Remark on the Complexified Iwasawa Decomposition

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Communicated by K.-H. Neeb

Abstract. Let $G_\mathbb{R}$ be a real form of a complex semisimple Lie group $G_\mathbb{C}$. We identify the complexification $K_\mathbb{C}A_\mathbb{C}N_\mathbb{C} \subset G_\mathbb{C}$ of an Iwasawa decomposition $G_\mathbb{R} = K_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ as \{ $g \in G_\mathbb{C} \mid gB \in \Omega$ \} where $B \subset G_\mathbb{C}$ is a Borel subgroup of $G_\mathbb{C}$ that contains $A_\mathbb{R}N_\mathbb{R}$ and $\Omega$ is the open $K_\mathbb{C}$–orbit on $G_\mathbb{C}/B$. This is done in the context of subsets $K_\mathbb{C}R_\mathbb{C} \subset G_\mathbb{C}$, where $R_\mathbb{C}$ is a parabolic subgroup of $G_\mathbb{C}$ defined over $\mathbb{R}$, and the open $K_\mathbb{C}$–orbits on complex flag manifolds $G_\mathbb{C}/Q$.

1. Details

Let $G_\mathbb{R}$ be a real form of a complex semisimple Lie group $G_\mathbb{C}$. For simplicity we assume that $G_\mathbb{C}$ is connected and that the adjoint representation of $G_\mathbb{R}$ maps $G_\mathbb{R}$ into the group of inner automorphisms of the Lie algebra of $G_\mathbb{C}$. Choose an Iwasawa decomposition $G_\mathbb{R} = K_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ and a corresponding minimal parabolic subgroup $P_\mathbb{R} = M_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ of $G_\mathbb{R}$. Every parabolic subgroup of $G_\mathbb{R}$ is conjugate to one that contains $P_\mathbb{R}$. Let $R_\mathbb{C} \subset G_\mathbb{C}$ be the complexification of a (real) parabolic subgroup $R_\mathbb{R} \subset G_\mathbb{R}$. Passing to a $G_\mathbb{R}$–conjugate we may, and do, assume $P_\mathbb{R} \subset R_\mathbb{R}$. Let $K_\mathbb{C} \subset G_\mathbb{C}$ be the complexification of the maximal compactly embedded subgroup $K_\mathbb{R} \subset G_\mathbb{R}$. Then $G_\mathbb{R} = K_\mathbb{R}R_\mathbb{R}$. In particular the Lie algebras satisfy $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{r}_\mathbb{R}$, so their complexifications satisfy $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{r}_\mathbb{C}$, and thus $K_\mathbb{C}R_\mathbb{C}$ is open in $G_\mathbb{C}$. Since the projection $G_\mathbb{C} \to G_\mathbb{C}/R_\mathbb{C}$ is an open map, $\Omega_X := K_\mathbb{C}(1R_\mathbb{C})$ is an open $K_\mathbb{C}$–orbit in $X = G_\mathbb{C}/R_\mathbb{C}$. If $g \in G_\mathbb{C}$ then $g \in K_\mathbb{C}R_\mathbb{C}$ if and only if $gR_\mathbb{C} \in K_\mathbb{C}(1R_\mathbb{C}) \subset G_\mathbb{C}/R_\mathbb{C}$.

Lemma 1.1. An element $g \in G_\mathbb{C}$ is contained in $K_\mathbb{C}R_\mathbb{C}$ if and only if $gR_\mathbb{C} \in \Omega_X$.

Let $Q$ be a parabolic subgroup of $G_\mathbb{C}$ and write $\Omega_Z$ for the open orbit $K_\mathbb{C}(1Q)$ on $Z = G_\mathbb{C}/Q$. The argument of Lemma 1.1 shows:

Lemma 1.2. Suppose $K_\mathbb{C}Q = K_\mathbb{C}R_\mathbb{C}$. Then $g \in K_\mathbb{C}R_\mathbb{C}$ if and only if $gQ \in \Omega_Z$.

There is a unique closed $G_\mathbb{R}$–orbit on $X$, and the other $G_\mathbb{R}$–orbits are finite in number and higher in dimension. See [4]. By $(G_\mathbb{R}, K_\mathbb{C})$–duality ([3]; see [2] for a geometric proof), there is a unique open $K_\mathbb{C}$–orbit on $X$ (which thus must be $\Omega_X$), and the other $K_\mathbb{C}$–orbits are finite in number and lower in dimension. Thus $\Omega_X$ is a dense open subset of $X$. Now Lemma 1.1 gives us

ISSN 0949–5932 / $2.50 © Heldermann Verlag
Lemma 1.3. \( K_CR \) is a dense open subset of \( G_C \).

Let \( M_C \), \( A_C \) and \( N_C \) denote the respective complexifications of \( M_R \), \( A_R \) and \( N_R \). Here \( M_C \) is the centralizer of \( A_C \) in \( K_C \). Let \( B \) denote a Borel subgroup of \( G_C \) that contains \( A_R N_R \); in other words \( B = B_M A_C N_C \) where \( B_M \) is a Borel subgroup of \( M_C \). The phenomenon \( K_C Q = K_C R \) of Lemma 1.2 occurs, for example, when \( R \) is the minimal parabolic subgroup \( P_R \) of \( G_R \) and \( Q = B \). In that case \( M_R \subset K_R \), so \( B_M \subset M_C \subset K_C \), and thus \( K_C = K_C B_M \). As \( B = B_M A_C N_C \) now \( K_C A_C N_C = K_C B \). We have proved

Proposition 1.4. Let \( B \) be a Borel subgroup of \( G_C \) that contains \( A_R N_R \). Let \( g \in G_C \). Then \( g \) is contained in \( K_C A_C N_C \) if and only if \( gB \) belongs to the open \( K_C \)-orbit on \( G_C/B \). In particular \( K_C A_C N_C \) is a dense open subset of \( G_C \).

Proposition 1.4 describes the complexified Iwasawa decomposition set \( K_C A_C N_C \) in terms of the open \( K_C \)-orbit on the full flag manifold \( G_C/B \). Usually it is easy to decide whether a given group element belongs to \( K_C A_C N_C \), and that then describes the open \( K_C \)-orbit on \( G_C/B \):

Example 1.5. Let \( G_R = SL_n(\mathbb{R}) \) and \( G_C = SL_n(\mathbb{C}) \). For \( g \in G_C \), let \( s_k \) be the \( k \times k \) submatrix in the upper left corner of \( tgg \). Then \( g \in K_C A_C N_C \) if and only if \( \det(s_k) \neq 0 \) for \( 1 \leq k \leq n \).

Added in proof. After this note was accepted for publication, we learned that there is a nontrivial overlap with material in [1, Appendix B], specifically [1, Theorem B.1.2] and some of the immediately preceding discussion.

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Received December 6, 2001  
and in final form February 2, 2002