *tangential to* $\mathfrak{v}$ *and* $\mathfrak{z}$ *respectively, and let* $F, G \in \tilde{\mathcal{F}}$*, with* $F = f + \alpha$*,* $f \in \mathcal{F}$*,* $\alpha \in \Lambda^2\mathfrak{z}^*$*. Then:*

(a) $\phi(H'_F, X) = X \cdot F$ *and* $\langle H''_F, Z \rangle = Z \cdot \alpha$

(b) $f \in \mathcal{E} \;\Leftrightarrow\; D_Z(H_f) = 0$

(c) $H_{H_\alpha f} = D_{H_\alpha}(H_f) + A_\alpha(H_f)$

(d) $[H_F, H_G] = H_{\{F,G\}}$

(e) $\tilde{\mathcal{F}}$ *is a Lie algebra under* $\{\cdot,\cdot\}$ *and* $\mathcal{E}$*,* $\tilde{\mathcal{E}}$*,* $\tilde{\mathcal{E}}^{(\cdot,1)}$ *are subalgebras.*

(f) $\mathcal{E}^{(1,1)} \oplus \mathcal{E}^{(0,1)}$ *is a subalgebra, isomorphic to* $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ *as a graded Lie algebra.*

(g) $\mathcal{E}^{(2,1)}$ *and* $\tilde{\mathcal{E}}^{(2,1)}$ *are subalgebras, with* $\mathfrak{sp}_o(\Phi) \cong \mathcal{E}^{(2,1)}$ *and* $\mathfrak{sp}(\Phi) \cong \tilde{\mathcal{E}}^{(2,1)}$

(h) $\mathrm{Der}(\mathfrak{n}) \cong \tilde{\mathcal{E}}^{(2,1)} \oplus \mathcal{F}^{(1,1)} \oplus \mathbb{R}\delta$ *and, with the identification in (f), the first two terms act on* $\mathfrak{n}$ *by inner derivations of the Poisson algebra.*

Proof. Define the bilinear map $\mathrm{Vect}(\mathfrak{n}) \times \tilde{\mathcal{F}} \to \mathcal{F}$ by

$$Y \cdot (f + \alpha) = D_Y f + Y \cdot \alpha$$

where $f \in \mathcal{F}$, $\alpha \in \Lambda^2\mathfrak{z}^*$ and the last term is the function on $\mathfrak{n}$ defined by

$$(Y \cdot \alpha)(v,s) = -\phi_s(A_\alpha v, Y'_{(v,s)}) - \alpha(s, Y''_{(v,s)}) \tag{3.2}$$

which is clearly smooth. By definition, $Y \cdot \tilde{\mathcal{F}} \subset \mathcal{F}$ and $Y \cdot f = D_Y f = Yf$ for $f \in \mathcal{F}$.

Define the Hamiltonian vector field associated to a $F = f + \alpha \in \tilde{\mathcal{F}}$ as the element in $\mathrm{Vect}(\mathfrak{n})$ given by $H_{f+\alpha} = H_f + H_\alpha$, where H_f is determined by the conditions

$$\phi_s((H_f')_{(v,s)}, X) = D_X f(v,s), \qquad H_f'' = 0 \tag{3.3}$$

while

$$(H_\alpha)_{(v,s)} = -A_\alpha v - B_\alpha s. \tag{3.4}$$

If we set $f_z(v) = f(v,z)$, then (3.3) says that $(H_f)_{(\cdot,z)}$ is the usual Hamiltonian vector field of f_z with respect to the symplectic form ϕ_z. The Hamiltonian H_α of a 2-form, on the other hand, is an infinitesimal spin in $\mathfrak{v}$ and an infinitesimal rotation in $\mathfrak{z}$, namely, those determined by $-\alpha$.

Finally, define the Poisson bracket in $\tilde{\mathcal{F}}$ by

$$\{f+\alpha, g+\beta\} = \{f,g\} + H_\alpha \cdot g - H_\beta \cdot f + [\alpha,\beta], \tag{3.5}$$

where $[\alpha,\beta]$ denotes the Lie bracket in $\Lambda^2\mathfrak{z}^* \cong \mathfrak{so}(\mathfrak{z})$ and $\{f,g\} = H_g \cdot f$.

We will now prove that the objects just defined satisfy (a) to (h).

(a) By (3.2), (3.3) and (3.4), one has

$$\begin{aligned}
\phi_s((H_f)_{(v,s)}, X) &= D_X f(v,s),\\
\phi_s((H_\alpha')_{(v,s)}, X) &= X \cdot \alpha(v,s),\\
\langle (H_\alpha'')_{(v,s)}, Z\rangle &= -\langle B_\alpha s, Z\rangle = -\alpha(s,Z) = Z \cdot \alpha(v,s).
\end{aligned}$$

(b) Differentiate the equation $\phi_s((H_f)_{(v,s)}, X) = D_X f(v,s)$ with respect to s in the direction z. Since $s \mapsto \phi_s$ is linear and X can be assumed to be a constant vector field, one gets

$$\phi_z((H_f)_{(v,s)}, X) + \phi_s((D_z H_f)_{(v,s)}, X) = D_X D_z f(v,s). \tag{3.6}$$

By (2.1) with $u = X$ and $v = H_f$,

$$\phi_s(H_f, K_{s,z}X) = -|s|^2 \phi_z(H_f, X)$$

and therefore

$$K_{s,z}(X) \cdot f = \phi_s(H_f, K_{s,z}X) = -|s|^2\phi_z(H_f, X). \tag{3.7}$$

We see from (3.6), (3.7) and the non-degeneracy of ϕ_z, that

$$f \in \mathcal{E} \qquad \Leftrightarrow \qquad K_{s,z}(X)f + |s|^2 D_X D_z f = 0 \qquad \Leftrightarrow \qquad D_z H_f = 0.$$

(c) Differentiating the function $X \cdot f(v,s) = \phi_s((H_f)_{(v,s)}, X)$ with respect to v in the direction $(H_\alpha)_{(v,s)} = -A_\alpha v - B_\alpha s$ while taking $X \in \mathfrak{v}$ constant, we get

$$-\phi_{B_\alpha(s)}((H_f)_{(v,s)}, X) + \phi_s((D_{H_\alpha} H_f)_{(v,s)}, X)) = [H_\alpha, X] \cdot f(v,s) + XH_\alpha \cdot f(v,s)$$

On one hand,

$$-\phi_{B_\alpha(s)}(u,v) = -\langle B_\alpha(s), \Phi(u,v)\rangle = \langle s, B_\alpha(\Phi(u,v))\rangle.$$

Since $(A_\alpha, B_\alpha) \in \mathfrak{sp}(\Phi)$,

$$\langle s, B_\alpha(\Phi(u,v))\rangle = \langle s, \Phi(A_\alpha u, v)\rangle + \langle s, \Phi(u, A_\alpha v)\rangle = \phi_s(A_\alpha u, v) + \phi_s(u, A_\alpha v),$$

so that

$$-\phi_{B_\alpha(s)}((H_f)_{(v,s)}, X) = \phi_s(A_\alpha(H_f)_{(v,s)}, X) + \phi_s((H_f)_{(v,s)}, A_\alpha(X)).$$

On the other hand,

$$[H_\alpha, X] = D_{H_\alpha}(X) - D_X(H_\alpha) = 0 + A_\alpha(X).$$

Therefore

$$\begin{aligned}\phi_s(A_\alpha(H_f)_{(v,s)}, X) + \phi_s((H_f)_{(v,s)}, A_\alpha(X)) + \phi_s((D_{H_\alpha}H_f)_{(v,s)}, X)& \\ = (A_\alpha(X) + XH_\alpha)\cdot f(v,s).&\end{aligned}$$

The terms $\phi_s((H_f)_{(v,s)}, A_\alpha(X)) = A_\alpha(X)f(v,s)$ cancel out and $XH_\alpha \cdot f(v,s) = \phi_s((H_{H_\alpha f})_{(v,s)}, X)$, so the equation becomes

$$\phi_s((A_\alpha H_f + D_{H_\alpha}H_f)_{(v,s)}, X) = \phi_s((H_{H_\alpha f})_{(v,s)}, X),$$

proving the assertion.

(d) If $F = f$ and $G = g$ are in $\mathcal{F}$, the assertion reduces to the standard identity for ordinary symplectic forms, because of the remark after (3.4). If $F = \alpha$ and $G = \beta$ are in $\Lambda^2\mathfrak{z}^*$, (3.5) reduces to $\{\alpha, \beta\} = [\alpha, \beta]$. Since

$$(H_\alpha)_{(v,s)} = -A_\alpha v - B_\alpha s$$

and $\alpha \mapsto A_\alpha$ and $\alpha \mapsto B_\alpha$ are Lie algebra morphisms,

$$\begin{aligned}[H_\alpha, H_\beta]_{(v,s)} &= -[A_\alpha, A_\beta](v) - [B_\alpha, B_\beta](s)\\ &= -A_{[\alpha,\beta]}v - B_{[\alpha,\beta]}s = (H_{[\alpha,\beta]})_{(v,s)} = (H_{\{\alpha,\beta\}})_{(v,s)}.\end{aligned}$$

Finally, if $F = \alpha \in \Lambda^2\mathfrak{z}^*$ and $G = g \in \mathcal{F}$,

$$\begin{aligned}[H_\alpha, H_g] &= D_{H_\alpha}(H_g) - D_{H_g}(H_\alpha)\\ &= H_{H_\alpha g} - A_\alpha(H_g) + A_\alpha(H_g) = H_{H_\alpha g} = H_{\{\alpha, g\}}.\end{aligned}$$

The first equality is just the definition of the commutator of two vector fields. The second follows from

$$D_{H_\alpha}(H_g) = H_{H_\alpha g} - A_\alpha(H_g),$$

which is (c), and from

$$(D_{H_g}H_\alpha)_{(v,s)} = D_{H_g}(-A_\alpha v - B_\alpha s) = -A_\alpha(H_g),$$

while the third equality follows from the definition of $\{\alpha, g\}$.

(e) Our Poisson bracket is clearly bilinear and skew-symmetric, so we must prove that it satisfies Jacobi's identity. This identity holds in $\mathcal{F}$ because of the corresponding classical statement for scalar-valued forms, while $\Lambda^2\mathfrak{z}^*$ is already a Lie algebra. Therefore we need to verify it in the cases $\{\alpha,\{f,g\}\}$ and $\{f,\{\alpha,\beta\}\}$ $(\alpha,\beta\in\Lambda^2\mathfrak{z}^*,\ f,g\in\mathcal{F})$.

In the first case, we have

$$\{\alpha,\{f,g\}\}(v,s)=H_\alpha\cdot\{f,g\}(v,s),\qquad \{f,g\}(v,s)=\phi_s((H_f)_{(v,s)},(H_g)_{(v,s)})$$

so that

$$\begin{aligned}\{\alpha,\{f,g\}\}(v,s)&=H_\alpha\cdot\{f,g\}(v,s)\\&=\phi_{-B_\alpha(s)}((H_f)_{(v,s)},(H_g)_{(v,s)})+\phi_s((D_{H_\alpha}H_f)_{(v,s)},(H_g)_{(v,s)})\\&\qquad+\phi_s((H_f)_{(v,s)},(D_{H_\alpha}H_g)_{(v,s)}).\end{aligned}$$

But

$$\begin{aligned}\phi_{-B_\alpha(s)}((H_f)_{(v,s)},(H_g)_{(v,s)})&=-\langle B_a(s),\Phi((H_f)_{(v,s)},(H_g)_{(v,s)})\rangle\\&=\langle s,B_\alpha\Phi((H_f)_{(v,s)},(H_g)_{(v,s)})\rangle\end{aligned}$$

which can be written as

$$\begin{aligned}&\langle s,\Phi((A_\alpha H_f)_{(v,s)},(H_g)_{(v,s)})+\Phi((H_f)_{(v,s)},(A_\alpha H_g)_{(v,s)})\rangle\\&=\phi_s((A_\alpha H_f)_{(v,s)},(H_g)_{(v,s)})+\phi_s((H_f)_{(v,s)},(A_\alpha H_g)_{(v,s)}).\end{aligned}$$

Therefore

$$\begin{aligned}\{\alpha,\{f,g\}\}&=\phi(A_\alpha H_f,H_g)+\phi(H_f,A_\alpha H_g)+\phi(D_{H_\alpha}H_f,H_g)+\phi(H_f,D_{H_\alpha}H_g)\\&=\phi(A_\alpha H_f,H_g)+\phi(H_f,A_\alpha H_g)+\phi(H_{H_\alpha f}-A_\alpha H_f,H_g)+\\&\qquad\phi(H_f,H_{H_\alpha g}-A_\alpha H_g)\\&=\phi(H_{H_\alpha f},H_g)+\phi(H_f,H_{H_\alpha g})\\&=\phi(H_{\{\alpha,f\}},H_g)+\phi(H_f,H_{\{\alpha,g\}})\\&=\{\{\alpha,f\},g\}+\{f,\{\alpha,g\}\}.\end{aligned}$$

In the other case,

$$\begin{aligned}\{f,\{\alpha,\beta\}\}&=\{f,[\alpha,\beta]\}=-H_{[\alpha,\beta]}\cdot f=-[H_\alpha,H_\beta]\cdot f\\&=-H_\alpha H_\beta\cdot f+H_\beta H_\alpha\cdot f=-H_\alpha\cdot\{\beta,f\}+H_\beta\cdot\{\alpha,f\}\\&=-\{\alpha,\{\beta,f\}\}+\{\beta,\{\alpha,f\}\}=\{\{f,\alpha\},\beta\}+\{\alpha,\{f,\beta\}\}.\end{aligned}$$

We now prove that $\mathcal{E},\tilde{\mathcal{E}}\subset\tilde{\mathcal{F}}$ are subalgebras. From (b), we can deduce that $\mathcal{E}$ is a subalgebra if and only if $D_zH_{\{f,g\}}=0$ for $z\in\mathfrak{z}$ and $f,g\in\mathcal{E}$. The last equation follows from (d) and the fact that D_z is a derivation on vector fields. Since $H_{\{\alpha,f\}}=H_{H_\alpha f}$, it follows from (c) and (b) that $D_z(H_{\{\alpha,f\}})=0$ for all $z\in\mathfrak{z}$. Because of (b), $\{\alpha,f\}\in\mathcal{E}$. So, $\tilde{\mathcal{E}}$ is a subalgebra as well.

For $f \in \mathcal{E}^{(\cdot,1)}$ define $f_o \in C^\infty(\mathfrak{v}, \mathfrak{z})$ by $f(v,z) = \langle f_o(v), z\rangle$. Let now $f, g \in \mathcal{E}^{(\cdot,1)}$ and $\alpha \in \Lambda^2\mathfrak{z}^*$. Because of (b), H_f and H_g depend only on $v \in \mathfrak{v}$. Therefore, the functions

$$\{f,g\}(v,s) = \phi_s((H_f)_v, (H_g)_v)$$

and

$$\begin{aligned}\{\alpha, f\}(v,s) &= H_\alpha \cdot f(v,s) = -D_{A_\alpha v} f(v,s) - D_{B_\alpha s} f(v,s)\\ &= -\langle D_{A_\alpha v} f_o(v), s\rangle - \langle B_\alpha s, f_o(v)\rangle\end{aligned}$$

are both smooth in v and linear in s and therefore lie in $\mathcal{E}^{(\cdot,1)}$. Hence both $\mathcal{E}^{(\cdot,1)}$ and $\tilde{\mathcal{E}}^{(\cdot,1)}$ are subalgebras.

(f) For $u \in \mathfrak{v}$ and $z \in \mathfrak{z}$ define the real-valued functions on $\mathfrak{n}$:

$$q_u(v,s) = \langle s, \Phi(u,v)\rangle, \qquad c_z(v,s) = \langle s, z\rangle. \tag{3.8}$$

Then the map $\Theta : u + z \mapsto q_u + c_z$ determines a Lie isomorphism

$$N \cong \mathcal{E}^{(1,1)} \oplus \mathcal{E}^{(0,1)}.$$

Indeed, we easily see that $H_{q_u} = u$ and $H_{c_z} = 0$, so, from (b), $q_u + c_z \in \mathcal{E}^{(1,1)} \oplus \mathcal{E}^{(0,1)}$. Also,

$$\{q_u, q_{u'}\}(v,s) = (H_{q_{u'}} \cdot q_u)(v,s) = \langle s, \Phi(u,u')\rangle = c_{\Phi(u,u')}(v,s),$$

while $\{q_u, c_z\} = (H_{c_z} \cdot q_u) = 0$ and $\{c_z, c_{z'}\} = 0$. Since the bracket in $\mathfrak{n}$ is given by $[u+z, u'+z'] = \Phi(u,u')$, we have

$$\begin{aligned}\Theta([u+z, u'+z']) &= c_{\Phi(u,u')} = \{q_u, q_{u'}\} = \{q_u + c_z, q_{u'} + c_{z'}\}\\ &= \{\Theta(u+z), \Theta(u'+z')\},\end{aligned}$$

so Θ is a Lie morphism. To see that it is surjective, let $g \in \mathcal{E}^{(1,1)}$. Since g is bilinear, there exist $T \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$ such that

$$g(v,s) = g_T(v,s) := \langle s, Tv\rangle, \quad \text{for } v \in \mathfrak{v}, s \in \mathfrak{z}. \tag{3.9}$$

Because $g \in \mathcal{E}$, H_g is constant along $\mathfrak{z}$. Therefore

$$\langle s, \Phi((H_g)_v, X)\rangle = \phi_s((H_g)_v, X) = X \cdot g(v,s) = \langle s, TX\rangle.$$

We conclude that $\Phi((H_g)_v, X) = TX$ and, consequently, H_g is also independent of v. Letting $u = H_g \in \mathfrak{v}$,

$$g(v,s) = \langle s, Tv\rangle = \langle s, \Phi(u,v)\rangle,$$

from which $g = q_u$. On the other hand, if $g \in \mathcal{E}^{(0,1)}$, then $g(v,s) = \langle s, z\rangle$ with $z \in \mathfrak{z}$, hence $g = c_z$, showing that the map is onto.

(g) We must prove that the operators $G \mapsto \{F, G\}$ with $F \in \mathcal{E}^{(2,1)}$ realize, upon restriction to $\mathcal{E}^{(1,1)} \oplus \mathcal{E}^{(0,1)} \cong \mathfrak{n}$, all of $\mathfrak{sp}_o(\Phi)$, viewed as subalgebra of $\mathrm{Der}(\mathfrak{n})$.

Let $f \in \mathcal{E}^{(2,1)} = \mathcal{E} \cap \mathcal{F}^{(2,1)}$, i.e., $f(v,z)$ is a homogeneous polynomial of degree 2 in v and of degree 1 in $\mathfrak{z}$, satisfying the differential equation (3.1). As we have already observed, its Hamiltonian H_f is just the classical Hamiltonian relative to ϕ_s and it is independent of $s \in \mathfrak{z}$. Therefore

$$\phi_s((H_f)_v, v') + \phi_s(v, (H_f)_v) = 0,$$

showing that $H_f \in \mathfrak{sp}_o(\Phi)$. Since $H_g f = \{f, g\}$, we conclude that $\mathcal{E}^{(2,1)}$ acts on $\mathfrak{n}$ as $\mathfrak{sp}_o(\Phi)$.

To see that $\mathcal{E}^{(2,1)}$ is isomorphic to $\mathfrak{sp}_o(\Phi)$ define, for any $Q \in \mathfrak{sp}_o(\Phi)$, the function

$$p_Q(v,s) := \frac{1}{2}\langle s, \Phi(Qv, v)\rangle \quad v \in \mathfrak{v}, s \in \mathfrak{z} \tag{3.10}$$

and prove that $H_{p_Q} = Q$. Indeed, $\phi_s(Qu, v) + \phi_s(u, Qv) = 0$, so

$$\phi_s((H_{p_Q})_{(v,s)}, X) = X \cdot p_Q(v,s) = \frac{1}{2}\phi_s(QX, v) + \frac{1}{2}\phi_s(Qv, X) = \phi_s(Qv, X).$$

Therefore $p_Q \in \mathcal{E}^{(2,1)}$. Moreover, two functions in $\mathcal{E}^{(2,1)}$ with the same Hamiltonian are equal, therefore $f \mapsto H_f$ is the inverse map of $Q \mapsto P_Q$. Because of (d), $F \mapsto H_F$ is a Lie morphism. From Proposition 2.1 and the fact that $\tilde{\mathcal{E}}^{(2,1)} = \mathcal{E}^{(2,1)} \oplus \Lambda^2\mathfrak{z}^*$, we conclude that $\mathfrak{sp}(\Phi) \cong \tilde{\mathcal{E}}^{(2,1)}$ and, therefore, that $\tilde{\mathcal{E}}^{(2,1)}$ acts as $\mathfrak{sp}(\Phi)$.

(h) Identifying $\mathfrak{n}$ with $\mathcal{E}^{(1,1)} \oplus \mathcal{E}^{(0,1)}$, $\mathfrak{sp}_0(\Phi)$ with $\mathcal{E}^{(2,1)}$ and $\mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$ with $\mathcal{F}^{(1,1)}$, we must prove that the action of $\mathrm{Der}(\mathfrak{n})$ on $\mathfrak{n}$ is by inner derivations of the Poisson bracket. Equivalently, that the functions q_u, c_z, p_Q and g_T, as defined in (3.8), (3.9) and (3.10), satisfy the following commutation relations:

(*i*) $\{p_Q, q_u\} = q_{Qu}$

(*ii*) $\{\alpha, q_u + c_z\} = q_{A_\alpha u} + c_{B_\alpha z}$

(*iii*) $\{g_T, q_u\} = c_{Tu}$

(*iv*) $\{g_T, c_z\} = 0$

for all $u \in \mathfrak{v}$, $z \in \mathfrak{z}$, $\alpha \in \Lambda^2\mathfrak{z}^*$, $T \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$ and $Q \in \mathfrak{sp}_o(\Phi)$.

To prove (*i*), just compute

$$q_u(v,s) = -H_{p_Q} q_u(v,s) = -\langle s, \Phi(u, Qv)\rangle = \phi_s(Qu, v) = q_{Qu}(s,v).$$

For (*ii*),

$$\begin{aligned}
\{\alpha, q_u\}(v,s) &= H_\alpha \cdot q_u(v,s) = -D_{A_\alpha v} q_u - D_{B_\alpha s} q_u \\
&= -\langle s, \Phi(u, A_\alpha v)\rangle - \langle B_\alpha s, \Phi(u,v)\rangle \\
&= -\langle s, \Phi(u, A_\alpha v)\rangle + \langle s, B_\alpha \Phi(u,v)\rangle \\
&= -\langle s, \Phi(u, A_\alpha v)\rangle + \langle s, \Phi(A_\alpha u, v)\rangle + \langle s, \Phi(u, A_\alpha v)\rangle = \langle s, \Phi(A_\alpha u, v)\rangle \\
&= q_{A_\alpha u}(v,s),
\end{aligned}$$

and, similarly,

$$\{\alpha, c_z\}(v,s) = H_\alpha \cdot c_z(v,s) = -D_{B_\alpha s} c_z = -\langle B_\alpha s, z\rangle = \langle s, B_\alpha z\rangle = c_{B_\alpha z}(v,s).$$

Finally, (iii) follows from

$$\{g_T, q_u\}(v,s) = H_{q_u} \cdot g_T(v,s) = D_u g_T(v,s) = \langle s, Tu\rangle = c_{Tu}(v,s),$$

and (iv) from

$$\{q_T, c_z\}(v,s) = H_{c_z} \cdot g_T(v,s) = 0.$$ ■

Remark 3.2. $\mathcal{F}^{(\cdot,1)}$ is *not* closed under $\{\ ,\ \}$. Also, in general, $\{F,G\} \neq H_F \cdot G$; instead one has the identity (3.5). For example, let $F = f$ and $G = \alpha$. Then $\{f,\alpha\} = -H_\alpha f = (A_\alpha v)f + (B_\alpha s)f$ while $(H_f)_{(v,s)} \cdot \alpha = -\phi_s(A_\alpha v, H_f') - \alpha(s, H_f'') = \phi_s(H_f, A_\alpha v) = (A_\alpha v)f$.

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