Moduli for Spherical Maps and Minimal Immersions of Homogeneous Spaces

Gabor Toth

Communicated by F. Knop

Abstract. The DoCarmo-Wallach theory studies isometric minimal immersions \( f: G/K \to S^n \) of a compact Riemannian homogeneous space \( G/K \) into Euclidean \( n \)-spheres for various \( n \). For a given domain \( G/K \), the moduli space of such immersions is a compact convex body in a representation space for the Lie group \( G \). In 1971 DoCarmo and Wallach gave a lower bound for the (dimension of the) moduli for \( G/K = S^m \), and conjectured that the lower bound was achieved. In 1997 the author proved that this was true. The DoCarmo-Wallach conjecture has a natural generalization to all compact Riemannian homogeneous domains \( G/K \). The purpose of the present paper is to show that for \( G/K \) a nonspherical compact rank 1 symmetric space this generalized conjecture is false. The main technical tool is to consider spherical functions of subrepresentations of \( C^\infty(G/K) \), express them in terms of Jacobi polynomials, and use a recent linearization formula for products of Jacobi polynomials.

1. Introduction and Statement of Results

Let \( M = G/K \) be a Riemannian homogeneous space, where \( G \) is a compact Lie group and \( K \) a closed subgroup. Then \( G \) acts on the space \( C^\infty(M) \) of (real valued) functions on \( M \) in a natural way: \( g \cdot \xi = \xi \circ g^{-1}, \ g \in G, \ \xi \in C^\infty(M) \). This action preserves the \( L^2 \)-scalar product on \( C^\infty(M) \) defined by the volume element \( v_M \). Let \( \mathcal{H} \subset C^\infty(M) \) be a \( G \)-submodule. We call a map \( f: M \to S_V \) into the unit sphere \( S_V \) of a Euclidean vector space \( V \) a spherical \( \mathcal{H} \)-map if its components \( \alpha \circ f, \ \alpha \in V^* \), belong to \( \mathcal{H} \). The Dirac delta \( \delta: M \to S_{\mathcal{H}^*} \) [5] defined by evaluating the elements of \( \mathcal{H} \) on points of \( M \) is the universal example of a spherical \( \mathcal{H}^* \)-map. (The scalar product on \( \mathcal{H}^* \) is induced by the \( L^2 \)-scalar product on \( \mathcal{H} \) suitably scaled.)

Remark. If \( M = G/K \) is naturally reductive, and \( \mathcal{H} \subset C^\infty(M) \) is irreducible then \( \mathcal{H} \) is contained in an eigenspace \( V_\lambda \) of the Laplacian \( \Delta^M \) for some eigenvalue \( \lambda \) [25]. In particular, the components of an \( \mathcal{H} \)-map \( f: M \to S_V \) are eigenfunctions of the Laplacian with a common eigenvalue. Thus an \( \mathcal{H} \)-map is a \( \lambda \)-eigenmap in the sense of Eells-Sampson [8], a harmonic map with constant energy density \( \lambda/2 \).
In general, a DoCarmo-Wallach type argument [6] shows that the set of (congruence classes of) full spherical $H$-maps $f: M \to S_V$, for various $V$, can be parametrized by a moduli space $\mathcal{L}(H)$, a compact convex body in a $G$-submodule $\mathcal{E}(H)$ of the symmetric square $S^2(H)$ (Propositions 3.1-3.2 in Section 3 below).

(The map $f$ is full if its image is not contained in a proper great sphere of the range [3], and congruent maps differ by an isometry between the ranges.) In what follows, we identify $S^2(H)$ with the space of linear endomorphisms of $H$. Then the moduli space is given by

$$\mathcal{L}(H) = \{ C \in \mathcal{E}(H) \mid C + I \geq 0 \},$$

where $\geq$ means positive semidefinite, and $I$ is the identity. The origin of $\mathcal{E}(H)$ is in the interior of $\mathcal{L}(H)$, and it corresponds to $\delta$.

The $G$-module homomorphism

$$\Psi^0: S^2(H) \to C^\infty(M)$$

given by multiplication has image $H \cdot H \subset C^\infty(M)$ consisting of (finite) sums of products of functions in $H$. The DoCarmo-Wallach parametrization of $\mathcal{L}(H)$ implies that the kernel of $\Psi^0$ is $\mathcal{E}(H)$ (Proposition 3.3). We thus have

$$\mathcal{E}(H) = S^2(H)/(H \cdot H),$$

as $G$-modules. To determine $\mathcal{E}(H)$ (and thereby to compute $\dim \mathcal{L}(H) = \dim \mathcal{E}(H)$) amounts to decomposing $H \cdot H$ into irreducible components.

Let $M = G/K$ be a compact rank 1 symmetric space. Then $M$ is the Euclidean $m$-sphere $S^m$, one of the projective spaces $\mathbb{R}P^m$, $\mathbb{C}P^m$, $\mathbb{H}P^m$, or the Cayley projective plane $\mathcal{C}aP^2$ [2]. It is well-known that $C^\infty(M)$ has a multiplicity one decomposition into irreducible components, and each component $H \subset C^\infty(M)$ is the full eigenspace $V_\lambda$ of the Laplacian corresponding to an eigenvalue $\lambda$ [15-17].

Our first result is the following:

**Theorem.** A Let $M = G/K$ be a compact rank 1 symmetric space, $\mathcal{H} \subset C^\infty(M)$ an irreducible $G$-submodule. We write $\mathcal{H} = V_\lambda$, where $\lambda_p$ is the $p$-th eigenvalue of the Laplacian on $\mathcal{H}$. Then we have

$$V_\lambda \cdot V_\lambda = \begin{cases} \sum_{j=0}^p V_{\lambda_{2j}} & \text{if } M = S^m \\ \sum_{j=0}^{2p} V_{\lambda_{j}} & \text{otherwise.} \end{cases}$$

In particular, the dimension of the moduli space is given by

$$\dim \mathcal{L}(V_\lambda) = \begin{cases} n(\lambda_p)(n(\lambda_p) + 1)/2 - \sum_{j=0}^p n(\lambda_{2j}) & \text{if } M = S^m, \\ n(\lambda_p)(n(\lambda_p) + 1)/2 - \sum_{j=0}^{2p} n(\lambda_{j}) & \text{otherwise,} \end{cases}$$

where $n(\lambda_p) = \dim V_{\lambda_p}$.

Since $n(\lambda_p)$ is known for each case of $M$ (second table in Section 2), an explicit formula can be derived for the dimension of $\mathcal{L}(V_\lambda)$. If the dimension is zero then the moduli space reduces to a point, and we have rigidity. This means that the corresponding spherical $V_\lambda$-maps are rigid in the sense that any full $f$ is congruent to the Dirac delta $\delta$. 
**Corollary.** Let $M$ and $\mathcal{H} = V_{\lambda p}$ be as in Theorem A. Then the cases when \(\dim \mathcal{L}(V_{\lambda p})\) is trivial are summarized in the following table:

<table>
<thead>
<tr>
<th>$M$</th>
<th>$m$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^m$</td>
<td>$m \geq 2$</td>
<td>$p = 1$</td>
</tr>
<tr>
<td>$S^m, \mathbb{R}P^m$</td>
<td>$m = 2$</td>
<td>$p \geq 1$</td>
</tr>
<tr>
<td>$\mathbb{C}P^m, \mathbb{H}P^m, \mathbb{C}aP^2$</td>
<td>$m = 2$</td>
<td>$p = 1$</td>
</tr>
</tbody>
</table>

**Remark.** Rigidity of a spherical $V_{\lambda 1}$-map $f : S^m \to S_V$ is obvious since $f$ is the restriction of a linear map, and thereby it is an isometry. Rigidity of spherical $V_{\lambda p}$-maps $f : M \to S_V$ for $M = S^2, \mathbb{R}P^2$ is due to Calabi [3] (stated only for minimal immersions). (In general, a spherical $V_{\lambda p}$-map $f : \mathbb{R}P^m \to S_V$ is a spherical $V_{\lambda 2p}$-map $\tilde{f} : S^m \to S_V$ factored through the twofold projection $S^m \to \mathbb{R}P^m$.)

A rigidity result of DoCarmo-Wallach [6,25] asserts that a minimal immersion $f : M \to S_V$ of a compact analytic manifold $M$ is rigid among minimal immersions, if the (geometric) degree of $f$ is $< 4$. For $M = \mathbb{C}P^m, \mathbb{H}P^m, \mathbb{C}aP^2$ as in the corollary, the degree of $\delta : M \to V_{\lambda p}$ is $2p$ [18]. Notice however that the corollary gives rigidity among all spherical $V_{\lambda 1}$-maps not just minimal immersions.

We now return to the general setting. Let $G$ be a compact Lie group. An orthogonal $G$-module $\mathcal{H}$ is a Euclidean vector space on which $G$ acts linearly via orthogonal transformations. In other words, $\mathcal{H}$ is a representation space for $G$, and it is endowed with a $G$-invariant scalar product.

Let $K$ be a closed subgroup. A class 1 representation of $(G, K)$ is an irreducible orthogonal $G$-module $\mathcal{H}$ so that there is a nonzero vector $\chi_0 \in \mathcal{H}$ fixed by $K$. It is well known that, for $M = G/K$ Riemannian homogeneous, the irreducible components of $C^\infty(M)$ are class 1 representations of $(G, K)$.

Since all components of $C^\infty(M)$ are class 1 with respect to $(G, K)$ it is natural to ask whether $\mathcal{H} \cdot \mathcal{H} \subset C^\infty(M)$ contains all class 1 components of $S^2(\mathcal{H})$.

We reformulate this by introducing $\tilde{\mathcal{E}}(\mathcal{H})$ as the sum of those irreducible $G$-submodules of $S^2(\mathcal{H})$ that are not class 1 with respect to $(G, K)$. The very existence of the homomorphism $\Psi^0$ above implies that

$$\tilde{\mathcal{E}}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}),$$

and the question is whether equality holds. For $M = S^m$, the answer is yes, and it follows from the (multiplicity one) decomposition for $S^2(V_{\lambda p})$ derived by DoCarmo and Wallach [6]. (For a simple proof, see also [14].)

Our next result shows that the answer is negative for $M = \mathbb{C}P^m$.

**Theorem.** Let $(G, K) = (U(m + 1), U(m) \times U(1))$, and $M = \mathbb{C}P^m$, $m \geq 2$, the complex projective $m$-space. Let $\mathcal{H} = V_{\lambda p}$, $p \geq 2$. Then $\tilde{\mathcal{E}}(V_{\lambda p})$ contains class 1 submodules with respect to $(U(m + 1), U(m) \times U(1))$. Equivalently

$$\tilde{\mathcal{E}}(V_{\lambda p}) \neq \mathcal{E}(V_{\lambda p}),$$

More precisely, we have

$$\sum_{q=2}^{2p-2} \frac{1}{2} \left( \min(q, 2p - q) + \frac{(-1)^q - 1}{2} \right) V_{\lambda q} \subset \tilde{\mathcal{E}}(V_{\lambda p}).$$
Assume now that $M = G/K$ is isotropy irreducible. This means that $K$ acts on
the tangent space $T_o(M)$, $o = \{K\}$, irreducibly by the isotropy representation.
Then, for an irreducible $G$-submodule $\mathcal{H} \subset C^\infty(M)$, the Dirac delta
$\delta : M \to S_{\lambda^\prime}$ is a minimal immersion inducing the $\lambda$/dim $M$-multiple of the original Riemannian
metric on $M$ [25].

DoCarmo and Wallach proved that the set of (congruence classes of) full minimal
immersions $f : M \to S_V$, for various $V$, and with induced Riemannian metric the $\lambda$/dim $M$-multiple of the original, can be parametrized by a moduli space $\mathcal{M}(\mathcal{H})$, a compact convex body in a $G$-submodule $\mathcal{F}(\mathcal{H})$ of $S^2(\mathcal{H})$ (Proposition 3.4). The moduli space is given by

$$\mathcal{M}(\mathcal{H}) = \{ C \in \mathcal{F}(\mathcal{H}) \mid C + I \geq 0 \};$$

where $\geq$ means positive semidefinite.

We now recall the definition of induced representations [25]. If $\mathcal{W}$ is a $K$-module
then $\text{Ind}_K^G(\mathcal{W})$ denotes the linear space of continuous maps $\phi : G \to \mathcal{W}$ which
satisfy $\phi(kg) = k \cdot \phi(g)$, $k \in K$, $g \in G$. The action of $G$ on $\text{Ind}_K^G(\mathcal{W})$
given by $g \cdot \phi(g') = \phi(gg')$, $g, g' \in G$, defines a $G$-module structure on $\text{Ind}_K^G(\mathcal{W})$. We call $\text{Ind}_K^G(\mathcal{W})$ the $G$-module induced from the $K$-module $\mathcal{W}$.

DoCarmo and Wallach constructed a homomorphism

$$\Psi : S^2(\mathcal{H}) \to \text{Ind}_K^G(S^2(p));$$

where $K$-module $S^2(p)$ is the symmetric square of the isotropy representation of
$M = G/K$. The kernel of $\Psi$ is $\mathcal{F}(\mathcal{H})$.

Let $\tilde{\mathcal{F}}(\mathcal{H})$ denote the sum of those components of $S^2(\mathcal{H})$ that, when restricted to
$K$, do not contain any irreducible $K$-submodules of $S^2(p)$. Frobenius reciprocity
[25] says that

$$\tilde{\mathcal{F}}(\mathcal{H}) \subset \mathcal{F}(\mathcal{H}).$$

(1)

Thus, once the irreducible decomposition of $S^2(\mathcal{H})$ is known, this gives a lower
bound on the dimension of the moduli $\mathcal{M}(\mathcal{H})$.

DoCarmo and Wallach carried this out for $M = S^m$, and $\mathcal{H} = V_{\lambda_p}$. Identifying the irreducible components of $\tilde{\mathcal{F}}(V_{\lambda_p})$, for $m \geq 3$ and $p \geq 4$, they obtained the lower estimate

$$\dim \mathcal{M}(V_{\lambda_p}) = \dim \mathcal{F}(V_{\lambda_p}) \geq \dim \tilde{\mathcal{F}}(V_{\lambda_p}) \geq \dim \tilde{\mathcal{F}}(V_{\lambda}) \geq 18.$$

They conjectured that equality holds in (1). This has been resolved by the author
in [21] using different methods. (For a recent algebraic proof, see [26].) For the
lowest dimensional moduli space, $\mathcal{M}(V_{\lambda_4})$ with $m = 3$, see [23].

Once again, it is natural to ask whether equality holds in (1) in general, or at least
for compact rank 1 symmetric spaces $M = G/K$. Our last result is to show that
the answer is negative for $M = CP^m$, and $\mathcal{H} = V_{\lambda_p}$.
Theorem. \( C \) Let \( m \geq 3, (G, K) = (U(m + 1), U(m) \times U(1)), \ M = U(m + 1)/(U(m) \times U(1)) = \mathbb{CP}^m, \) and \( \mathcal{H} = V_{\lambda_p}. \) Then, for \( p = 3 \) and \( m \not\equiv 1 \pmod{4}, \) or for \( p \geq 4, \) we have
\[
\mathcal{F}(V_{\lambda_p}) \neq \mathcal{F}(V_{\lambda_p}).
\]

The striking difference between the spherical and complex projective cases is that \( S^2(V_{\lambda_p}), \ V_{\lambda_p} \subset C^\infty(S^m), \) has a multiplicity one decomposition into irreducible components, but according to the multiplicity formulas developed by Barbasch [22], this fails for \( S^2(V_{\lambda_p}), \ V_{\lambda_p} \subset C^\infty(\mathbb{CP}^m). \)

2. Zonal Spherical Functions and Jacobi Polynomials

In this section we describe the main idea of the proof of Theorem A as well as assemble some preliminary facts.

Let \( M = G/K \) be a compact rank 1 symmetric space. As noted above, an irreducible \( G \)-submodule \( \mathcal{H} \subset C^\infty(M) \) is class 1 with respect to the pair \( (G, K). \) We call a \( K \)-fixed vector \( \chi_0 \in \mathcal{H} \) a zonal spherical function [15, 25]. It is well-known that a zonal spherical function is unique up to a constant multiple [2].

Let \( \chi_0 \) be a zonal spherical function of \( \mathcal{H}. \) Its square \( \chi_0^2 \in \mathcal{H} \cdot \mathcal{H} \) is also fixed by \( K. \) Since \( C^\infty(M) \) has a multiplicity one decomposition into irreducible components, as an element of \( C^\infty(M), \) \( \chi_0^2 \) decomposes into a sum
\[
\chi_0^2 = \sum_{j=1}^{n} \chi_j,
\]
where each \( \chi_j \) belongs to a unique irreducible component \( \mathcal{H}_j \subset C^\infty(M). \) Clearly, \( \chi_j \) is a zonal spherical function of \( \mathcal{H}_j. \) Since \( \chi_j \in \mathcal{H}_j \) is a component of \( \chi_0^2 \in \mathcal{H} \cdot \mathcal{H}, \) by Schur’s lemma, \( \mathcal{H}_j \) projects nontrivially to \( \mathcal{H} \cdot \mathcal{H}, \) and we obtain
\[
\sum_{j=1}^{n} \mathcal{H}_j \subset \mathcal{H} \cdot \mathcal{H}.
\]

In Section 4 we will prove Theorem A by showing that equality holds here. We now illustrate this in a different setting by a simple example.

Example. Let \( G \) be a compact Lie group viewed as a symmetric space \( G \times G/G^* \) of compact type, where \( G^* \subset G \times G \) is the diagonal [15, 16, 24]. (The map \( (g_1, g_2)G^* \mapsto g_1 g_2^{-1}, \ g_1, g_2 \in G, \) identifies \( G \times G/G^* \) with \( G). \) The space \( C^\infty(G \times G/G^*, \mathbb{C}) \) of complex valued smooth functions on \( G \times G/G^* \) has a multiplicity one decomposition into irreducible components. A component, a complex irreducible \( G \times G \)-submodule of \( C^\infty(G \times G/G^*, \mathbb{C}), \) has the form \( \mathcal{H}^* \otimes \mathcal{H}, \) where \( \mathcal{H} \) is a complex irreducible \( G \)-module. The \( G^* \)-fixed vectors in \( \mathcal{H}^* \otimes \mathcal{H} \) can be identified with the (multiples of a normalized) character \( \chi_0 \) of \( \mathcal{H} \) [24]. Given \( \chi_0, \) according to our procedure, we need to decompose the square \( \chi_0^2 \) into a sum of (nonzero) characters
\[
\chi_0^2 = \sum_{j=1}^{n} c_j \chi_j.
\]
By elementary character theory, this decomposition corresponds to the decomposition of the tensor product

\[ \mathcal{H} \otimes \mathcal{H} = \sum_{j=1}^{n} c_{j} \mathcal{H}_{j} \]

as a $G$-module, where $\chi_{j}$ is the character of $\mathcal{H}_{j}$ and $c_{j} \in \mathbb{N}$ is the multiplicity of $\mathcal{H}_{j}$ in $\mathcal{H} \otimes \mathcal{H}$. Since $\chi_{0}^{2} \in (\mathcal{H}^{*} \otimes \mathcal{H}) \cdot (\mathcal{H}^{*} \otimes \mathcal{H})$, Schur’s lemma tells us that

\[ \sum_{j=1}^{n} \mathcal{H}_{j}^{*} \otimes \mathcal{H}_{j} \subset (\mathcal{H}^{*} \otimes \mathcal{H}) \cdot (\mathcal{H}^{*} \otimes \mathcal{H}) \]

as $G \times G$-modules. We claim that equality holds here. Indeed, consider the natural extension of $\Psi^{0}$ above

\[ \Psi^{0}: (\mathcal{H}^{*} \otimes \mathcal{H}) \otimes (\mathcal{H}^{*} \otimes \mathcal{H}) \rightarrow (\mathcal{H}^{*} \otimes \mathcal{H}) \cdot (\mathcal{H}^{*} \otimes \mathcal{H}) \]

given by multiplication. The domain of $\Psi^{0}$, as a $G \times G$-module, can be decomposed as

\[ (\mathcal{H}^{*} \otimes \mathcal{H}) \otimes (\mathcal{H}^{*} \otimes \mathcal{H}) = (\mathcal{H}^{*} \otimes \mathcal{H}^{*}) \otimes (\mathcal{H} \otimes \mathcal{H}) \]

\[ = \left( \sum_{j=1}^{n} c_{j} \mathcal{H}_{j}^{*} \right) \otimes \left( \sum_{l=1}^{n} c_{l} \mathcal{H}_{l} \right) = \sum_{j,l=1}^{n} c_{j} c_{l} (\mathcal{H}_{j}^{*} \otimes \mathcal{H}_{l}). \]

Finally, by Schur’s lemma again $\mathcal{H}_{j}^{*} \otimes \mathcal{H}_{l}$ contains a $G^{*}$-fixed vector if and only if $j = l$ [24]. The claim follows.

We now return to our compact rank 1 symmetric space $M = G/K$. As noted in Section 1, the full eigenspace $V_{\lambda}$ of the Laplacian $\triangle^{M}$ corresponding to an eigenvalue $\lambda$ is an irreducible $G$-module. Moreover, if $\{\lambda_{p}\}_{p=0}^{\infty}$ denotes the sequence of eigenvalues of $\triangle^{M}$ in increasing order, then we have

\[ C^{\infty}(M) = \sum_{p=0}^{\infty} V_{\lambda_{p}}. \]

By the above, $V_{\lambda_{p}} \subset C^{\infty}(M)$ contains a zonal spherical function $\chi_{0}$, unique up to a constant multiple.

We now recall that, for fixed $\alpha, \beta > -1$, the Jacobi polynomials $P_{n}^{(\alpha,\beta)}$, $n \geq 0$, form an orthogonal series on $[-1,1]$ with respect to the weight function $(1-x)^{\alpha}(1-x)^{\beta}$ [1]. The polynomial $P_{n}^{(\alpha,\beta)}$ can be defined by

\[ (1-x)^{\alpha}(1-x)^{\beta}P_{n}^{(\alpha,\beta)}(x) = \frac{(-1)^{n}}{2^{n}n!} \frac{d^{n}}{dx^{n}}[(1-x)^{n+\alpha}(1-x)^{n+\beta}]. \]

With a suitable choice of parameters on $M$, the zonal function $\chi_{0}$ of a component $V_{\lambda_{p}}$ of $C^{\infty}(M)$ is a constant multiple of $P_{n}^{(\alpha,\beta)}$ with $\alpha$, $\beta$, $n$ depending on $V_{\lambda_{p}}$ [4,10,24]. (For example, $n = p$ in all cases but $M = \mathbb{R}P^{m}$ for which $n = 2p$.) The classification of compact rank 1 symmetric spaces $M$, the eigenvalues of $\triangle^{M}$ [2], and the Jacobi polynomials corresponding to the zonal spherical harmonics $\chi_{0}$ are summarized in the following table:
A decomposition of the square \((G, K)\) is a parametrization, the zonals are Jacobi polynomials, we need to obtain that 
\[
\lambda \p_p^n \geq 2.
\]
The multiplicities \(\lambda \p_p^n\) are given as follows:

<table>
<thead>
<tr>
<th>(M)</th>
<th>(n(\lambda_p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S^m)</td>
<td>(\binom{p+m}{m} - \binom{p+m-2}{m})</td>
</tr>
<tr>
<td>(R^m)</td>
<td>(\binom{p+m}{m} - \binom{p+m-2}{m})</td>
</tr>
<tr>
<td>(C^m)</td>
<td>(\binom{p+m}{m} - \binom{p+m-1}{m})</td>
</tr>
<tr>
<td>(H^m)</td>
<td>(\binom{p+m}{m} - \binom{p+m-1}{m})</td>
</tr>
<tr>
<td>(CaF^2)</td>
<td>(\binom{p+11}{1280} \binom{p+10}{7} \binom{p+1}{7})</td>
</tr>
</tbody>
</table>

To simplify the treatment and to avoid some overlapping cases, we will assume that \(m \geq 2\).

Since, up to parametrization, the zonals are Jacobi polynomials, we need to obtain a decomposition of the square \((P_p^{(\alpha, \beta)})^2\) into a sum of Jacobi polynomials:

\[
(P_p^{(\alpha, \beta)})^2 = \sum_{j=0}^{2p} c(j, p; \alpha, \beta) P_j^{(\alpha, \beta)}.
\]

(2)

More generally, a formula of the type

\[
P_p^{(\alpha, \beta)} P_q^{(\alpha, \beta)} = \sum_{j=|p-q|}^{p+q} c(j, p, q; \alpha, \beta) P_j^{(\alpha, \beta)}.
\]

(3)

is usually called “linearization of the product.”

For \(\alpha = \beta\), the Jacobi polynomial \(P_p^{(\alpha, \beta)}\) is, up to normalization, the ultraspherical (or Gegenbauer) polynomial \(C_p^{\nu}\), where \(\nu - 1/2 = \alpha = \beta\). (The precise formula is given in (20) below.) Linearization of ultraspherical polynomials dates back to the early twentieth century, and the coefficients \(c(j, p, q; \lambda - 1/2, \lambda - 1/2)\) have been calculated explicitly \([1,7,24]\). For our purposes, we need only that \(c(j, p, q; \lambda - 1/2, \lambda - 1/2)\) is positive if and only if \(|p-q| \leq j \leq p+q\) and \(j \equiv p+q \pmod{2}\).

For Jacobi polynomials in general linearization proved to be much more difficult and the exact decomposition formula is fairly recent \([19]\). A general and sharp positivity result for the coefficients \(c(j, p, q; \alpha, \beta)\) (covering the remaining cases in the table above for \(m \geq 2\)) is due to Gasper \([11,12]\). It states that if \(\alpha, \beta > -1\), \(a = \alpha + \beta + 1\), \(b = \alpha - \beta\), then \(c(j, p, q; \alpha, \beta) > 0\) provided that \((\alpha, \beta)\) is in the interior of the set

\[
V = \{((\alpha, \beta) \mid \alpha \geq \beta, a(a+5)(a+3)^2 \geq (a^2 - 7a - 24)b^2}.
\]

Note that Theorem 1 in \([12]\) states nonnegativity of the coefficients for \((\alpha, \beta) \in V\). As Professor Gasper communicated to the author [13], a closer inspection of his proof of Theorem 1 in [12], pp. 585-591, shows strict positivity of the coefficients if \((\alpha, \beta)\) is in the interior of \(V\). Another proof of the positivity follows by using the \(\{\}}_9F_8\) series representations for the linearization coefficients in \([19]\) (formula (3.9)).
3. Generalities on the Moduli

Let $G$ be a compact Lie group and $\mathcal{H}$ an orthogonal $G$-module. We define

$$\mathcal{K}(\mathcal{H}) = \{ C \in S^2(\mathcal{H}) \mid C + I \geq 0 \}.$$  

We write $\mathcal{K} = \mathcal{K}(\mathcal{H})$ if there is no danger of confusion. $\mathcal{K}$ is a $G$-invariant set in $S^2(\mathcal{H})$, where the $G$-module structure on $S^2(\mathcal{H})$ is extended from that of $\mathcal{H}$.

Since $C + I \geq 0$ is a convex condition, $\mathcal{K}$ is a convex set. The interior of $\mathcal{K}$ consists of those endomorphisms $C$ that satisfy $C + I > 0$. It follows that $\mathcal{K}$ has a nonempty interior, and hence it is a convex body in $S^2(\mathcal{H})$. Notice that $\mathcal{K}$ is noncompact since the multiples $\lambda I$, $\lambda \geq -1$, are contained in $\mathcal{K}$. We call $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the general moduli space for $\mathcal{H}$.

We let $S^2_0(\mathcal{H})$ denote the $G$-submodule of $S^2(\mathcal{H})$ comprised of the traceless symmetric endomorphisms of $V$. We define

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap S^2_0(\mathcal{H}) = \{ C \in S^2_0(\mathcal{H}) \mid C + I \geq 0 \}.$$  

The eigenvalues of the symmetric endomorphisms in $\mathcal{K}$ are greater or equal to $-1$. Hence the eigenvalues of the endomorphisms in $\mathcal{K}_0$ are contained in $[-1, \dim \mathcal{H} - 1]$. It follows that $\mathcal{K}_0$ is compact, and a convex body in $S^2_0(\mathcal{H})$. We call $\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H})$ the reduced moduli space for $\mathcal{H}$.

We now give an interpretation of the moduli as parameter spaces for certain maps. We let $M$ be a compact Riemannian manifold, and $G$ a compact Lie group of isometries of $M$. ($G$ is a closed subgroup of the full isometry group of $M$.) The space $C^\infty(M)$ of smooth functions on $M$ is a representation space for $G$, where $g \in G$ acts on $\xi \in C^\infty(M)$ by $g \cdot \xi = \xi \circ g^{-1}$. We fix a finite dimensional $G$-submodule $\mathcal{H} \subset C^\infty(M)$. We endow $\mathcal{H}$ with the scaled $L^2$-scalar product

$$\langle \chi_1, \chi_2 \rangle = \frac{\dim \mathcal{H}}{\text{vol}(M)} \int_M \chi_1 \chi_2 v_M, \quad \chi_1, \chi_2 \in \mathcal{H},$$  

where $v_M$ is the Riemannian volume form on $M$, and $\text{vol}(M) = \int_M v_M$ is the volume of $M$. With this scalar product $\mathcal{H}$ becomes an orthogonal $G$-module.

A smooth map $f: M \rightarrow V$ into a Euclidean vector space $V$ is said to be full if the image of $f$ spans $V$. A component of $f$ is $\alpha \circ f \in C^\infty(M)$, where $\alpha \in V^*$. The space of components of $f$ is defined as

$$V_f = \{ \alpha \circ f \mid \alpha \in V^* \} \subset C^\infty(M).$$

The map $f$ is full if and only if the linear map $f^*: V^* \rightarrow V_f$, given by precomposition with $f$, is an isomorphism. Since $V$ is Euclidean we also have $V \cong V^* \cong V_f$.

Note that any map can be made full by restricting its range to the linear span of the image.

Two maps $f_1: M \rightarrow V_1$ and $f_2: M \rightarrow V_2$ are said to be congruent if there is a linear isometry $U: V_1 \rightarrow V_2$ such that $f_2 = U \circ f_1$.

With $\mathcal{H}$ as above, $f: M \rightarrow V$ is said to be an $\mathcal{H}$-map if $V_f \subset \mathcal{H}$. Note that any smooth map $f: M \rightarrow V$ is an $\mathcal{H}$-map for $\mathcal{H}$ the smallest $G$-invariant linear subspace in $C^\infty(M)$ that contains $V_f$.

The Dirac delta as a map $\delta_\mathcal{H}: M \rightarrow \mathcal{H}^*$ is defined in the usual way

$$\delta_\mathcal{H}(x)(\chi) = \chi(x), \quad x \in M, \ \chi \in \mathcal{H}.$$
The component of $\delta_H$ corresponding to $\chi \in H = \mathcal{H}^*$ is $\langle \delta_H, \chi \rangle = \chi$. Hence, $V_{\delta_H} = H$ and $\delta_H$ is a full $H$-map.

In what follows we will identify $H$ with its dual $H^*$ via the scalar product on $H$. With respect to an orthonormal basis $\{\chi^j\}_{j=0}^N \subset H$, $\dim H = N + 1$, the Dirac delta as a map $\delta_H: M \to H$ can be written as

$$\delta_H(x) = \sum_{j=0}^N \chi^j(x)\chi^j, \quad x \in M. \tag{5}$$

Indeed, for $\chi \in H$, we have

$$\langle \delta_H(x), \chi \rangle = \chi(x) = \sum_{j=0}^N \langle \chi, \chi^j \rangle \chi^j(x) = \left\langle \sum_{j=0}^N \chi^j(x)\chi^j, \chi \right\rangle.$$ 

The Dirac delta $\delta_H$ is equivariant with respect to the homomorphism $\rho_H: G \to O(H)$ that defines the orthogonal $G$-module structure on $H \cong \mathcal{H}^*$.

For a full $H$-map $f: M \to V$, we have $f = A \circ \delta_H$, where $A: H \to V$ is a surjective linear map. We associate to $f$ the symmetric linear endomorphism $\langle f \rangle = A^*A - I \in S^2(H)$.

It is clear that $\langle f \rangle$ depends only on the congruence class of $f$. Since $A^*A$ is always positive semidefinite, we also have $\langle f \rangle \in K(H)$. A DoCarmo-Wallach type argument shows that $f \mapsto \langle f \rangle$ gives rise to a one-to-one correspondence between the set of congruence classes of full $H$-maps and the general moduli space $K(H)$ [6,25].

Let $f: M \to V$ be a full $H$-map. With respect to an orthonormal basis in $V$, $f$ can be written in components as $f = (f^0, \ldots, f^n)$, $\dim V = n + 1$. With the orthonormal basis $\{\chi^j\}_{j=0}^N \subset H$ as above, $A: H \to V$ becomes an $(n+1) \times (N+1)$-matrix with entries $a_{kj}$, $k = 0, \ldots, n$, $j = 0, \ldots, N$. In components, $f = A \circ \delta_H$ can be written as

$$f^k = \sum_{j=0}^N a_{kj} \chi^j, \quad k = 0, \ldots, n.$$ 

We now calculate

$$\text{trace} \left( \langle f \rangle + I \right) = \text{trace} A^*A = \sum_{k=0}^n \sum_{j=0}^N a_{kj}^2 = \sum_{k=0}^n |f^k|^2.$$ 

We conclude that, in terms of the scaled $L^2$-scalar product (4) on $H$, the parameter point $\langle f \rangle \in K(H)$ is traceless if and only if

$$\int_M \sum_{k=0}^n (f^k)^2 v_M = \text{vol}(M). \tag{6}$$

We call $f$ normalized if (6) is satisfied. Clearly, $\delta_H$ is normalized.

It is also clear that, by suitable scaling, any nontrivial map can be normalized.

Summarizing, we obtain the following:
Proposition 3.1. Let $M$ be a compact Riemannian manifold with compact group $G$ of isometries. Given a finite dimensional $G$-submodule $H$ of $C^\infty(M)$, the set of congruence classes of full $H$-maps $f : M \to V$ can be parametrized by the general moduli space $K(H)$. The reduced moduli $K_0(H)$ parametrizes the normalized $H$-maps.

An $H$-map $f : M \to V$ is called spherical if the image of $f$ is contained in the unit sphere $S^V$ of $V$. A finite dimensional $G$-module $H \subset C^\infty(M)$ is called $\delta$-spherical if $\delta H$ is spherical. Due to the scaling of the $L^2$-scalar product in (4), $H$ is $\delta$-spherical if and only if

$$\sum_{j=0}^N (\chi_j^2) = 1$$

on $M$, where $\{\chi_j\}_{j=0}^N \subset H$ is an orthonormal basis.

If $M = G/K$ is homogeneous then any $H \subset C^\infty(M)$ is $\delta$-spherical. This is because $\delta H$ is equivariant, and thereby its image is a $G$-orbit in $H$ necessarily contained in $S_H$.

Let $H$ be a $\delta$-spherical $G$-module. A full $H$-map $f : M \to V$ is spherical if and only if

$$|f(x)|^2 - |\delta H(x)|^2 = \langle (A^* A - I)\delta H(x), \delta H(x) \rangle = \langle \langle f \rangle, \delta H(x) \circ \delta H(x) \rangle = 0,$$

for all $x \in M$. Here $\circ$ denotes the symmetric tensor product. We define

$$E(H) = \{\delta H(x) \circ \delta H(x) \mid x \in M\}^\perp \subset S^2(H).$$

(7)

The previous computation shows that an $H$-map $f : M \to V$ is spherical if and only if $\langle f \rangle \in E(H)$.

Once again, since $\delta H$ is equivariant, $E(H) \subset S^2(H)$ is a $G$-submodule.

We obtain the following:

Proposition 3.2. Let $M$ be a compact Riemannian manifold with compact group $G$ of isometries, and $H \subset C^\infty(M)$ a $\delta$-spherical $G$-submodule. Then the set of congruence classes of full spherical $H$-maps $f : M \to S^V$ can be parametrized by the moduli space

$$L(H) = K(H) \cap E(H).$$

Moreover $L(H)$ is a compact convex body in $E(H)$.

Compactness follows since spherical maps are automatically normalized:

$$L(H) \subset K_0(H) \Rightarrow E(H) \subset S^2_0(H),$$

so that

$$L(H) = K_0(H) \cap E(H).$$

Remark. Let $M = G/K$ be a compact naturally reductive Riemannian homogeneous space, and $V_\lambda \subset C^\infty(M)$ the eigenspace of $\triangle^M$ corresponding to an eigenvalue $\lambda$. Recall from Section 1 that a $\lambda$-eigenmap $f : M \to S_V$ is a spherical $V_\lambda$-map.
Let $\mathcal{H} \subset C^\infty(M)$ be a finite dimensional $G$-submodule. Then $\mathcal{H} \subset V_\lambda$ for some $\lambda$. Proposition 3.2 asserts that $L(\mathcal{H})$ parametrizes the congruence classes of full $\lambda$-eigenmaps $f: M \to S_V$ with components in $\mathcal{H} \subset V_\lambda$. In particular, $L(V_\lambda)$ parametrizes the congruence classes of all full $\lambda$-eigenmaps $f: M \to S_V$.

Returning to the general situation, let $\mathcal{H} \subset C^\infty(M)$ be a $\delta$-spherical $G$-module. We define 

$$\Psi^0 = \Psi_\mathcal{H}^0: S^2(\mathcal{H}) \to C^\infty(M)$$

by

$$\Psi^0(C)(x) = \langle C\delta_\mathcal{H}(x), \delta_\mathcal{H}(x) \rangle = \langle C, \delta_\mathcal{H}(x) \circ \delta_\mathcal{H}(x) \rangle, \quad x \in M.$$ 

Since $\delta_\mathcal{H}$ is equivariant, $\Psi^0$ is a homomorphism of $G$-modules. By (7), we have

$$\ker \Psi^0 = \mathcal{E}(\mathcal{H}).$$

We claim that the image of $\Psi^0$ is the $G$-submodule

$$\mathcal{H} \cdot \mathcal{H} = \text{span} \{\chi_1\chi_2 \mid \chi_1, \chi_2 \in \mathcal{H} \} \subset C^\infty(M).$$

Indeed, using (5) in the definition of $\Psi^0$, we obtain

$$\Psi^0(C) = \sum_{j,l=0}^N c_{jl} \chi^j \chi^l,$$

where $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$ is an orthonormal basis, and the $c_{jl}$'s are the matrix entries of $C \in S^2(\mathcal{H})$. The claim follows.

Note that $\mathcal{H} \cdot \mathcal{H}$ always contains the trivial $G$-module, a consequence of $\delta$-sphericality.

We obtain the following:

**Proposition 3.3.** Let $\mathcal{H} \subset C^\infty(M)$ be a $\delta$-spherical $G$-module. Then the $G$-module homomorphism

$$\Psi^0: S^2(\mathcal{H}) \to \mathcal{H} \cdot \mathcal{H}$$

is onto, and has kernel $\mathcal{E}(\mathcal{H})$. In particular, $\mathcal{H} \cdot \mathcal{H}$ is (isomorphic to) a $G$-submodule of $S^2(\mathcal{H})$ and we have

$$\mathcal{E}(\mathcal{H}) \cong S^2(\mathcal{H})/(\mathcal{H} \cdot \mathcal{H})$$

as $G$-modules.

Let $K \subset G$ be a closed subgroup. Recall that an irreducible orthogonal $G$-module $V$ is called class 1 with respect to the pair $(G,K)$ if $V$ contains a nonzero $K$-fixed vector, or equivalently, if $V|_K$ contains the trivial representation.

We now assume that $M = G/K$ is Riemannian homogeneous. As noted in Section 1, any irreducible $G$-submodule of $C^\infty(M)$ is class 1 with respect to $(G,K)$. Conversely, any class 1 $G$-module $V$ with respect to $(G,K)$ is isomorphic to an irreducible $G$-submodule of $C^\infty(M)$ [25].

Let $\mathcal{H} \subset C^\infty(M)$ be a $\delta$-spherical $G$-submodule. We define $\mathcal{E}(\mathcal{H}) \subset S^2(\mathcal{H})$ as the
sum of those irreducible $G$-submodules in $S^2(\mathcal{H})$ that are not class 1 with respect to $(G, K)$. By the description of class 1 modules above and (8)-(9), we see that

$$\bar{E}(\mathcal{H}) \subset E(\mathcal{H}).$$

Equality holds if and only if the sum of all irreducible $G$-submodules in $S^2(\mathcal{H})$ that are class 1 with respect to $(G, K)$, is isomorphic to $H \cdot H$.

A map $f: M \to V$ is said to be conformal if

$$\langle f_*(X), f_*(Y) \rangle = c \langle X, Y \rangle, \quad X, Y \in T(M),$$

where $c > 0$ is a constant. Then $c$ is called the conformality constant of $f$. We say that a finite dimensional $G$-module $\mathcal{H} \subset C^\infty(M)$ is $\delta$-conformal if $\delta_H$ is conformal.

Using (5), we have

$$(\delta_H)_* (X) = X \delta_H = \sum_{j=1}^{N} X(\chi^j) \chi^j, \quad X \in T(M).$$

Thus, $\mathcal{H}$ is $\delta$-conformal if and only if

$$\sum_{j=0}^{N} X(\chi^j)Y(\chi^j) = c \langle X, Y \rangle, \quad X, Y \in T(M),$$

holds for any orthonormal basis $\{\chi^j\}_{j=0}^{N} \subset \mathcal{H}$.

Let $f: M \to V$ be a conformal map as above, and assume that $V_f \subset V_\lambda$ for some eigenvalue $\lambda$ of $\Delta^M$. Then $f: M \to V$ is an isometric immersion with respect to $c$ times the original metric on $M$. By Takahashi’s theorem [20] $f$ maps into a sphere $rS_V$ for some $r$. Calculating $\Delta^M(|f|^2)$, we obtain $c = r^2 \lambda / \dim M$. If $f$ is normalized then $r = 1$ and we get $c = \lambda / \dim M$. Again by Takahashi, we obtain that $f: M \to S_V$ is an isometric minimal immersion of the $\lambda / \dim M$-multiple of the metric on $M$.

Let $\mathcal{H} \subset V_\lambda$ be a $\delta$-conformal $G$-submodule. By definition, $\delta_\mathcal{H}: M \to \mathcal{H}$ is conformal with $V_{\delta_\mathcal{H}} = \mathcal{H} \subset V_\lambda$ so that the argument above applies. Since $\delta_\mathcal{H}$ is automatically normalized, we obtain that $\delta_\mathcal{H}: M \to S_H$ is an isometric minimal immersion of the $\lambda / \dim M$-multiple of the metric on $M$. In particular, $\mathcal{H}$ is $\delta$-spherical.

**Remark.** Let $M = G/K$ be isotropy irreducible. Then any irreducible $G$-submodule $\mathcal{H} \subset C^\infty(M)$ is $\delta$-conformal. Indeed, (12) holds because $\sum_{j=0}^{N} d\chi^j \odot d\chi^j$ is a $G$-invariant bilinear form on $\mathcal{H}$. Its coordinate representation in (5) shows that $\delta_\mathcal{H}: M \to S_H^r$ is the standard minimal immersion [6,25].

A DoCarmo-Wallach type argument gives the following:
Proposition 3.4. Let $\mathcal{H} \subset V_\lambda \subset C^\infty(M)$ be a $\delta$-conformal $G$-submodule. Then the congruence classes of isometric minimal $\mathcal{H}$-immersions $f: M \to S_V$ (with respect to the $\lambda/\dim M$-multiple of the metric on $M$) are parametrized by the compact convex body

$$\mathcal{M}(\mathcal{H}) = \mathcal{K}_0(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}),$$

in the $G$-module

$$\mathcal{F}(\mathcal{H}) = \{ X\delta_H \circ Y\delta_H \mid X, Y \in T(M) \}^\perp \subset S^2(\mathcal{H}).$$

We also have

$$\mathcal{F}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}),$$

so that

$$\mathcal{M}(\mathcal{H}) = \mathcal{L}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}).$$

Let $M = G/K$ be a naturally reductive homogeneous space with orthogonal decomposition $g = k \oplus p$, where $g$ and $k$ are the Lie algebras of $G$ and $K$, and $p$ is identified with the tangent space $T_o(M)$. The subgroup $K$ acts on its Lie algebra $k$ by the adjoint representation, and, under the identification $p \cong T_o(M)$, this action corresponds to the action of $K$ on $T_o(M)$ via the isotropy representation.

As noted in Section 1, if $W$ is any (finite dimensional) orthogonal $K$-module then the induced $G$-module $\text{Ind}_K^G(W)$ is comprised of continuous maps $f: G \to W$ that satisfy $f(kg) = k \cdot f(g)$, $g \in G$, $k \in K$. Precomposition of these maps by right multiplication on $G$ defines the $G$-module structure on $\text{Ind}_K^G(W)$. In addition, integration with respect to the Haar measure on $G$ and the scalar product on $W$ define a $G$-invariant scalar product on $\text{Ind}_K^G(W)$.

By Frobenius reciprocity, we have

$$\text{Hom}_G(V, \text{Ind}_K^G(W)) = \text{Hom}_K(V|_K, W),$$

where $V$ is an orthogonal $G$-module and $W$ is an orthogonal $K$-module.

Let $\mathcal{H} \subset C^\infty(M)$ be a $\delta$-conformal $G$-module. Restricting the differential of $\delta_H$ to $p = T_o(M)$ gives a $K$-equivariant linear imbedding $(\delta_H)_*: p \to \mathcal{H}$. We identify the $K$-module $p$ with the image, and think of $p$ as a $K$-submodule of $\mathcal{H}|_K$. Notice that this can also be thought of as the inclusion $p \subset \mathcal{H}^* \cong \mathcal{H}$ given by the action of the tangent vectors at $o$ to $M$ on the elements of $\mathcal{H}$ by directional differentiation.

We define

$$\Psi: S^2(\mathcal{H}) \to \text{Ind}_K^G(S^2(p))$$

as follows. For $C \in S^2(\mathcal{H})$, we let $\Psi(C): G \to S^2(p)$ be the map defined by $\Psi(C)(g) = \pi(g \cdot C)$, where $\pi: S^2(\mathcal{H}) \to S^2(p)$ is the orthogonal projection, a homomorphism of $K$-modules.

We have

$$\ker \Psi = \mathcal{F}(\mathcal{H}).$$
Indeed, for $C \in S^2(\mathcal{H})$, $\Psi(C) = 0$ if and only if $\langle g \cdot C, S^2(p) \rangle = 0$ for all $g \in G$, if and only if $\langle C, g \cdot S^2(p) \rangle = 0$ for all $g \in G$. In view of the identification $p \subset \mathcal{H}_{|K}$, this holds if and only if
\[
\langle C, S^2((\delta_\mathcal{H})_*(T_x(M))) \rangle = 0, \quad x \in M.
\]
This is equivalent to $C \in \mathcal{F}(\mathcal{H})$.

4. Proofs of Theorems A-C.

**Proof of Theorem A.** We first let $(G, K) = (SO(m + 1), S(m))$, $SO(m) = SO(m) \oplus [1] \subset SO(m + 1)$, with $M = G/K = S^m$ the Euclidean $m$-sphere, and $\mathcal{H} = V_\lambda$. The eigenspace $V_\lambda$ corresponding to $\lambda_p = p(p + m - 1)$ is $\mathcal{H}_p$, the irreducible $SO(m + 1)$-module of spherical harmonics of order $p$ on $S^m$.

We let $\mathcal{P}^p$ denote the $SO(m + 1)$-module of homogeneous polynomials on $\mathbb{R}^{m+1}$ of degree $p$ (with the usual action $g \cdot \xi = \xi \circ g^{-1}$, $g \in SO(m + 1)$, $\xi \in \mathcal{P}^p$). By homogeneity, a polynomial in $\mathcal{P}^p$ is uniquely determined by its restriction to $S^m \subset \mathbb{R}^{m+1}$.

We also think of a spherical harmonic $\chi$ of order $p$ on $S^m$ as a harmonic homogeneous polynomial on $\mathbb{R}^{m+1}$ of degree $p$. (The equivalence of these two representations is given by restriction from $\mathbb{R}^{m+1}$ to $S^m$, and comparison of the Euclidean and spherical Laplacians. We suppress the restriction if there is no danger of confusion.) This way $\mathcal{H}^p$ becomes an $SO(m + 1)$-submodule of $\mathcal{P}^p$. We have the orthogonal decomposition
\[
\mathcal{P}^p = \mathcal{H}^p \oplus \mathcal{P}^{p-2} \cdot \rho^2 = \sum_{k=0}^{[p/2]} \mathcal{H}^{p-2k} \cdot \rho^{2k}, \tag{17}
\]
where $\rho(x) = |x|$, $x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$ [15,24].

Since $\mathcal{H}^p \subset \mathcal{P}^p$, we have
\[
\mathcal{H}^p \cdot \mathcal{H}^p \subset \mathcal{P}^{2p} = \sum_{j=0}^{p} \mathcal{H}^{2j}
\]
as $SO(m + 1)$-modules. Theorem A for $M = S^m$ states that equality holds, and this is what we need to show.

We define the harmonic projection operator as the orthogonal projection $H : \mathcal{P}^p \to \mathcal{H}^p$ with kernel $\ker H = \mathcal{P}^{p-2} \cdot \rho^2$ [24]. It is given explicitly by
\[
H(\xi) = \xi + \sum_{j=1}^{[p/2]} (-1)^j(p - 1) \cdots (p - j) j! \lambda_2(p-1) \cdots \lambda_2(p-j) \Delta^j \xi \cdot \rho^{2j}, \quad \xi \in \mathcal{P}^p. \tag{18}
\]
Since $SO(m)$ fixes $x_m$, a zonal spherical harmonic in $\mathcal{H}^p$ is $H(x_m^p)$. By (18), it is given by
\[
H(x_m^p) = x_m^p + \sum_{j=1}^{[p/2]} (-1)^j(p - 1) \cdots (p - j) p(p - 1) \cdots (p - 2j + 1) x_m^{p-2j} \rho^{2j}.
\]
Rewriting the coefficients in terms of the Gamma function, we obtain
\[
H(x_m^p) = \frac{p!}{2^p \Gamma \left( p + \frac{m-1}{2} \right)} \sum_{j=0}^{[p/2]} \frac{(-1)^j \Gamma \left( p + \frac{m-1}{2} - j \right)}{j! (p - 2j)!} (2x_m)^{p-2j} \rho^{2j}.
\]
Up to a normalizing factor, this is the ultraspherical polynomial \( C_p^\nu \) with \( \nu = (m-1)/2 \) [1,24]:
\[
H(x_m^p) = \frac{p! \Gamma \left( \frac{m-1}{2} \right)}{2^p \Gamma \left( p + \frac{m-1}{2} \right)} p^p \rho^{p(m-1)/2} \cos \theta,
\]
where \( x_m/\rho = \cos \theta \). In terms of the Jacobi polynomials, we have
\[
C_p^\nu = \frac{(2\nu)_p}{(\nu + 1/2)_p} p^p \rho^{p-1/2,\nu-1/2}.
\]
where \((a)_p = \Gamma(a + p)/\Gamma(a)\). The choice of the zonal spherical harmonic \( \chi_0 \) for \( M = S^m \) specified in the first table of Section 2 follows. The linearization of the product formula for ultraspherical polynomials [7] reads as
\[
C_p^\nu C_q^\nu = \sum_{k=0}^{\min(p,q)} \frac{(p + q + \nu - 2k)}{(p + q + \nu - k)} \frac{(\nu)_k (\nu - k)_{p-k} (2\nu)_{p+q-k}}{k! (p-k)! (q-k)! (2\nu)_{p+q-2k}} C_{p+q-2k}^\nu.
\]
We now let \( p = q = (m-1)/2 \). In view of (20), the linearization formula above reduces to (2) with \( \alpha = \beta = m/2 - 1 \), and we also obtain an explicit formula for the linearization coefficients. This immediately shows that \( c(j,p;m/2 - 1,m/2 - 1) \) is nonzero if and only if \( 0 \leq j \leq 2p \) is even. By (19), evaluating (2) on \( \cos \theta \), the Jacobi polynomials become zonal spherical harmonics.

Suppressing the argument \( \cos \theta \), by definition, \((P_p^{m/2 - 1,m/2 - 1})^2 \in \mathcal{H}_p \cdot \mathcal{H}_p\). The restriction of the orthogonal projection \( \mathcal{P}^{2p} \to \mathcal{H}_{2j}^2 \), \( j = 0, \ldots, p \), to \( \mathcal{H}_p \cdot \mathcal{H}_p \subset \mathcal{P}^{2p} \) maps \((P_p^{m/2 - 1,m/2 - 1})^2 \) to a nonzero constant multiple of \( F_{2j}^{m/2 - 1,m/2 - 1} \) since \( c(2j,p;m/2 - 1,m/2 - 1) \) is nonzero. Schur’s lemma implies that \( \mathcal{H}_{2j} \) must be a component of \( \mathcal{H}_p \cdot \mathcal{H}_p \) for \( j = 0, \ldots, p \). Theorem A follows for \( M = S^m \).

For \( M = \mathbb{R}P^m \), the real projective \( m \)-space, the eigenspace \( \mathcal{V}_{\lambda} \) corresponding to the \( p \)-th eigenvalue \( \lambda_p = 2p(2p + m - 1) \) of the Laplacian \( \Delta_{\mathbb{R}P^m} \) can be identified with \( \mathcal{H}_{2p} \). Theorem A follows from the spherical case above.

Next we let \( (G, K) = (U(m + 1), U(m) \times U(1)) \) with \( \mathcal{C}P^m = U(m + 1)/(U(m) \times U(1)) \), the complex projective \( m \)-space. Let \( \mathcal{P}^{p,q} \) denote the space of complex homogeneous polynomials of bidegree \((p,q)\) on \( \mathbb{C}^{m+1} \). An element \( \xi \in \mathcal{P}^{p,q} \) is a complex valued homogeneous polynomial that has degree \( p \) in the variables \( z_0, \ldots, z_m \in \mathbb{C} \) and degree \( q \) in the variables \( \bar{z}_0, \ldots, \bar{z}_m \in \mathbb{C} \). By homogeneity, \( \xi \) can be thought of as a function on the unit sphere \( S^{2m+1} \subset \mathbb{C}^{m+1} \).

The space \( \mathcal{P}^{p,p} \) is the complexification of a real \( U(m+1) \)-submodule, and this real submodule is also denoted by the same symbol. An element in \( \mathcal{P}^{p,p} \) can be thought of as a function on \( \mathbb{C}P^m \).

The decomposition in (17) gives
\[
\mathcal{P}^{p,q} = \mathcal{H}^{p,q} \oplus \mathcal{P}^{p-1,q-1} \cdot \rho^2 = \sum_{k=0}^{\min(p,q)} \mathcal{H}^{p-k,q-k} \cdot \rho^{2k},
\]
where $\rho = |z|$, $z = (z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$, and $\mathcal{H}^{p,q}$ is the space of complex harmonic homogeneous polynomials of bidegree $(p, q)$ on $\mathbb{C}^{m+1}$. Then $\mathcal{H}^{p,q}$ is a complex irreducible $U(m+1)$-module. For real valued polynomials we also have

$$\mathcal{P}^{p,p} = \sum_{k=0}^{p} \mathcal{H}^{-k,p-k} \cdot \rho^{2k},$$

as real $U(m+1)$-modules. Here $\mathcal{P}^{p,p}$ is the space of real valued homogeneous polynomials of bidegree $(p, p)$ on $\mathbb{C}^{m+1}$, and $\mathcal{H}^{p,p}$ is the space of real valued homogeneous polynomials of bidegree $(p, p)$ on $\mathbb{C}^{m+1}$. Then $\mathcal{H}^{p,p}$ is a complex irreducible $U(m+1)$-module. For real valued polynomials we also have

$$\mathcal{P}^{p,p} = \sum_{j=0}^{2p} \mathcal{H}^{j,j}$$

as real $U(m+1)$-modules. To prove Theorem A for $M = \mathbb{CP}^m$, it remains to show that equality holds.

Since $U(m)$ fixes $z_m$ and the center $U(1)$ acts on $\mathcal{H}^{p,p}$ trivially, a zonal spherical harmonic in $\mathcal{H}^{p,p}$ is $H(|z_m|^2)$. Here the harmonic projection operator $H$ is the restriction of the harmonic projection for the spherical case above. We have

$$H(|z_m|^2) = \frac{p!(p+m-1)!}{(2p+m-1)!} \rho^{2p} P_{2p}^{(m-1,0)}(\cos(2\theta)),$$

where $|z_m|/\rho = \cos \theta$. (See also [24], formula (5') in Chapter 11.3.2, Vol. 2.) In the linearization of the square $(P_{2p}^{(m-1,0)})^2$ all coefficients $c(j, p; m-1, 0)$, $j = 0, \ldots, 2p$, are positive for $m \geq 2$. As in the spherical case it follows that $\mathcal{H}^{j,j}$, $j = 0, \ldots, 2p$, are $U(m+1)$-submodules of $\mathcal{H}^{p,p}$. The equality in (21) follows.

The cases of the quaternionic projective space $\mathbb{HP}^m$ and the Cayley projective plane $\mathbb{CaP}^2$ are entirely analogous [10]. The zonal spherical functions for $\mathbb{HP}^m$ are explicitly derived in [24] (cf. formula (14) in Chapter 11.7.4, Vol. 2). Another approach for the Cayley projective plane is to determine the highest weights of the class 1 modules with respect to the pair $(F_4, Spin(9))$ and to use the Weyl dimension formula for the multiplicities.

**Proof of Theorem B.** One of the principal results of [22] (Theorem 4.1, p. 136) gives the multiplicity $m$ of $\mathcal{H}^{q,q}$ in $S^2(\mathcal{H}^{p,p})$ as follows:

$$m [\mathcal{H}^{q,q} : S^2(\mathcal{H}^{p,p})] = \frac{1}{2} \left[ \min(q, 2p - q) + 1 + \frac{1 + (-1)^q}{2} \right].$$

By the definition of $\bar{\mathcal{E}}(\mathcal{H}^{p,p})$, we thus have

$$S^2(\mathcal{H}^{p,p}) = \sum_{q=0}^{2p} \frac{1}{2} \left( \min(q, 2p - q) + 1 + \frac{1 + (-1)^q}{2} \right) \mathcal{H}^{q,q} \oplus \bar{\mathcal{E}}(\mathcal{H}^{p,p}).$$

On the other hand, by Proposition 3.3 and Theorem A, we have

$$\mathcal{E}(V_{\lambda_p}) = S^2(V_{\lambda_p})/(V_{\lambda_p} \cdot V_{\lambda_p}) = S^2(V_{\lambda_p})/\left( \sum_{q=0}^{2p} V_{\lambda_q} \right).$$
Since $V_\lambda = H^{p,q}$, combining these two formulas, Theorem B follows.

**Proof of Theorem C.** The proof is based on comparing the multiplicities of some irreducible components of the domain and the image of $\Psi$ in (16). To do this, we first complexify, and consider

$$\Psi: S^2(H^{p,p}) \otimes_R C \to \text{Ind}_{U(m) \times U(1)}^{U(m+1)}(S^2(C^m) \otimes_R C),$$

(22)

where $C^m = T_o(CP^m)$ with $U(m) \times U(1)$-module structure given by $U(m)$ acting on $C^m$ by matrix multiplication, and the center $U(1) \subset U(m+1)$ acting trivially. For the irreducible decompositions, recall that a complex irreducible $U(m+1)$-module $V$ is given by its highest weight which, with respect to the standard maximal torus in $U(m+1)$, is an $(m+1)$-tuple with integral coefficients. We write $V = V^\rho_{U(m+1)}$, where $\rho = (\rho_1, \ldots, \rho_{m+1}) \in \mathbb{Z}^{m+1}$ with

$$\rho_1 \geq \rho_2 \geq \ldots \geq \rho_{m+1}.$$

The center $U(1) \subset U(m+1)$ acts by the weight $\sum_{j=1}^{m+1} \rho_j$.

For example, we have $H^{p,q} = V^\rho_{U(m+1)}$, where $\rho = (p, 0, \ldots, 0, -q)$.

The branching rule for restrictions from $U(m+1)$ to $U(m)$ takes the form

$$V^\rho_{U(m+1)} |_{U(m)} = \sum_\sigma V^\sigma_{U(m)},$$

where the summation runs over all $\sigma \in \mathbb{Z}^m$ for which

$$\rho_1 \geq \sigma_1 \geq \rho_2 \geq \ldots \geq \rho_m \geq \sigma_m \geq \rho_{m+1}.$$

The decomposition of the domain in (22) into irreducible components is one of the technical results in [22] (Theorem 4.1 on p. 136). For $m \geq 3$, we have

$$S^2(H^{p,p}) \otimes_R C \cong \sum_{b=0}^{2p} \sum_{c=0}^{\min(b,2p-b)} \sum_{d=0}^{\min(b,p,e)} n_0(b,c,d) + m_0(b,c,d) \frac{2}{2} \times V^{(b,c,0,\ldots,0,-d,d-b-c)}.$$

(23)

Here $e = \left\lceil \frac{b+c}{2} \right\rceil$, and

$$n_0(b,c,d) = \min(b-c, b-d, p-c, p-d, b+c-2d, 2p-b-c) + 1.$$

$m_0(b,c,d) = 0$ for $b \not\equiv c \pmod{2}$, and for $b \equiv c \pmod{2}$, we have

$$m_0(b,c,d) = \begin{cases} -1 & \text{if } b, d \text{ are odd and } m \equiv 1 \pmod{4} \\ 1 & \text{otherwise.} \end{cases}$$

We now fix a component $V^{(b,c,0,\ldots,0,-d,d-b-c)}$ in $S^2(H^{p,p}) \otimes_R C$. We need to determine the multiplicity

$$m \left[ V^{(b,c,0,\ldots,0,-d,d-b-c)} : \text{Ind}_{U(m) \times U(1)}^{U(m+1)}(S^2(C^m) \otimes_R C) \right].$$

(24)
First note that the multiplicity in (24) is the dimension of the module
\[
\text{Hom}_{U(m)} \left( V^{(b,c,0,...,0,-d,d-b-c)}|_{U(m)}, S^2(C^m) \otimes R C \right).
\] (25)

This follows by Frobenius reciprocity along with the fact that $U(1)$ acts trivially.
In particular, the multiplicity in (24) is nonzero if and only if $V^{(b,c,0,...,0,-d,d-b-c)}$
is disjoint from $\bar{F}(\mathcal{H}^{op})$.
As a real $SO(2m)$-module
\[
S^2(C^m) = S^2(R^{2m}) = \mathcal{H}^0 \oplus \mathcal{H}^2.
\]
Complexifying, and restricting to $U(m) \subset SO(2m)$, we obtain
\[
S^2(C^m) \otimes R C = \mathcal{H}^0|_{U(m)} \oplus \mathcal{H}^2|_{U(m)} = \mathcal{H}^{0,0} \oplus \sum_{j=0}^{2} \mathcal{H}^{2-j,j}.
\]
Thus (25) can be written as
\[
\text{Hom}_{U(m)} \left( V^{(b,c,0,...,0,-d,d-b-c)}|_{U(m)}, \mathcal{H}^{0,0} \oplus \sum_{j=0}^{2} \mathcal{H}^{2-j,j} \right).
\]

The dimension of this module is equal to
\[
m\left[ \mathcal{H}^{0,0} : V^{(b,c,0,...,0,-d,d-b-c)}|_{U(m)} \right] + \sum_{j=0}^{2} m\left[ \mathcal{H}^{2-j,j} : V^{(b,c,0,...,0,-d,d-b-c)}|_{U(m)} \right].
\]
By the branching rule, the first multiplicity is 1 if and only if $c = d = 0$ and
zero otherwise. The remaining multiplicities can be obtained similarly using the
branching rule. For $0 \leq j \leq 2$, we obtain
\[
m\left[ \mathcal{H}^{2-j,j} : V^{(b,c,0,...,0,-d,d-b-c)}|_{U(m)} \right] = \begin{cases} 1 & \text{if } b \geq 2 - j \geq c \text{ and } -d \geq -j \geq d - b - c \\ 0 & \text{otherwise}. \end{cases}
\]
Comparing this with (23), we see that, for $m \not\equiv 1 \pmod{4}$, $p = 3$, $b = 3$, $c = d = 1$, the multiplicity of the component $V^{(3,1,0,...,0,-1,-3)}$ is 2 in the domain
of $\Psi$ and 1 in the image of $\Psi$. The same holds for $p = 4$, $b = 4$, $c = d = 1$.
Theorem C follows.

Acknowledgment. The author wishes to thank Professor Gasper for pointing
out positivity of the linearizing coefficients in products of Jacobi polynomials. The
author is indebted to Professor Knop for various discussions on representation
theory, and for providing the multiplicity formulas for the eigenvalues in the
quaternionic and Cayley cases. The author is also indebted to the referee for
the careful reading of the manuscript and for pointing out various improvements
of the text.

References


Gabor Toth  
Department of Mathematical Sciences  
Rutgers University  
Camden, New Jersey, 08102  
gtoth@camden.rutgers.edu

Received August 3, 2001  
and in final form November 29, 2001