Resolutions of Singularities of Varieties of Lie Algebras of Dimensions 3 and 4

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Abstract. We will determine the singular points and a resolution of singularities of each irreducible component of the varieties of the Lie algebras of dimension 3 and 4 over \( \mathbb{C} \).

1. Introduction

Let \( \mathcal{L}_n \) be the projective variety of the Lie algebras of dimension \( n \) over \( \mathbb{C} \). In some recent papers many results on the irreducible components of \( \mathcal{L}_n \) were found for small values of \( n \). In [2] Carles and Diakité determined the open orbits and described the irreducible components of \( \mathcal{L}_n \) as orbit closures for \( n \leq 7 \). In [6] Kirillov and Neretin determined the number of irreducible components of \( \mathcal{L}_n \) and their dimension for \( n \leq 6 \); they also determined representatives of the generic orbits of any component of \( \mathcal{L}_4 \). In [1] Burde and Steinhoff gave a classification of any orbit closure of \( \mathcal{L}_4 \). The variety \( \mathcal{L}_3 \) has two irreducible components and one of them is a linear variety; the variety \( \mathcal{L}_4 \) has four irreducible components.

In this paper we will determine the singular points and find a resolution of singularities of each irreducible component of \( \mathcal{L}_3 \) and \( \mathcal{L}_4 \). By using the classification of the Lie algebras of dimension 3 and 4 over \( \mathbb{C} \), we will describe each irreducible component by giving algebraic equations of it. The first classification is well known (see [3]); the second one may be deduced from [8] and from [9] (see [1]); nevertheless we will give a short proof of it. Each resolution of singularities is a subvariety of the product of the irreducible component with a suitable grassmannian or is a resolution of singularities of a variety of this type. We observe that the results of this paper are also true over any algebraically closed field \( K \) such that \( \text{char } K \neq 2 \).

2. Preliminaries

For any \( n \in \mathbb{N} \) let \( \mathcal{L}_n \) be the subvariety of the projective space

\[ \mathbb{P}(\text{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n)) \]

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of all $[\alpha]$ such that $\alpha(x \wedge \alpha(y \wedge z)) + \alpha(y \wedge \alpha(z \wedge x)) + \alpha(z \wedge \alpha(x \wedge y)) = 0$ for any $x, y, z \in \mathbb{C}^n$, which we regard as the variety of all the Lie algebras over $\mathbb{C}$ of dimension $n$. For any $[\alpha] \in L_n$ let $L_\alpha$ be the Lie algebra defined by $\alpha$. The group $GL(n, \mathbb{C})$ acts on $\text{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n)$ by the relation $\alpha \cdot G(\mathbb{C}x \wedge \mathbb{C}y) = G(\alpha(x \wedge y))$, for any $G \in GL(n, \mathbb{C})$, $\alpha \in \text{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n)$, $x, y \in \mathbb{C}^n$ and this induces an action of $GL(n, \mathbb{C})$ on $L_n$; the orbits of this action are the classes of isomorphic Lie algebras. For any $n, n' \in \mathbb{N}$ let $M_{n \times n'}$, $M_n$ and $S_n$ be the vector spaces of all $n \times n'$ matrices, of all $n \times n$ matrices and of all $n \times n$ symmetric matrices respectively over $\mathbb{C}$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{C}^n$ and let us order the set $\{e_i \wedge e_j : i, j = 1, \ldots, n, i < j\}$, writing it as $\{E_1, \ldots, E_m\}$.

For any $\alpha \in \text{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n)$ let $A_\alpha \in M_{n \times m}$ be the matrix of $\alpha$ with respect to the previous bases; then $A_{\alpha G} = G A_\alpha \tilde{G}$ where $\tilde{G} \in GL(m, \mathbb{C})$ is the matrix whose $(h, k)$ entry is the determinant of the $2 \times 2$ submatrix of $G^{-1}$ obtained by choosing the rows $i, j$ with $E_h = e_i \wedge e_j$ and the columns $i', j'$ with $E_k = e_{i'} \wedge e_{j'}$. If $n = 3$ we set $E_1 = e_2 \wedge e_3$, $E_2 = e_3 \wedge e_1$, $E_3 = e_1 \wedge e_2$ and we get $A_{\alpha G} = (\det G)^{-1} G A_\alpha G'$. Then we have

$$L_3 = \{[\alpha] \in \mathbb{P}(\text{Hom}(\mathbb{C}^3 \wedge \mathbb{C}^3, \mathbb{C}^3)) : \text{cof} A_\alpha \in S_3\}$$

where for any $A = (a_{ij}) \in M_n$ cof $A$ is the matrix whose $(i, j)$ entry is the algebraic complement of $a_{ij}$.

We recall that, up to isomorphisms, we have the following non-abelian Lie algebras of dimension 3 over $\mathbb{C}$ ([3]), which may also be obtained as in the proof of theorem 4.1:

- $l_3 : [e_1, e_2] = e_2$, $[e_1, e_3] = a e_3$, $[e_2, e_3] = 0$, $a \in \mathbb{C}$,
- $n_3 : [e_1, e_2] = [e_1, e_3] = 0$, $[e_2, e_3] = e_1$,
- $r_3 : [e_1, e_2] = e_2$, $[e_1, e_3] = e_2 + e_3$, $[e_2, e_3] = 0$,
- $\text{sl}(2, \mathbb{C}) : [e_1, e_2] = e_3$, $[e_1, e_3] = -2 e_1$, $[e_2, e_3] = 2 e_2$,

where the only pairs of isomorphic Lie algebras are $\{l_3, l_3^{-1}\}$, $a \neq 0, a^{-1}$, and $n_3$, the Heisenberg Lie algebra, is the only nilpotent one. Hence the following subvarieties:

$$\mathcal{W}_1 = \{[\alpha] \in L_3 : A_\alpha \in S_3\}$$
$$= \{[\alpha] \in L_3 : \text{for any } v \in L_\alpha \text{ \text{tr ad} } v = 0\}$$

which is isomorphic to $\mathbb{P}(S_3)$, and

$$\mathcal{W}_2 = \{[\alpha] \in L_3 : \text{rank } A_\alpha \leq 2\}$$
$$= \{[\alpha] \in L_3 : L_\alpha \text{ has an abelian ideal of dimension } 2\}$$

that is the subvariety of the solvable Lie algebras, are the irreducible components of $L_3$.

For any $n, n' \in \mathbb{N}$ let $G_{n', n}$ be the grassmannian of all the subspaces of $\mathbb{C}^n$ of dimension $n'$.

3. **The variety of the Lie algebras of dimension 3**

We identify $\alpha$ with $A_\alpha$ and we set $A = (a_{ij})$ for any $A \in M_3$. 

Lemma 3.1. We have \( \mathcal{W}_2 = \{ [A] \in \mathbb{P}(M_3) : \dim(\ker A \cap \ker A^i) \geq 1 \} \).

Proof. Since both subsets are stable with respect to the action of \( GL(3, \mathbb{C}) \) it is sufficient to show that if \( A \) is such that \( a_{j1} = 0, \ j = 1, 2, 3 \), the condition \( \text{cof} A \in S_3 \) is equivalent to the condition \( \dim(\ker A \cap \ker A^i) \geq 1 \). But in this case both these conditions are equivalent to the following one: \( \text{rank} A \leq 1 \) or \( a_{ij} = 0, \ j = 2, 3 \); hence we get the claim. The result also follows from the classification of the Lie algebras of dimension 3 over \( \mathbb{C} \).

Let

\[ \mathcal{W}_2' = \{(H,[A]) \in \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}(M_3) : H \subseteq \ker A \cap \ker A^i \} \]

and let \( \pi, \pi' \) be the canonical projections of \( \mathcal{W}_2' \) onto \( \mathbb{P}^2(\mathbb{C}) \) and \( \mathcal{W}_2 \) respectively.

Proposition 3.2. \( \mathcal{W}_2 \) is irreducible, \( \dim \mathcal{W}_2 = 5 \) and \( \pi' \) is a resolution of singularities of \( \mathcal{W}_2 \). The set of the singular points of \( \mathcal{W}_2 \) is \( \mathcal{Z} = \{ [A] \in \mathbb{P}(M_3) : \dim(\ker A \cap \ker A^i) = 2 \} \), that is the orbit of \( \mathfrak{n}_3 \), and \( \dim \mathcal{Z} = 2 \).

Proof. For \( i = 1, 2, 3 \) let \( \mathcal{U}_i \) be the open subset of \( \mathbb{P}^2(\mathbb{C}) \) given by the condition that the \( i \)-th coordinate doesn’t vanish and let \( \mathcal{F}_i \) be the subset of \( \mathbb{P}(M_3) \) of all \([A]\) such that the \( i \)-th row and column of \( A \) vanish. Let \( G_i \in GL(3, \mathbb{C}) \) be such that \( G_i(e_i) = (e_i) \) and let \( G_1^i, G_2^i, G_3^i \) be the columns of \( G_i \). Let \( \phi_i : \mathcal{U}_i \rightarrow GL(3, \mathbb{C}) \) be such that for any \( H = \langle (x_1, x_2, x_3) \rangle \in \mathcal{U}_i \) the \( i \)-th column of \( \phi_i(H) \) is \( G_i^j - \sum_{j \neq i} x_j (x_i)^{-1} G_i^j \), the others are equal to those of \( G_i \); then \( \phi_i(H)(H) = \langle e_i \rangle \). If \( \mathcal{A}_i = \pi^{-1}(\mathcal{U}_i) \) the map \( (H, [A]) \mapsto (H, [(\phi_i(H)^{-1})^t A \phi_i(H)^{-1}]) \) from \( \mathcal{A}_i \) to \( \mathcal{U}_i \times \mathcal{F}_i \) is an isomorphism. Hence \( \mathcal{W}_2' \), with the map \( \pi \), is a vector bundle on \( \mathbb{P}^2(\mathbb{C}) \) with fibers isomorphic to \( \mathbb{P}(M_2) \).

The map \( ([A]) \mapsto (\ker A \cap \ker A^i, [A]) \) from \( \mathcal{W}_2 \) to \( \mathcal{W}_2' \) is regular except in the points of \( \mathcal{Z} \), where the fibers of \( \pi' \) have dimension 1, and is a birational inverse of \( \pi' \). Let \( \mathcal{Z}' = \{(H,[A]) \in G_{2,3} \times \mathbb{P}(M_3) : H \subseteq \ker A \cap \ker A^i \} \). If \( \pi_1 \) and \( \pi_2 \) are the canonical projections of \( \mathcal{Z}' \) on \( G_{2,3} \) and \( \mathcal{Z} \) respectively, \( \pi_2 \) is a birational morphism and the fibers of \( \pi_1 \) have only one point. Hence \( \mathcal{Z}' \) and \( \mathcal{Z} \) are irreducible of dimension 2 and \( (\pi')^{-1}(\mathcal{Z}) \) is irreducible of dimension 3. Then by Theorem 2 of chap. II, §4 of [10] we get the claim.

Corollary 3.3. The set of the singular points of \( \mathcal{L}_3 \) is \( \mathcal{W}_1 \cap \mathcal{W}_2 \), that is the union of the orbits of \( \mathfrak{n}_3 \) and \( \mathfrak{l}_{-1} \).

For any \([\alpha] \in \mathcal{L}_n \) the tangent space in \([\alpha] \) to \( \mathcal{L}_n \) is \( \mathbb{P}(V_\alpha) \), where \( V_\alpha \) is the vector space of 2-cocycles in the cohomology of \( L_\alpha \) as \( L_\alpha \)-module ([5]). By the equations of the space of 2-cocycles of a Lie algebra we have found that the dimensions of the tangent spaces to \( \mathcal{L}_3 \) in \( \mathfrak{n}_3 \) and \( \mathfrak{l}_{-1} \) are 7 and 6 respectively.

4. Classification of the Lie algebras of dimension 4 over \( \mathbb{C} \)

For any \((\beta, \gamma) \in \mathbb{C}^2 \) let \([[(\beta, \gamma)]\) and \([[(\beta)]\) be the orbit in \( \mathbb{P}^2(\mathbb{C}) \) of \([1, \beta, \gamma] \) and \([1, \beta, 1-\beta] \) respectively with respect to the action of the group of the permutations of the coordinates of \( \mathbb{P}^2(\mathbb{C}) \).
Theorem 4.1. We have $[\alpha] \in L_4$ if and only if $L_\alpha$ is isomorphic to one and only one of the following Lie algebras (where we omit $[e_i, e_j]$, $i, j \in \{1, \ldots, 4\}$, if it is 0):

$$
\begin{align*}
g_{[[\beta, \gamma]]} &: \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = \beta e_2, \quad [e_4, e_3] = \gamma e_3, \quad \beta, \gamma \in \mathbb{C}; \\
g_{[[\beta]]} &: \quad [e_2, e_3] = e_1, \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = \beta e_2, \\
& \quad [e_4, e_3] = (1 - \beta)e_3, \quad \beta \in \mathbb{C}; \\
g_c &: \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = e_2 + e_3, \quad c \in \mathbb{C}; \\
a_1 &: \quad [e_2, e_3] = e_1, \quad [e_4, e_1] = 2e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = e_2 + e_3; \\
a_2 &: \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = e_1 + e_2, \quad [e_4, e_3] = e_2 + e_3; \\
a_3 &: \quad [e_3, e_2] = e_2, \quad [e_4, e_1] = e_1; \\
an &: \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_1, e_2] = e_4; \\
a_6 &: \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = e_3; \\
a_7 &: \quad [e_2, e_3] = e_1; \\
a_8 &: \quad [e_2, e_3] = e_1, \quad [e_4, e_3] = e_2.
\end{align*}
$$

Proof. Let $L$ be a Lie algebra over $\mathbb{C}$ of dimension 4. Let $H$ be a Cartan subalgebra of $L$, $h \in H$ be such that $H = L_0(\text{ad} \ h) = \{v \in L : \exists \ n \in \mathbb{N} : (\text{ad} \ h)^n v = 0\}$ and ad $h$, if not nilpotent, has the eigenvalue 1. 

$H'$ be a subspace of $L$ such that $H \oplus H' = L$, $[h, H'] = H'$. Let $\dim H = 1$. Then $H' = [L, L]$. Let $\{x, y, z\}$ be a basis of $H'$ such that the matrix of $\text{ad} H' h$ with respect to it is in Jordan canonical form. From the Jacobi's relations between $h$ and the pairs of elements of $\{x, y, z\}$, when $\text{ad} H' h$ is represented by a diagonal matrix with diagonal entries 1, $\beta, \gamma$ respectively, $\beta, \gamma \neq 0$, we get

$$
(\beta + 1)[x, y] = [h, [x, y]], \quad (\gamma + 1)[x, z] = [h, [x, z]], \quad (\beta + \gamma)[y, z] = [h, [y, z]],
$$

hence either $H'$ is abelian or, permuting $x, y, z$ and multiplying them by a scalar if necessary, $\beta + \gamma = 1$ and $H'$ is a Heisenberg Lie algebra with $x = [y, z]$. We get the Lie algebras $g_{[[\beta, \gamma]]}, \beta, \gamma \neq 0$, and $g_{[[\beta]]}, \beta \neq 0, 1$, respectively. If $\text{ad} H' h$ is represented by two Jordan blocks, the first one of order 2 and eigenvalue 1, the second one of eigenvalue $c \neq 0$, we get

$$
(c + 1)[z, x] = [h, [z, x]], \quad [z, x] + (c + 1)[z, y] = [h, [z, y]], \quad 2[x, y] = [h, [x, y]],
$$

hence $[z, x] = [z, y] = 0$ and either $H'$ is abelian or $c = 2$ and $H'$ is a Heisenberg Lie algebra, with (multiplying $x$ and $y$ by a scalar) $[x, y] = z$. We get the Lie algebras $g_c, c \neq 0$, and $a_1$ respectively. If $\text{ad} H, h$ is represented by only one Jordan block we get

$$
2[x, y] = [h, [x, y]], \quad 2[x, z] = [h, [x, z]] - [x, y], \quad 2[y, z] = [h, [y, z]] - [x, z],
$$

hence $H'$ is abelian and we get the Lie algebra $a_2$.

Let $\dim H = 2$. Then, since $H$ is abelian, $\text{ad}_L H$ is abelian and $H' = [H, H'] = [H, L]$. Let $\{x, y\}$ be a basis of $H'$ such that the matrix of $\text{ad} H' h$ with respect to $\{x, y\}$ is in Jordan canonical form. We have to require

$$
[h, [x, y]] = [x, [h, y]] + [y, [x, h]] = (\text{tr} \ \text{ad} H' h)[x, y],
$$
hence either \([x, y] = 0\) or for any \(v \in H\) \(\text{ad}_H v\) has the eigenvalues 1, -1 and \(\dim \text{ad} H \leq 1\). If \(\dim \text{ad} H = 2\) and there exist in \(H\) elements \(v\) such that \(\text{ad}_H v\) has two different eigenvalues, we may choose \(w, z \in H\) such that with respect to the basis \({x, y}\) \(\text{ad}_H w\) and \(\text{ad}_H z\) are represented by two diagonal matrices with diagonal entries 1, 0 and 0, 1 respectively, hence we get the Lie algebra \(a_3\). If \(\dim \text{ad} H = 2\) but for any \(v \in H\) \(\text{ad}_H v\) has only one eigenvalue we may choose \(h, z \in H\) such that with respect to the basis \({x, y}\) \(\text{ad}_H h\) and \(\text{ad}_H z\) are represented respectively by the identity matrix and by the nilpotent Jordan block of order 2, hence we get the Lie algebra \(g_{[[0, 0]]}\). If \(\dim \text{ad} H = 1\) let \(z \in H \setminus \{0\}\) be such that \(\text{ad} z = 0\). If the Jordan form of \(\text{ad}_H h\) is diagonal and \([x, y] = 0\) we get the Lie algebras \(g_{[[0, \gamma]]}, \gamma \in \mathbb{C} \setminus \{0\}\). If the Jordan form of \(\text{ad}_H h\) is diagonal and \([x, y] \notin \langle z \rangle\) we may assume \(h = [x, y]\) and we get the Lie algebra \(a_4\). If the Jordan form of \(\text{ad}_H h\) is diagonal and \([x, y] \in \langle z \rangle \setminus \{0\}\) we may assume \([x, y] = z\) getting the Lie algebra \(a_5\). If the Jordan form of \(\text{ad}_H h\) has only one Jordan block we get the Lie algebra \(g_0\).

Let \(\dim H = 3\). If \(H\) is abelian, since \(\dim \text{ad} H = 1\) there exist \(y, z \in H\) linearly independent such that \(\text{ad} y = \text{ad} z = 0\), hence we get the Lie algebra \(g_{[[0, 0]]}\). If \(H\) is a Heisenberg Lie algebra, since the subset of all \(v \in H\) such that \(H = L_0(\text{ad} v)\) is open in \(H\), we may assume \(H = \langle h, y, z \rangle\) with \([h, y] = z, [h, x] = x, x \notin H\). Since \(\text{ad} h\) and \(\text{ad} z\) commute, \([z, x] \in \langle x \rangle\). Since \(\text{ad} y\) and \(\text{ad} z\) commute, if we had \([z, x] \neq 0\) we would have \([y, x] \in \langle x \rangle\) and then, since \(\text{ad}_H y\) and \(\text{ad}_H h\) commute, \(\text{ad} y\) and \(\text{ad} h\) would commute; but this holds if and only if \(\text{ad} z = 0\). Hence \([z, x] = 0\) and \([y, x] \in \langle x \rangle\). Since \(\dim[H, x] = 1\) we may choose \(y\) such that \([y, x] = 0\); we get the Lie algebra \(a_6\).

Let \(\dim H = 4\), that is \(L\) nilpotent. If \(L\) isn’t abelian there exists \(x \neq 0\) such that \(x \in Z(L) \cap [L, L]\). If \(x = [y, z]\), since \(H'' = \langle x, y, z \rangle\) is a nilpotent subalgebra, \(\dim H'' = 3\) and \(H''\) is a Heisenberg Lie algebra. Since \(L\) is nilpotent \([h, H''] \subseteq H''\) for any \(h \in L\). Since \([h, x] = 0\) it is possible to choose \(h, x, y, z\) such that \(h \notin H''\), the matrix of \(\text{ad}_H h\) with respect to the basis \({x, y, z}\) is in Jordan canonical form and \([h, y] = 0\) (in fact if \([h, y] = x\) then \([h + z, y] = 0\)). We get the Lie algebras \(a_7\) and \(a_8\).

5. The variety of the Lie algebras of dimension 4

For any Lie algebra \(L\) let \(Z(L)\) be the center of \(L\).

**Proposition 5.1.** \(\mathcal{L}_4\) is the union of the following closed subsets:

\[
\begin{align*}
C_1 & = \{ [\alpha] \in \mathcal{L}_4 : L_\alpha \text{ has an abelian ideal of dimension 3} \}, \\
C_2 & = \{ [\alpha] \in \mathcal{L}_4 : L_\alpha \text{ has a nilpotent ideal } J_\alpha \text{ of dimension 3 such that } \frac{1}{2} \text{tr} \text{ad} v \text{ is eigenvalue of } \text{ad}_{J_\alpha} v \text{ for any } v \in L_\alpha \}, \\
C_3 & = \{ [\alpha] \in \mathcal{L}_4 : \dim[L_\alpha, L_\alpha] \leq 2, \text{ ad}_{[L_\alpha, L_\alpha]} L_\alpha \text{ is abelian} \}, \\
C_4 & = \{ [\alpha] \in \mathcal{L}_4 : Z(L_\alpha) \neq \{0\}, \text{ tr} \text{ad} v = 0 \text{ for any } v \in L_\alpha \}
\end{align*}
\]

and \(C_i \nsubseteq \bigcup_{j \neq i} C_j \text{ for } i, j = 1, \ldots, 4\).
Proof. Since by Theorem 4.1 each one of these subsets is the union of the orbits of the following Lie algebras:

\[ C_1 : \mathfrak{g}_{[[0,1]]}, \mathfrak{g}_2, \mathfrak{a}_2, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8 \]
\[ C_2 : \mathfrak{g}_{[[0,1]]}, \mathfrak{g}_2, \mathfrak{g}_0, \mathfrak{a}_1, \mathfrak{a}_5, \mathfrak{a}_7, \mathfrak{a}_8 \]
\[ C_3 : \mathfrak{g}_{[[0,1]]}, \mathfrak{g}_{[[0]]}, \mathfrak{g}_0, \mathfrak{a}_3, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8 \]
\[ C_4 : \mathfrak{g}_{[[0,1]]}, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_7, \mathfrak{a}_8 \]

where \( \beta, \gamma, c \in \mathbb{C} \), we get the claim.

For any \( i = 1, \ldots, 4 \) let \( \mathcal{A}_i = \{ J \in G_{3,4} : e_i \not\in J \} \) and let \( \{ i_1, i_2, i_3 \} = \{ 1, \ldots, 4 \} \setminus \{ i \} \), \( i_1 < i_2 < i_3 \). If \( J \in \mathcal{A}_i \) let \( J = \langle e_{i_1}, e_{i_2}, e_{i_3}^J \rangle \), where, with respect to the basis \( \{ e_{i_1}, e_{i_2}, e_{i_3}, e_i \} \), for \( j = 1, 2, 3 \) the \( j \)-th coordinate of \( e_{i_j}^J \) is 1 and for \( k \in \{ 1, 2, 3 \}, k \neq j \) the \( k \)-th coordinate of \( e_{i_k}^J \) is 0. Let

\[ C'_1 = \{(J, [\alpha]) \in G_{3,4} \times C_1 : J \text{ is an abelian ideal of } L_\alpha \} \]

and let \( p_1, p'_1 \) be the canonical projections of \( C'_1 \) onto \( G_{3,4} \) and \( C_1 \) respectively.

**Proposition 5.2.** \( C_1 \) is irreducible, \( \dim C_1 = 11 \) and \( p'_1 \) is a resolution of singularities of \( C_1 \). The set of the singular points of \( C_1 \) is \( Z_1 = \{ [\alpha] \in C_1 : L_\alpha \) is nilpotent and \( \dim [L_{a_7}, L_\alpha] \leq 1 \}, \) that is the orbit of \( a_7 \), and \( \dim Z_1 = 5 \).

**Proof.** Let \( \mathcal{A} : \mathfrak{g}_\mathcal{A} \times \mathbb{P}(M_3) \rightarrow \mathcal{A}' \)
defined by \( \xi_i : \mathcal{A} \times \mathbb{P}(M_3) \rightarrow \mathcal{A}' \)
defined by \( \xi_i(J, [\alpha]) = \langle J, [\alpha] \rangle \) where \( [\alpha] \) is such that in \( L_\alpha \) \( \text{ad}_J e_i \) is represented by \( A \) with respect to the basis \( \{ e_{i_1}, e_{i_2}, e_{i_3} \} \) is an isomorphism, hence \( C'_1 \), with the map \( p_1 \), is a vector bundle and \( \dim C'_1 = 11 \).

The map \( p'_1 \) is birational and \( (p'_1)^{-1} \) is regular in the open subset of \( C_1 \) of all \( [\alpha] \) such that \( L_\alpha \) is not nilpotent or there exists \( x \in L_\alpha \) such that \( \dim [x, L_\alpha] = 2 \) (we set \( (p'_1)^{-1}(J, [\alpha]) = \langle J, [\alpha] \rangle \) where \( J \) is the subspace of all the nilpotent elements of \( L_\alpha \) such that \( \dim [x, L_\alpha] = 1 \). It isn’t regular in the points of \( Z_1 = \{ [\alpha] \in C_1 : L_\alpha \) is nilpotent and \( \dim [L_{a_7}, L_\alpha] = 1 \}, \) that is the orbit of \( a_7 \), since the fibers of \( p'_1 \) on the elements of \( Z_1 \) have dimension 1. The variety \( Z'_1 := (p'_1)^{-1}(Z_1) \), with the map \( p_1|Z'_1 \), is a bundle on \( G_{3,4} \) whose fibers are isomorphic to \( \mathbb{P}(N'_3) \), where \( N'_3 \) is the variety of all the nilpotent \( 3 \times 3 \) matrices over \( \mathbb{C} \) of rank less or equal 1; hence it is irreducible of dimension 6, which shows the claim.

Let

\[ C'_2 = \{(J, [\alpha]) \in G_{3,4} \times C_2 : J \text{ is a nilpotent ideal of } L_\alpha \} \]

and for any \( v \in L_\alpha \) \( \frac{1}{2} \text{tr} \text{ad} v \) is eigenvalue of \( \text{ad}_J v \).

**Lemma 5.3.** If \( (J, [\alpha]) \in C'_2 \) and \( v \in L_\alpha \) then \( [J, J] \) is contained in the eigenspace of \( \text{ad}_J v \) corresponding to \( \frac{1}{2} \text{tr} \text{ad} v \).

**Proof.** Let \( y \neq 0 \) belong to the previous eigenspace but \( [J, J] \not\subseteq \langle y \rangle \). Then we may choose a basis \( \{ y, x, z \} \) of \( J \) such that \( [J, J] \subseteq \langle x \rangle \). Since \( \{ x, v \} \subseteq \langle x \rangle \) (in fact \( 0 = [x, [y, v]] = [y, [x, v]] \), hence \( [x, v] \in \langle x, y \rangle \), in the same way \( [x, v] \in \langle x, z \rangle \), there exist \( a, b, c, d \in \mathbb{C} \) such that \( [v, y] = ay, [v, x] = bx, [v, z] = (a-b)z + cx + dy \), hence by the condition \( [y, [z, v]] = [z, [y, v]] + [v, [z, y]] \) we get \( a = b \).
Let \( S' = \{ (H, [(A, B)]) \in \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}(S_3 \times M_3) : \text{Im} \ A \subseteq H, H \subseteq \ker(\nu - (\frac{1}{2} \text{tr} B)I_3) \}; \)

let \( S \) be the image of the canonical projection of \( S' \) on \( \mathbb{P}(S_3 \times M_3) \) and let \( s, s' \) be the canonical projections of \( S' \) on \( \mathbb{P}^2(\mathbb{C}) \) and \( S \) respectively.

**Lemma 5.4.** \( S \) is irreducible, \( \dim S = 8 \) and \( s' \) is a resolution of singularities of \( S \). The set of the singular points of \( S \) is

\[ \hat{S} = \{ [(A, B)] \in S : A = 0, \dim \ker(\nu - (\frac{1}{2} \text{tr} B)I_3) \geq 2 \}, \]

which is irreducible of dimension 4.

**Proof.** The variety \( S' \) with the map \( s \) is a vector bundle on \( \mathbb{P}^2(\mathbb{C}) \) with fibers of dimension 6. The map \( s' \) is birational and \( (s')^{-1} \) is regular in the open subset of all \( [(A, B)] \) such that \( A \neq 0 \) or \( \dim \ker(\nu - (\frac{1}{2} \text{tr} B)I_3) = 1 \). It isn’t regular in the points of \( \hat{S} \), where the generic fiber of \( s' \) has dimension 1, and \( \hat{S} := (s')^{-1}(\hat{S}) \) is irreducible of dimension 5 (the fiber of \( s|_{\hat{S}} \) in \( H \) is birational to \( \{(V, [B]) \in G_{2,3} \times \mathbb{P}(M_3) : H \subseteq V \subseteq \ker(\nu - (\frac{1}{2} \text{tr} B)I_3) \} \), hence has dimension 3), which shows the claim.

Let \( p_2 \) and \( p'_2 \) be the canonical projections of \( C'_2 \) on \( G_{3,4} \) and \( C_2 \) respectively.

**Lemma 5.5.** \( C'_2 \), with the map \( p_2 \), is a bundle on \( G_{3,4} \) with fibers isomorphic to \( S \).

**Proof.** Let \( U_i = (p_2)^{-1}(A_i), i = 1, \ldots, 4 \). For any \( (J, [\alpha]) \in C'_2 \) let \( \alpha_J \in \text{Hom}(J \wedge J, J) \) be defined by \( \alpha_J(v \wedge v') = \alpha_J(v \wedge v') \) for any \( v, v' \in J \). The map \( \nu_J : A_i \times S \to U_i \) such that \( \nu_J((J, [(A, B)])) = (J, [\alpha]) \) where \( \alpha \) is such that the matrix of \( \alpha_J \) with respect to the bases \( \{e^j_{i2} \wedge e^j_{i3}, e^j_{i1} \wedge e^j_{i2}, e^j_{i3}\} \) and \( \{e^j_{i1}, e^j_{i2}, e^j_{i3}\} \) is \( A \) and in \( L_\alpha \) the matrix of \( \text{ad} J e_i \) with respect to the basis \( \{e^j_{i1}, e^j_{i2}, e^j_{i3}\} \) is \( B \) is an isomorphism, which shows the claim.

For any \( i = 1, \ldots, 4 \) and \( J \in A_i \) let \( B^J_i = \{e^j_{i1}, e^j_{i2}, e^j_{i3}\} \). Let \( J \in A_i \cap A_{i'} \) and let \( G_J \) be the matrix whose columns are the coordinates of the elements of \( B^J_i \) with respect to \( B^J_{i'} \). Let \( \delta : S_3 \times M_3 \to M_{3 \times 6} \) be the isomorphism such that, by regarding \( \delta((A, B)) \) as a block matrix, we have

\[ \delta((A, B)) = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}. \]

Then, by using the notations of the proof of Lemma 5.5, we have that the automorphism \( (\nu_i)^{-1} \circ \nu_J \) of \( (A_i \cap A_{i'}) \times S \) is given by

\[ (\nu_i)^{-1} \circ \nu_J((J, [(A, B)])) = (J, \delta^{-1}(G_J \delta((A, B)) G_{J'})). \]

Let \( C''_2 \) be the vector bundle on \( G_{3,4} \) which is the union of open subsets \( U'_i, i = 1, \ldots, 4 \), with isomorphisms \( \nu'_i : A_i \times S' \to U'_i \) such that

\[ (\nu'_i)^{-1} \circ \nu'_J(J, ([A, B])) = (J, (H_J, [\delta^{-1}(G_J \delta((A, B)) G_{J'})))). \]

where if \( H = [h_1, h_2, h_3] \) then \( H_J = [h^J_1, h^J_2, h^J_3] \) is such that \( (h^J_1, h^J_2, h^J_3, 0) = G_J^{-1}(h_1, h_2, h_3, 0) \). Let \( \nu''_i : C''_2 \to C'_2 \) be the morphism such that \( p''_i(U'_i) = U_i \) and, if \( p''_i \) is \( p''_i|_{U'_i} \) as map onto \( U_i \), we have \( \nu_i \circ (\text{id}_{A_i} \times s') = p''_i \circ \nu'_i \) for any \( i = 1, \ldots, 4 \). Then \( p''_i \) is a resolution of singularities of \( C'_2 \).
Lemma 5.7. Let $r$ be the canonical projection of $\mathbb{P}^2$ onto $\mathbb{P}^1$. We may set $p_2 \circ p''$ is a resolution of singularities of $C_2$. The set of the singular points of $C_2$ is $Z = \hat{Z}_1 \cup \hat{Z}_2$, where $\hat{Z}_2 = \{[\alpha] \in C_2 : L_\alpha$ is nilpotent$\}$ and $\hat{Z}_2 = \{[\alpha] \in C_2 : L_\alpha$ has an abelian ideal of dimension 3 and for any $v \in L_\alpha$ $\dim\Im(\text{ad} - (\frac{1}{2}\text{tr} v)1) \leq 1\}$. We have that $\hat{Z}_2$ is irreducible of dimension 8 and is the union of the orbits of $\alpha_7$ and $\alpha_8$; $\hat{Z}_2$ is irreducible of dimension 7 and is the union of the orbits of $g_{[0,1]}$, $g_0$ and $\alpha_7$.

Proof. For any $C$ is the set of the singular points of $C_2$, we have that $\hat{Z}_2$ is irreducible by Proposition 5.5. Let $\hat{Z}_2 := (p_2)^{-1}(\hat{Z}_2')$ and $\hat{Z}_2' = (p_2)^{-1}(\hat{Z}_2)$. If we set $\hat{Z}_2 = \{(A,B) \in S : B$ is nilpotent$\}$ we have that $\hat{Z}_2$ is irreducible and $\dim \hat{Z}_2 = 6$ (in fact, by Lemma 5.4, $(s')^{-1}(\hat{Z}_2)$ has these properties). Since the fibers of $p_2|_{\hat{Z}_2'}$ are isomorphic to $\hat{Z}_2$ we get that $\hat{Z}_2$ is irreducible of dimension 9 and $\hat{Z}_2$ is irreducible of dimension 8, hence by Theorem 2 of chap. II, §4 of [10] the points of $\hat{Z}_2$ are singular for $C_2$. By Lemma 5.4 and Lemma 5.5 $\hat{Z}_2$ is irreducible of dimension 7 and the points of $\hat{Z}_2 \setminus \hat{Z}_2$ are singular for $C_2$, hence we get the claim.

For any $n \in \mathbb{N}$ let $C_n = \{(A,B) \in M_n \times M_n : [A,B] = 0\}$. If $(x_0, \ldots, x_7)$ are coordinates of $\mathbb{C}^8$, we set

$$A = \begin{pmatrix} x_0 & x_2 \\ x_4 & x_0 + x_6 \end{pmatrix}, \quad B = \begin{pmatrix} x_1 & x_3 \\ x_5 & x_1 + x_7 \end{pmatrix}$$

and we regard $C_2$ as a subvariety of $\mathbb{C}^8$. Let $\mathcal{V}' = \{(x_0, \ldots, x_7) \in C_2 : (x_2, \ldots, x_7) \neq (0, \ldots, 0)\}$; then the map $u : \mathcal{V}' \rightarrow \mathbb{P}^5(\mathbb{C})$ such that $u((x_0, \ldots, x_7)) = [x_2, \ldots, x_7]$ is a morphism. Let $\mathcal{V} = u(\mathcal{V}')$, let:

$$\mathcal{W} = \{((x_0, \ldots, x_7), [z_2, \ldots, z_7]) \in C_2 \times \mathcal{V} : x_i z_j = z_i x_j, i, j = 2, \ldots, 7\}$$

and let $r$ be the canonical projection of $\mathcal{W}$ on $C_2$.

Lemma 5.7. $C_2$ is irreducible, $\dim C_2 = 6$ and $\mathcal{V} = \{(A,B) \in C_2 : A, B \in \langle I_2 \rangle\}$ is the set of the singular points of $C_2$. The variety $\mathcal{W}$ is irreducible and $r$ is a resolution of singularities of $C_2$.

Proof. For any $n \in \mathbb{N}$ $C_n$ is irreducible of dimension $n^2 + n$ ([7], [4]). If $X = (x_{ij})$, $Y = (y_{ij})$ are the coordinates of $M_n \times M_n$ and $(A,B) \in C_n$ then $[A,X] + [B,Y] = 0$ are equations of the tangent space to $C_n$ in $(A,B)$. Hence the points $(A,B)$ such that $A$ or $B$ is regular, that is has centralizer of minimum dimension $n$, are non singular for $C_n$, which shows the first claim. Since $\mathcal{V}$ and $C_2$ have the same equations, $\mathcal{V}$ is an irreducible nonsingular variety of dimension 3. The map $r$ is birational, since for any $(x_0, \ldots, x_7) \in C_2$ such that $(x_2, \ldots, x_7) \neq (0, \ldots, 0)$ we may set $r^{-1}((x_0, \ldots, x_7)) = ((x_0, \ldots, x_7), [x_2, \ldots, x_7])$; if $(x_2, \ldots, x_7) = (0, \ldots, 0)$ we have $r^{-1}((x_0, \ldots, x_7)) = \{(x_0, \ldots, x_7)\} \times \mathcal{V}$. Since for any $x_0, x_1, t \in \mathbb{C}$ and $[z_2, \ldots, z_7] \in \mathcal{V}$ we have that $((x_0, x_1, t z_2, \ldots, t z_7), [z_2, \ldots, z_7] \in \mathcal{W}$, $\mathcal{W}$ is irreducible. Since $\mathcal{V}$ has the same equations as $C_2$ the tangent space to $\mathcal{W}$ in a point such that $(x_2, \ldots, x_7) = (0, \ldots, 0)$ has the same dimension as in a point of $\mathcal{W} \setminus r^{-1}(\mathcal{V})$, hence we get the claim.
Let 
\[ G' = \{ ([y_1, y_2, x_0, \ldots, x_7], [z_2, \ldots, z_7]) \in \mathbb{P}^9(\mathbb{C}) \times \mathbb{P}^5(\mathbb{C}) : \}
\begin{align*}
& : ([x_0, \ldots, x_7], [z_2, \ldots, z_7]) \in W; \\
& (x_0, \ldots, x_7), [z_2, \ldots, z_7] \in W; \\
& (x_0, \ldots, x_7), [z_2, \ldots, z_7] \in W; \\
& \}
\end{align*}

let \( G \) be the image of the canonical projection of \( G' \) onto \( \mathbb{P}^9(\mathbb{C}) \) and let \( r' \) be the canonical projection of \( G' \) on \( G \).

**Corollary 5.8.** The map \( r' \) is a resolution of singularities of \( G \).

Let 
\[ C'_3 = \{(W, [\alpha]) \in G_{2,4} \times C_3 : [L_\alpha, L_\alpha] \subseteq W, \ \text{ad}_W L_\alpha \text{ is abelian}\}, \]
and let \( p_3, p'_3 \) be the canonical projections of \( C'_3 \) on \( G_{2,4} \) and \( C_3 \) respectively.

For any \( i, j \in \{1, \ldots, 4\}, i < j \) let \( A_{ij} = \{ W \in G_{2,4} : W \cap \langle e_i, e_j \rangle = \{0\}\} \). Let 
\[ \{i_0, j_0\} = \{1, \ldots, 4\} \setminus \{i, j\}, \ i_0 < j_0; \text{ if } W \in A_{ij} \text{ let } W = \langle e_{i_0}^W, e_{j_0}^W \rangle \]
where the first two coordinates of \( e_{i_0}^W \) and \( e_{j_0}^W \) with respect to the basis \( \{e_{i_0}, e_{j_0}, e_i, e_j\} \) are 1, 0 and 0, 1 respectively.

**Lemma 5.9.** \( C'_3 \) with the map \( p_3 \) is a bundle on \( G_{2,4} \) with fibers isomorphic to \( G \).

**Proof.** Let \( U_{ij} = (p_3)^{-1}(A_{ij}), i, j \in \{1, \ldots, 4\}, i < j \). The map \( \eta_{ij} : A_{ij} \times G \to U_{ij} \) defined by \( \eta_{ij}(W, [(y_1, y_2, A, B)]) = (W, [\alpha]) \) where \( \alpha \) is such that in \( L_\alpha \) \( e_i e_j = y_1 e_{i_0}^W + y_2 e_{j_0}^W \) and \( \text{ad}_W e_i, \ \text{ad}_W e_j \) are represented, with respect to the basis \( \{e_{i_0}^W, e_{j_0}^W\} \), respectively by \( A \) and \( B \) is an isomorphism, which shows the claim.

For any \( i, j \in \{1, \ldots, 4\}, i < j, \) and \( W \in A_{ij} \) let \( B_{ij}^W = \{ e_{i_0}^W, e_{j_0}^W, e_i, e_j \} \) let \( W \in A_{ij} \cap A_{i', j'} \) and let \( G_W \) be the matrix whose columns are the coordinates of the elements of \( B_{ij}^W \) with respect to \( B_{i', j'}^W \). Let \( \zeta : C^2 \times M_2 \times M_2 \to M_{4 \times 6} \) be the isomorphism such that, by regarding \( \zeta((y_1, y_2, A, B)) \) as a block matrix, we have:

\[ \zeta((y_1, y_2, A, B)) = \begin{pmatrix} 0 & Y & A & B \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \]

Then, by using the notations of the proof of Lemma 5.9, we have that the automorphism \( (\eta_{i', j'})^{-1} \circ \eta_{ij} \) of \( (A_{ij} \cap A_{i', j'}) \times G \) is given by

\[ (\eta_{i', j'})^{-1} \circ \eta_{ij} (W, [y_1, y_2, x_0, \ldots, x_7]) = \\
(W, \zeta^{-1}(G_W \zeta((y_1, y_2, x_0, \ldots, x_7))G_W)). \]

Let \( \pi : C^2 \times Y' \to Y \) be defined by \( \pi((y_1, y_2, x_0, \ldots, x_7)) = [x_2, \ldots, x_7] \). Let \( C''_3 \) be the vector bundle on \( G_{2,4} \) which is the union of open subsets \( U'_{ij}, i, j \in \{1, \ldots, 4\}, \ i < j \), with isomorphisms \( \eta'_{ij} : A_{ij} \times G' \to U'_{ij} \), such that

\[ (\eta'_{i', j'})^{-1} \circ \eta'_{ij} (W, ([y_1, y_2, x_0, \ldots, x_7], [z_2, \ldots, z_7])) = \\
(W, ([\zeta^{-1}(G_W \zeta((y_1, y_2, x_0, \ldots, x_7))G_W)], \\
\pi \circ \zeta^{-1}(G_W \zeta((0, \ldots, 0, z_2, \ldots, z_7))G_W)) \]

and let \( q'' : C''_3 \to C'_3 \) be the morphism such that \( q''(U'_{ij}) = U_{ij} \) and, if \( q'' \) as map onto \( U_{ij} \), we have \( q''(id_{A_{ij}} \times r') = q'' \circ \eta'_{ij} \) for any \( i, j \in \{1, \ldots, 4\}, \ i < j \).

Then \( q'' \) is a resolution of singularities of \( C'_3 \).
Proposition 5.10. $C_3$ is irreducible, $\dim C_3 = 11$ and $p'_3 \circ q''$ is a resolution of singularities of $C_3$. The set of the singular points of $C_3$ is $Z_3 = \{[\alpha] \in C_3 : \text{ad}_{\{L_\alpha, L_a\}} L_\alpha \subseteq \langle \text{id} \rangle \}$, that is the union of the orbits of $g_{[[0,0]]}, g_{[[0,1]]}$ and $a_7$, which is irreducible of dimension 7.

Proof. The map $p'_3$ is birational and the subset of $C_3$ in which $(p'_3)^{-1}$ isn’t regular is $\hat{Z}_3 := \{[\alpha] \in C_3 : \dim [L_\alpha, L_a] < 2 \}$ (since $(p'_3)^{-1}([\alpha]) = ([L_\alpha, L_a], [\alpha])$ and the generic fiber of $p'_3$ on $\hat{Z}_3$ has dimension 2). By Theorem 4.1 we have $\hat{Z}_3 \subset Z_3$ and by Lemma 5.7 and Lemma 5.9 the points of $Z_3 \setminus \hat{Z}_3$ are singular for $C_3$. If $Z'_3 := (p'_3)^{-1}(Z_3)$, the fibers of $p_3|Z'_3$ are isomorphic to $\mathbb{P}^3(C)$, hence $Z_3$ is irreducible of dimension 7. Since the subset of the singular points is closed this shows the claim.

Let

$$C'_4 = \{(T, [\alpha]) \in \mathbb{P}^3(C) \times C_4 : T \subseteq Z(L_a)\}$$

and let $p'_4$ be the canonical projections of $C'_4$ on $C_4$.

Proposition 5.11. $C_4$ is irreducible, $\dim C_4 = 11$ and $p'_4$ is a resolution of singularities of $C_4$. The set of the singular points of $C_4$ is $Z_4 = \{[\alpha] \in C_4 : \dim Z(L_a) \geq 2\}$, that is the orbit of $a_7$.

Proof. Let $C'_4 = \{(J, T, [\alpha]) \in G_{3,4} \times C'_4 : J$ is an ideal of $L_a\}$ and let $q_1, q_2$ be the canonical projections of $C'_4$ on $G_{3,4} \times \mathbb{P}^3(C)$ and on $C'_4$ respectively. If $(J, T) \in G_{3,4} \times \mathbb{P}^3(C)$ is such that $T \not\subset J$ the fiber of $q_1$ in $(J, T)$ is isomorphic to $\mathbb{P}(S_3)$. If $(J, T) \in G_{3,4} \times \mathbb{P}^3(C)$ is such that $T \subset J$ then $J$ is a nilpotent ideal such that $[J, J] \subset T$ for any $L_a$ such that $(J, T, [\alpha]) \in C'_4$, hence the fiber of $q_1$ in $(J, T)$ is also a projective subspace of dimension 5. This proves that $C'_4$ is irreducible and $\dim C'_4 = 11$, since $q_2$ is birational ($(q_2)^{-1}$ is regular in the open subset of all the elements $(T, [\alpha])$ such that $\dim [L_a, L_a] = 3$).

For any $i \in \{1, \ldots, 4\}$ let $A^i = \{T \in \mathbb{P}^3(C) : T \cap \langle e_{i1}, e_{i2}, e_{i3} \rangle = \{0\}\}$; for any $T \in A^i$ we set $T = (e^T)$ where the first coordinate of $e^T$ with respect to the basis $\{e_i, e_{i1}, e_{i2}, e_{i3}\}$ is 1. Let

$$U' = \{(T, [\alpha]) \in C'_4 : T \in A^i, \alpha(e_{i1} \wedge e_{i3}) \neq 0\},$$

$$U'' = \{[x_1, \ldots, x_8] \in \mathbb{P}^7(C) : (x_1, \ldots, x_4) \neq (0, \ldots, 0)\}.$$  

The map $\psi : A^i \times U'' \to U'$ defined by $\psi(T, [x_1, \ldots, x_8]) = (T, [\alpha])$ where $\alpha$ is such that $\alpha(e_{i1} \wedge e_{i3}) = x_5 e^{T} + x_7 e^{T} + x_6 e_{i3} + x_4 e_{i2} - x_1 e_{i3}, \alpha(e_{i2} \wedge e_{i3}) = x_7 e^{T} + x_6 e_{i1} - x_3 e_{i2}$ is an isomorphism, hence $C'_4$ is nonsingular. The map $p'_4$ is birational and the subset of $C_4$ in which $(p'_4)^{-1}$ isn’t regular is $Z_4 = \{[\alpha] \in C_4 : \dim Z(L_a) \geq 2\}$, that is the orbit of $a_7$, where the fibers of $p'_4$ have dimension 1. By Proposition 5.2 $(p'_4)^{-1}(Z_4)$ is irreducible of dimension 6, which shows the claim.

Corollary 5.12. The varieties $C_i, i = 1, \ldots, 4$, are the irreducible components of $L_4$ and the set of the singular points of $L_4$ is $\bigcup_{i \neq j} C_i \cap C_j, i, j = 1, \ldots, 4$, that is the union of the orbits of the following Lie algebras: $g_{[[0,0]]}, g_{[[0,1]]}, g_{[[\gamma, \gamma]]}, g_{[[\gamma+1, \gamma]]}, g_{[[c, c]]}, c = 0, 2, a_5, a_6, a_7, a_8$. 


By the equations of the space of 2-cocycles of a Lie algebra we have found that the dimension of the tangent space to $L_4$ in $g[[\beta, \gamma]]$, $\beta = 0$ or $\beta = \gamma + 1$, $[[\beta, \gamma]] \neq [[0, 1]], [[0, -1]], [[0, 0]]$, $g[[0]]$: $c = 0, 2$, $a_5$, $a_6$ is 12. It is 13 in $g[[0,1]], g[[0,-1]], g[[0,0]]$. In $a_7$ and $a_8$ it is 18 and 14 respectively.

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