An Invariant Symmetric Non-selfadjoint Differential Operator

Erik G. F. Thomas

Communicated by K.-H. Neeb

Abstract. Let $D$ be a symmetric left invariant differential operator on a unimodular Lie group $G$ of type $I$. Then we show that $D$ is essentially self-adjoint if and only if for almost all $\pi \in \hat{G}$, with respect to the Plancherel measure, the operator $\pi(D)$ is essentially self-adjoint. This, in particular, allows one to exhibit a left invariant symmetric differential operator on the Heisenberg group, which is not essentially self-adjoint.

Introduction

Let $X = G/H$ be a homogeneous space, having an invariant measure. If $D$ is an invariant differential operator on $X$ which is symmetric, it is often important to know whether $D$, with domain the space of test functions $\mathcal{D}(X) = C^\infty_c(X)$, is essentially self-adjoint in $L^2(X)$.

The simplest positive result in this regard, involving an individual operator, is perhaps the following: if $\tau$ is the quasi-regular representation of $G$ on $L^2(X)$, and $D$ is a symmetric element in the centre of the universal enveloping algebra, the operator $\tau(D)$ is an invariant differential operator on $X$, which, by a theorem of I.E. Segal [16], is essentially self-adjoint, at least on the Gårding domain. A result of E. Nelson and W.F. Stinespring shows that $\tau(D)$ is also essentially self-adjoint on the (smaller) domain $\mathcal{D}(X)$ (see [12] or the addendum to §1 below). In general not all invariant differential operators on $X$ are obtained in this way however.

Also, several types of homogeneous space are known with the property that every symmetric invariant differential operator on it is essentially self-adjoint. For instance, every compact homogeneous space (having an invariant measure) has this property. As another example we mention the hyperbolic spaces $U(p, q; \mathbb{R})/U(1; \mathbb{R}) \times U(p-1, q; \mathbb{C}), \mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (cf. [6] and [17] theorem C, example b). In particular, this justifies the assertion in Lemma 9 of [14]. E.P. van den Ban has shown that for every semi-simple symmetric pair $(G, \sigma, H)$, $G/H$ has this property [1]. This also includes cases where $(G, H)$ is not a generalised Gelfand pair [5].

Perhaps because of the abundance of these results no example was known (to the author) of a homogeneous space and a symmetric invariant differential
operator on it, which is not essentially self-adjoint. On the other hand, several examples are known of a Lie group \( G \), an irreducible representation \( \pi \) of \( G \), and a symmetric element \( D \) in the universal enveloping algebra, such that \( \pi(D) \) is not essentially self-adjoint. The best known example of this is probably the one due to J. von Neumann (unpublished cf. [12]) where \( G \) is the Heisenberg group.

We have proved the following theorem (corollary to Theorem 1.7):

**Selfadjointness Theorem** Let \( G \) be a unimodular Lie group of type 1. Let \( D \) be a symmetric element of the universal enveloping algebra. Then \( D \), viewed as a left invariant differential operator on \( G \), with domain \( \mathcal{D}(G) \), is essentially self-adjoint if and only if \( \pi(D) \) is essentially self-adjoint for almost all irreducible \( \pi \), with respect to the Plancherel measure.

This then allows one to convert von Neumann’s example into an example of a symmetric left invariant differential operator on the Heisenberg group, which is not essentially self-adjoint.

We have taken the opportunity to state some related results, such as Theorem 1.8, which connects strong commutation of two operators \( D_1 \) and \( D_2 \), with strong commutation of the operators \( \pi(D_1) \) and \( \pi(D_2) \), but the reader mainly interested in the counter-example could read as far as half way through the proof of proposition 1.4, and then turn directly to the third example in paragraph 2.

In the addendum to paragraph 1 we show that, for an arbitrary unitary representation \( U \), the operator \( U(D) \), on the \( C^\infty \)-vectors, and its restriction to the space of analytic vectors, always have the same closure.

## 1. Generalities

Let \( G \) be a unimodular Lie group which eventually we shall assume to be of type 1. We denote \( \mathcal{D}(G) \) the space of \( C^\infty \) functions with compact support, and \( \mathcal{D}'(G) \) the space of distributions on \( G \). Having chosen a Haar measure on \( G \) we identify the locally integrable functions with distributions as usual. Thus we have the inclusions:

\[
\mathcal{D}(G) \subset \subset L^2(G) \subset \subset \mathcal{D}'(G)
\]

More generally, let

\[
\mathcal{H} \subset \subset \mathcal{D}'(G)
\]

be any Hilbert subspace of \( \mathcal{D}'(G) \), i.e. a linear subspace equipped with a Hilbert space inner–product, such that the inclusion map, \( j \), is continuous. For any \( f \in \mathcal{D}'(G) \) and \( \phi \in \mathcal{D}(G) \) we use the notation \( \langle f, \phi \rangle = j(\phi) \). Let \( j^* : \mathcal{D} \longrightarrow \mathcal{H} \) be the adjoint of \( j \) defined by the equation:

\[
(f, j^*\phi) = \langle jf, \phi \rangle
\]

where the left–hand side stands for the inner product in \( \mathcal{H} \). (Note that in this equation \( j \) is usually omitted from the expression on the right–hand side).

The reproducing operator for \( \mathcal{H} \), analogous to the orthogonal projection on a closed subspace of a Hilbert space, is by definition the operator \( H = jj^* \). It is a continuous linear operator from \( \mathcal{D} \) to \( \mathcal{D}' \) which completely characterises \( \mathcal{H} \) (see [15] or the summary in [17]).
Every Hilbert subspace $\mathcal{H} \subset \mathcal{D}'$ possesses a privileged dense subspace, namely $j^*(\mathcal{D})$, or, somewhat incorrectly, $H(\mathcal{D})$. We shall denote it by $\mathcal{H}_0$. (In the particular case where $\mathcal{H}$ is a dense subspace of $\mathcal{D}'$, $j^*$ is injective, and one obtains a Gelfand triplet).

Let $R$ denote the right regular representation in $\mathcal{D}'(G)$. A Hilbert subspace $\mathcal{H}$ is said to be right invariant if $R(g)\mathcal{H} = \mathcal{H}$ and the restriction of each operator $R(g)$ to $\mathcal{H}$ is unitary. This happens if and only if $R(g)H = HR(g)$ for all $g$, i.e. $H$ intertwines the regular representations on $\mathcal{D}$ and $\mathcal{D}'$. Equivalently, $H$ is a convolution operator:

$$H(\phi) = K * \phi$$

(2)

where $K$ is some positive definite distribution on $G$. Let us denote $R^\mathcal{H}$ the restriction of $R$ to the space $\mathcal{H}$. It is a continuous unitary representation in $\mathcal{H}$.

To verify the continuity it is sufficient to check weak continuity on $\mathcal{H}_0$. We have in fact: $(j^*\phi, R^\mathcal{H}(g)j^*\psi) = (j^*\phi, j^*R(g)\psi) = \langle H\phi, R(g)\psi \rangle$ which is continuous with respect to $g$. Let us note also that the subspace $\mathcal{H}_0$, which is invariant under $R^\mathcal{H}$, is composed of regular distributions, in fact of functions of class $C^\infty$. This is a consequence of formula (2).

Similar considerations apply to left invariant spaces. A space which is both left and right invariant will be called bi-invariant. In the particular case where $\mathcal{H} = L^2(G)$, we have $H(\phi) = \phi$, and everything we have said is most familiar.

Let $\mathfrak{g}$ denote the Lie algebra of $G$, and $\mathcal{U}$ the universal enveloping algebra of its complexification. We identify $\mathcal{U}$ with the algebra of left invariant differential operators on $G$ ($\mathcal{U}_L$ if any confusion should arise).

These differential operators are viewed as acting on $\mathcal{D}(G)$, on $\mathcal{E}(G)$, the space of all functions of class $C^\infty$, and on $\mathcal{D}'(G)$. If $D^*$ denotes the formal adjoint of $D$, we have, $G$ being unimodular,

$$\langle D^*f, \phi \rangle = \langle f, D\phi \rangle$$

(3)

for all $\phi \in \mathcal{D}(G)$ and $f \in \mathcal{D}'(G)$.

If $U$ is any unitary representation of $G$, we denote $U_\infty(D)$ the operator corresponding to $D \in \mathcal{U}$, acting on the space of $C^\infty$-vectors for $U$. If $\mathcal{H}$ is a right invariant Hilbert subspace of $\mathcal{D}'(G)$, we denote $\mathcal{H}_\infty$ the space of $C^\infty$-vectors for $R^\mathcal{H}$.

**Proposition 1.1.** Let $\mathcal{H}$ be a right invariant Hilbert subspace of $\mathcal{D}'(G)$, and let $D \in \mathcal{U}$ be a left invariant differential operator.

a. $R_\infty(D)$ is the restriction to $\mathcal{H}_\infty$ of the operator $D : \mathcal{D}'(G) \to \mathcal{D}'(G)$, i.e.:

$$R_\infty^\mathcal{H}(D)f = Df \quad \forall f \in H_\infty$$

b. $\mathcal{H}_0 \subset \mathcal{H}_\infty$. If $R_0^\mathcal{H}(D)$ denotes the restriction of $R_\infty^\mathcal{H}(D)$ to $\mathcal{H}_0$, the operators $R_0^\mathcal{H}(D)$ and $R^\mathcal{H}(D)$ have the same closure.

c. Let $T$ denote either $R^\mathcal{H}_\infty(D)$ or $R^\mathcal{H}_0(D)$. Then the domain of $T^*$ is

$$\text{dom}(T^*) = \{f \in \mathcal{H} : D^*f \in \mathcal{H} \}$$

and $T^*f = D^*f$ for all $f$ in this domain.
Proof. a. It is sufficient to prove this for $X \in \mathfrak{g}$, an arbitrary $D \in \mathcal{U}$ being a linear combination of products of such elements. Now we have $R^H_{\infty}(X)f = \frac{d}{dt}R(\exp tX)f|_{t=0}$ in the space $\mathcal{H}$, and so a fortiori in $\mathcal{D}'(G)$. But $G$ being unimodular, we have $X^* = -X$, and so it is easy to see by transposition, that the above expression yields $Xf$.

b. Since the reproducing operator intertwines the regular representations in $\mathcal{D}$ and $\mathcal{D}'$, we have $R^H_{\infty}(g)j^*\phi = j^*R^H_{\infty}(g)\phi$. On the other hand, the map $j^*: \mathcal{D} \rightarrow \mathcal{H}$ is a continuous linear operator and $\phi$ is a $C^\infty$-vector for the regular representation in $\mathcal{D}$. Thus $j^*\phi$ is a $C^\infty$-vector for $R^H_{\infty}$. Next we need to show that for any $f \in \mathcal{H}_{\infty}$, there exists $f_n \in \mathcal{H}_0$ with $f_n \rightarrow f$ and $Df_n \rightarrow Df$ in $\mathcal{H}$. (The fact that $R^H_{\infty}(D)$ actually has a closure is well known, and besides an immediate consequence of a). Now it is known that $\mathcal{H}_{\infty}$ is in fact equal to the Gårding domain, i.e. the linear span of the elements of the form $R^H_{\infty}(\phi)h$, with $h \in \mathcal{H}$ and $\phi \in \mathcal{D}(G)$ ([3] Theorem 3.3). Thus we may assume $f = R^H_{\infty}(\phi)h$. Let $h_n \in \mathcal{H}_0$ tend to $h$ in the space $\mathcal{H}$. Then $f_n = R^H_{\infty}(\phi)h_n = h_n * \phi$ belongs to $\mathcal{H}_0$, $f_n$ converges to $f$, and $Df_n = R^H_{\infty}(D\phi)h_n$ converges to $R^H_{\infty}(D\phi)h = Df$.

c. To prove this it will be useful to first note the following

$$R^H_{\infty}(D)j^*\phi = j^*D\phi \quad \forall \phi \in \mathcal{D}(G)$$

or equivalently

$$DH(\phi) = H(D\phi) \quad \forall \phi \in \mathcal{D}(G)$$

which is proved, as before, first for $D = X$ by differentiation. Now, if $f = j^*\phi$, $h$ belongs to $\mathcal{H}$, and $T$ denotes $R^H_{\infty}(D)$, we have:

$$(h, Tf) = (h, Df) = (h, j^*(D\phi)) = \langle h, D\phi \rangle = \langle D^*h, \phi \rangle$$

If $D^*h$ belongs to $\mathcal{H}$, this equals $(D^*h, j^*\phi)$, and so we have

$$(h, Tf) = (D^*h, f)$$

for all $f \in \mathcal{H}_0$, which implies $h \in \text{dom}(T^*)$, and $T^*h = D^*h$. Conversely, if $h$ belongs to the domain of $T^*$, the above equalities show that there exists a constant $M$ such that,

$$|\langle D^*h, \phi \rangle| \leq M||j^*\phi|| \quad \forall \phi \in \mathcal{D}(G)$$

Thus, by the Riesz–Fréchet representation theorem, there exists an element $f \in \mathcal{H}$, such that $\langle D^*h, \phi \rangle = \langle f, j^*\phi \rangle = \langle f, \phi \rangle$ for all $\phi \in \mathcal{D}(G)$, which implies $D^*h = f \in \mathcal{H}$. Since by b. $R^H_{\infty}(D)$ and $R^H_{\infty}(D)$ have the same adjoint, the proof is complete.

A particular consequence of proposition 1.1 is that, when $D = D^*$, the operator $R^H_{\infty}(D)$ is essentially self-adjoint, i.e. has self-adjoint closure, if and only if $R^H_{\infty}(D)$ is essentially self-adjoint. From now on we shall describe the essential self-adjointness of these operators simply by saying that $R^H_{\infty}(D)$ is essentially self-adjoint. Similarly, if $U$ is a unitary representation of $G$ we say that $U(D)$ is essentially self-adjoint if the operator $U_{\infty}(D)$, with domain the $C^\infty$-vectors has this property. If $\pi$ is the equivalence class of $U$ we also describe this by saying that $\pi(D)$ is essentially self-adjoint. Similarly $\pi(\phi)$ stands for $U(\phi)$, etc.
Now let $\hat{G}_1$ denote the set of equivalence classes $\pi$ of irreducible unitary representations of $G$, such that, for each $\phi \in \mathcal{D}(G)$, $\pi(\phi)$ is an operator of trace class. Then, if for $\pi \in \hat{G}_1$ we put $\chi_\pi(\phi) = \text{trace } \pi(\phi)$, $\chi_\pi$ is a central positive definite distribution which determines $\pi$, the character of $\pi$. If we now put

$$H_\pi(\phi) = \chi_\pi * \phi = \phi * \chi_\pi$$

(6)

$H_\pi$ is the reproducing operator of a minimal bi-invariant Hilbert subspace of $\mathcal{D}'(G)$, with the property that $R^{H_\pi}$ is the $d(\pi)$-fold repetition of $\pi$, $d(\pi)$ being the degree of $\pi$. If we topologize $\hat{G}_1$ by making the map $\pi \to \chi_\pi$ a homeomorphism, $\hat{G}_1$ becomes a Suslin space whose Borel sets are Borel sets of $\hat{G}$ in the sense of Mackey [10]. Moreover, if $G$ is a group of type I, which we shall assume from now on, the map $\pi \to \chi_\pi$ is an admissible section for the set of extreme generators of the cone of central positive definite distributions on $G$, which is a lattice cone. Thus, there exists a unique measure $d\pi$ on $\hat{G}_1$, the Plancherel measure, such that:

$$\delta = \int \chi_\pi d\pi$$

(7)

Equivalently, (see [17] Theorem A), one has the direct integral decomposition:

$$L^2(G) = \int_{\hat{G}_1} \mathcal{H}_\pi d\pi$$

(8)

More generally, if $\mathcal{H}$ is any bi-invariant Hilbert subspace of $\mathcal{D}'(G)$, there exists a unique measure $m$ on $\hat{G}_1$ such that

$$\mathcal{H} = \int_{\hat{G}_1} \mathcal{H}_\pi dm(\pi)$$

(9)

(for details regarding this approach to Plancherel measure see [10] and [17].)

**Proposition 1.2.** Let $D = D^\ast$ be a symmetric element in $\mathcal{U}$, and let $\pi$ belong to $\hat{G}_1$. Then $\pi(D)$ is essentially self-adjoint if and only if $R^{H_\pi}(D)$ is essentially self-adjoint.

**Proof.** Recall that a densely defined symmetric operator $T$ fails to be essentially self-adjoint if and only if at least one of the equations $T^* f = \pm i f$ admits a solution $f \neq 0$. First assume that $\pi(D)$ is not essentially self-adjoint. Let $\mathcal{K}$ be a closed minimal right invariant subspace of $\mathcal{H}_\pi$ (which exists because $R^{H_\pi}$ is a factor representation of type I). Then $R^{\mathcal{K}}$ represents $\pi$, and so, by proposition 1.1, there exists a non zero solution of the equation $Df = if$ (say) in the space $\mathcal{K}$. But then $f$ belongs to $\mathcal{H}_\pi$ and so for the same reason $R^{H_\pi}(D)$ is not essentially self-adjoint. Conversely, assume $R^{H_\pi}(D) = T$ is not essentially self-adjoint. Then there exists an element $f \in \mathcal{H}_\pi$ such that, for instance $T^* f = if$, $f \neq 0$. Also, $\mathcal{H}_\pi$ being the orthogonal direct sum of minimal right invariant closed subspaces, there exists such a space, $\mathcal{K}$, such that the orthogonal projection $P_\mathcal{K} f$ of $f$ on $\mathcal{K}$ is not zero. We shall prove that the orthogonal projection operator $P_\mathcal{K}$ commutes with $T^\ast$. Then it will follow that $T^* P_\mathcal{K} f = iP_\mathcal{K} f$, and so, by proposition 1.1, the equation $Dk = ik$ has a non-zero solution in $\mathcal{K}$. Thus $R^{\mathcal{K}}(D)$ is not essentially
self-adjoint, which means that $\pi(D)$ is not essentially self-adjoint. To prove that $T^*$ commutes with $P_K$, note that $T^*$ commutes with the operators of left translation in $H$. Thus the bounded operators $B = (I + T^*T^*)^{-1}$ and $C = T^*B$ commute with the left translations. Therefore, by the Godement–Segal commutativity theorem ([8], [10]) the operators $B$ and $C$ belong to the Von Neumann algebra $R$ generated by the operators $R^H$ $(g)$. On the other hand, $P_K$ commutes with right translations, and so belongs to the commutant of $R$. Thus $P_K$ commutes with $B$ and $C$, and so also with the operator $T^*$, which can be recovered from $B$ and $C$, i.e. we have $P_K T^* \subset T^* P_K$. Thus if $T^* f = i f$, $k = P_K f$ belongs to the domain of $T^*$, and $T^* k = i k$ as was to be shown. □

**Remark 1.3.** We have obviously proved something slightly more precise than the statement of Proposition 1.2, namely, that the operators $\pi(D)$ and $R^H(D)$ have positive (resp. negative) deficiency indices differing from zero, simultaneously.

**Proposition 1.4.** Let $H$ be any bi–invariant Hilbert subspace of $D(G)$, and let $D \in \mathcal{U}$ be any left invariant differential operator. Let $K = \{f \in H : Df = 0\}$ and let $K_\pi = \{f \in H_\pi : Df = 0\}$. Then

$$K = \int K_\pi dm(\pi)$$

(10)

($m$ being the measure defined by equation (9))

The proof depends on the following lemma which will be proved, on another occasion.

**Lemma 1.5.** Let $E$ be a locally convex Hausdorff space such that its dual contains a countable subset separating the points of $E$. Let $F$ be a closed linear subspace of $E$. Let $\Lambda$ be a topological Hausdorff space equipped with a Radon measure $m$, and let $(H_\lambda)_{\lambda \in \Lambda}$ be an $m$–measurable family of Hilbert subspace of $E$. Also, let $K_\lambda = H_\lambda \cap F$, with the Hilbert space structure induced from $H_\lambda$. Then $(K_\lambda)_{\lambda \in \Lambda}$ is an $m$–measurable family of Hilbert subspaces of $E$.

If we apply this lemma with $E = D'(G)$ and $F = \{f \in D'(G) : Df = 0\}$, we see that the family $(K_\pi)_{\pi \in \hat{G}}$ is $m$–measurable. Thus the integral on the right-hand side of (10) exists as Hilbert subspace of $D'(G)$, and it is a closed subspace of $H$, which we denote as $W$. Every element $f \in W$ has an expansion in $D'(G)$:

$$f = \int f_\pi dm(\pi)$$

(11)

where $(f_\pi)_{\pi \in \hat{G}}$ is a square integrable field such that $f_\pi \in K_\pi$. Now, since $D : D'(G) \rightarrow D'(G)$ is a continuous linear operator equation (11) yields $Df = \int Df_\pi dm(\pi) = 0$, which proves the inclusion $W \subset K$. (This will be sufficient for the construction below of a non essentially self-adjoint left invariant differential operator).

To prove the opposite inclusion we need some further notations. By equation (9) every $f \in H$ has a unique expansion as in (11), with a square integrable field $(f_\pi)$ where $f_\pi \in H_\pi$. If $M \subset \hat{G}_1$ is a Borel subset, we put:

$$P_M f = \int_M f_\pi dm(\pi)$$

(12)
Also, let $\mathcal{L}$ (resp. $\mathcal{R}$) denote the Von Neumann algebra of operators in $\mathcal{H}$ generated by the left (resp. right) translations. Then it is known that $\mathcal{L} \cap \mathcal{R}$, which by the Godement–Segal commutativity theorem is the centre of $\mathcal{L}$ and of $\mathcal{R}$, is generated by the projections $P_M$, in fact (9) is the central decomposition of $\mathcal{H}$. Actually, we shall only need the fact that the projections $P_M$ commute with the operators $R^\mathcal{H}(g)$, which may be easily verified as follows: $R(g)$ being continuous in $\mathcal{D}'(G)$ we have from (11): $R(g)f = \int R(g)f_\pi \, dm(\pi)$, which may also be written:

$$R^\mathcal{H}(g)f = \int R^\mathcal{H}(g)f_\pi \, dm(\pi) \quad (13)$$

Now, since $R^\mathcal{H}(g)$ preserves the norm in $\mathcal{H}_\pi$, the right-hand side of (13) is the integral of a square integrable field, and so equation (13) is the decomposition of $R^\mathcal{H}(g)f$ corresponding to (9). Thus we have, by definition of $P_M$,

$$P_M R^\mathcal{H}(g)f = \int_M R^\mathcal{H}(g)f_\pi \, dm(\pi) = R^\mathcal{H}(g)P_M f$$

which shows that $P_M$ commutes with the right translations in $\mathcal{H}$, and so belongs to $\mathcal{R}'$, the commutant of $\mathcal{R}$.

Now let $T = R^\mathcal{H}_0(D^*)$. Then by Proposition 1.1, we have

$$\mathcal{K} = \{ f \in \text{dom}(T^*) : T^*f = 0 \} = \text{Ker}(T^*)$$

As in the proof of the previous proposition, we see that, since $T^*$ commutes with left translations, the corresponding operators $B$ and $C$ belong to $\mathcal{L}' = \mathcal{R}$, and so commute with $P_M$, which implies that $T^*$ commutes with $P_M$, i.e. we have $P_M T^* \subset T^* P_M$. Thus, in particular, if $f$ belongs to $K = \text{Ker}(T^*)$, $P_M f$ belongs to $K$. Therefore we have:

$$D P_M f = \int_M D f_\pi \, dm(\pi) = 0$$

for all Borel sets $M \subset \hat{G}_1$. This implies $D f_\pi = 0$ $m$-almost everywhere, i.e. $f_\pi \in \mathcal{K}_\pi$ $m$-almost everywhere, which means that $f$ belongs to $\mathcal{W}$. The proof is complete.

Now for any bi–invariant Hilbert subspace $\mathcal{H}$ of $\mathcal{D}'(G)$ and left differential operator $D$, let us put:

$$\mathcal{H}^\pm = \{ f \in \mathcal{H} : Df = \pm if \}$$

Then we have the following corollary of proposition 1.4:

**Corollary 1.6.** Under the same hypotheses as in proposition 1.4 we have:

$$\mathcal{H}^\pm = \int \mathcal{H}_\pi^\pm \, dm(\pi) \quad (14)$$

$m$ being the measure defined by equation (9).
Theorem 1.7. Let $\mathcal{H}$ be a bi-invariant Hilbert subspace of $\mathcal{D}'(G)$ and let $m$ be the measure on $\hat{G}$, defined by equation (9). Then, if $D$ is a symmetric left invariant differential operator on $G$, $R^H(D)$ is essentially self-adjoint if and only if $\pi(D)$ is essentially self-adjoint for $m$-almost all $\pi \in \hat{G}$.

This is entirely clear from the preceding result once it is recognized that

a. the space $\mathcal{H}^\pm$ in (14) is equal to the space (0) if and only if almost each space $\mathcal{H}^\pm_\pi$ equals (0), and

b. $\pi(D)$ fails to be essentially self-adjoint on a set of positive measure if and only if either $\mathcal{H}^+_\pi \neq (0)$ on a set of positive measure, or $\mathcal{H}^-_\pi \neq (0)$ on a set of positive measure.

If we let $\mathcal{H} = L^2(G)$, so that $m$ is the Plancherel measure and $\mathcal{H}_0 = \mathcal{D}(G)$, we obtain the theorem stated in the introduction.

Let us mention some related results with only summary indication of proof.

Theorem 1.8. Under the same preconditions as in Theorem 1.7, let $D_1$ and $D_2$ be left invariant symmetric differential operators such that $R^{H}(D_1)$ and $R^{H}(D_2)$ are essentially self-adjoint. Then $R^{H}(D_1)$ and $R^{H}(D_2)$ strongly commute if and only if $\pi(D_1)$ and $\pi(D_2)$ strongly commute for $m$-almost all $\pi \in \hat{G}$.

This will be a consequence of the following two propositions:

Proposition 1.9. Let $D$ be a symmetric left invariant differential operator such that $R^H(D)$ is essentially self-adjoint. Let $M_0$ be the set of elements $\pi \in \hat{G}$ such that $\pi(D)$ is essentially self-adjoint. Let $E$ be the spectral measure corresponding to the self-adjoint closure of $R^H(D)$, and for $\pi \in M_0$ let $E_\pi$ be the spectral measure belonging to the closure of $R^{H\pi}(D)$. Then we have:

$$E(\Delta) f = \int_{M_0} E_\pi(\Delta) f_d \pi dm(\pi)$$

for every Borel subset $\Delta \subset \mathbb{R}$ and $f \in \mathcal{H}$.

Proof. Let $T$ be the closure of $R^{H}(D)$ and $E$ its spectral measure. Then, since $T$ commutes with the projections $P_M$, the $E(\Delta)$ also commutes with the $P_M$. Hence there exist spectral measures $E_\pi$ in the spaces $\mathcal{H}_\pi$ such that formula (15) is valid ($M_0 = \{ \pi : \mathcal{H}^\pm_\pi = (0) \}$) is a measurable subset of $\hat{G}$ whose complement has measure 0 by Theorem 1.7). Let $T_\pi = \int \lambda E_\pi(d\lambda)$ be the corresponding self-adjoint operator. It can then be proved that for almost all $\pi$, $T_\pi$ is equal to the closure of $R^{H\pi}(D)$. We only indicate the principle of the proof. Let $G_\pi$ be the graph of $T_\pi$, $G_M$ the graph of $TP_M = P_M T$, and let $G$ be the graph in $\mathcal{D}'(G) \times \mathcal{D}'(G)$ of the operator $D$. Then we show that $(G_\pi)_{\pi \in M_0}$ is a measurable family of Hilbert subspaces of $\mathcal{D}' \times \mathcal{D}'$, and that $G_M = \int_{M_0} G_\pi dm(\pi)$ for all $M$. Now we know that $G_M$ is contained in $G$, a closed subspace of $\mathcal{D}' \times \mathcal{D}'$; this implies $G_\pi \subset G$ for almost all $\pi$. But for those $\pi$, $T_\pi$ is a restriction of the adjoint, or closure of $R^{H\pi}(D)$, and so being maximal symmetric, $T_\pi$ equals this closure. □

Remark 1.10. The relation between the graphs mentioned above, by projection on the first space, gives the following relation between the domains of the operator $T$ and $T_\pi$, viewed, with their graph norms, as Hilbert subspaces of $\mathcal{D}'(G)$:

$$D_T = \int_{M_0} D_{T_\pi} dm(\pi)$$

(16)
Next consider some abstract Hilbert space $K_\pi$ in which the representation $\pi$, or rather a member of $\pi$, takes place, and let $\overline{K_\pi}$ and $\overline{\pi}$ denote respectively the conjugate space and representation. Then there exists an isomorphism:

$$\Phi : \overline{K_\pi} \otimes K_\pi \to H_\pi$$

which transforms $\overline{\pi} \otimes \pi$ into the double representation $L_{H_\pi}^* R_{H_\pi}^*$.

**Proposition 1.11.** With the same conventions as in proposition 1.9, let, for $\pi \in M_0$, $F_\pi$ be the spectral measure in $K_\pi$ corresponding to the closure of $\pi(D)$, and let $I_\pi$ be the identity in $\overline{K_\pi}$. Then we have:

$$E_\pi(\Delta) = \Phi(I_\pi \otimes F_\pi(\Delta))\Phi^{-1}$$  \hspace{1cm} (17)

for all Borel sets $\Delta \subset \mathbb{R}$.

**Proof.** Choose an orthonormal basis in the space $\overline{K_\pi}$. Then the tensor product becomes a direct sum of copies of $K_\pi$, which is transformed by $\Phi$ into a direct sum:

$$H_\pi = \sum_k \otimes \mathcal{H}_\pi^k$$

such that each space $\mathcal{H}_\pi^k$ is minimal right invariant with $R_{\mathcal{H}_\pi^k} \in \pi$. Let $F^k$ denote the spectral measure in $\mathcal{H}_\pi^k$ corresponding to $F$. Then we should show that $E_\pi(\Delta)f = \sum_k F^k(\Delta)f_k$, $f_k$ being the orthogonal projection of $f$ on the space $\mathcal{H}_\pi^k$.

We shall do this, and simplify the notation by dropping the index $\pi$ throughout the remainder of the proof. Let $E'(\Delta)f = \sum_k F^k(\Delta)f_k$. Then $E'$ is a spectral measure in the space $H = H_\pi$. Let $T = \int \lambda E'(d\lambda)$ and let $T_k = \int \lambda F^k(d\lambda)$. Then $T_k$ is equal to the closure of $R_{\mathcal{H}_\pi^k}(D)$. The domain $D_T$ of $T$ is composed of the elements $f \in H$, such that for all $k$, $f_k$ belongs to the domain of $T_k$, and $
abla_k ||T_k f_k||^2 < +\infty$. Moreover, we then have $Tf = \sum_k T_k f_k$. Let $j$ and $j_k$ denote the inclusions of $H$, respectively $\mathcal{H}_\pi^k$, in $D'(G)$. Then, if $f = j^* \phi$ belongs to $H_0$, $f_k = j_k^* \phi \in H_0^k \subset D_{T_k}$, and $T_k f_k = j_k^* D \phi$ is the projection of $j^* D \phi$ onto $\mathcal{H}_\pi^k$. Thus $f$ belongs to the domain of $T$ and $Tf = j^* D \phi = Df$. This means that $T$ is an extension of the operator $R_{\mathcal{H}}(D)$, and so $T$ equals the closure of this operator and $E' = E$, as was to be shown.

**Addendum to §1**

The argument in the proof of 1.1, part b, only made use of the fact that $\mathcal{H}_0$ is invariant under the operators $R_{\mathcal{H}}(\phi)$. In particular, it can be applied to the analytic vectors. Let us state and prove the result explicitly in this case:

**Theorem 1.12.** Let $U$ be a unitary representation of $G$. Let $D \in \mathcal{U}$, and let $U_\omega(D)$ be the restriction of $U_\infty(D)$ to the space of analytic vectors for $U$. Then $U_\omega(D)$ and $U_\infty(D)$ have the same closure.
Proof. Let $\mathcal{H}, \mathcal{H}_{\omega}$ and $\mathcal{H}_{\infty}$ be the representation space and the subspaces of analytic and $C^\infty$-vectors respectively. If $f \in \mathcal{H}_{\omega}$ and $v(x) = U(x)f$ is the corresponding analytic function, we have $U(g)U(\phi)f = \int \phi(x)v(gx)dx$, which is an analytic function of $g$ by direct integration of the power series, at least if $\phi$ has its support in a sufficiently small coordinate patch. Thus $\mathcal{H}_{\omega}$ is invariant under the operators $U(\phi)$. Now let $f \in \mathcal{H}_{\infty}$. By the theorem of Dixmier and Malliavin ([3] Theorem 3.3) we may assume $f = U(\phi)h$ for some $h \in \mathcal{H}$, and $\phi \in \mathcal{D}(G)$. Let $h_n \in \mathcal{H}_{\omega}$ tend to $h$ (Nelson’s theorem [7]). Then $f_n = U(\phi)h_n$ belongs to $\mathcal{H}_{\omega}$, tends to $f$, and $U_{\omega}(D)f_n = U(D\phi)h_n$ tends to $U(D\phi)h = U_\infty(D)f$. Thus the closure of $U_{\omega}(D)$ extends $U_\infty(D)$, and so these two operators have the same closure. 

An analogous assertion and argument is obviously valid for any subspace $\mathcal{H}_0$ of $\mathcal{H}_{\infty}$ which is dense in $\mathcal{H}$ and invariant under the operators $U(\phi), \phi \in \mathcal{D}(G)$. For example, if, as in the introduction, $U = \tau$ is the quasi-regular representation in $L^2(X)$, we may take $\mathcal{H}_0 = \mathcal{D}(X)$.

Thus the theorem of Dixmier and Malliavin according to which the Gårding domain actually coincides with the space of $C^\infty$-vectors, entails some simplification in the situation as described by Nelson and Stinespring ([12] §1).

2. Examples on the Heisenberg group

Let $G$ now be the group of upper-triangular matrices

$$
\begin{bmatrix}
1, & x, & z \\
0, & 1, & y \\
0, & 0, & 1
\end{bmatrix}
$$

abbreviated $(x, y, z)$. The Lie algebra $\mathfrak{g}$ is identified as usual with the strictly upper triangular matrices, and we put:

$$
X = \begin{bmatrix}
0, & 1, & 0 \\
0, & 0, & 0 \\
0, & 0, & 0
\end{bmatrix}, \quad Y = \begin{bmatrix}
0, & 0, & 0 \\
0, & 0, & 1 \\
0, & 0, & 0
\end{bmatrix}, \quad Z = \begin{bmatrix}
0, & 0, & 1 \\
0, & 0, & 0 \\
0, & 0, & 0
\end{bmatrix}
$$

Let $U^r$ be the unitary representation on $L^2(\mathbb{R})$ defined by

$$
[U^r(x, y, z)f](t) = \exp \imath r[(z + ty)f(t + x)]
$$

Then, to summarise the relevant facts, $U^r$ is irreducible for every $r \in \mathbb{R} = \mathbb{R} \setminus \{0\}$, $\hat{G}_1 = \hat{G}$, the map $r \mapsto \chi_r$ which associates with $r \in \mathbb{R}$, the character of $U^r$, is a homeomorphism of $\mathbb{R}$ onto its image in $\mathcal{D}(G)$, and one has the formula:

$$
\delta = \int \chi_r|r|dr
$$

Thus $\mathbb{R}$ may be identified with a (Borel) subset of $\hat{G}$, and $|r|dr$ with the Plancherel measure [2], [9]. It is known moreover, that the space of $C^\infty$-vectors for $U^r$ is precisely the Schwartz space $\mathcal{S}(\mathbb{R})$ ([2] (1.4)). Since we have:

$$
U_{\infty}^r(X) = \frac{d}{dt}, \quad U_{\infty}^r(Y) = \imath rt, \quad U_{\infty}^r(Z) = \imath r
$$
we see that, as $D$ describes $\mathcal{U}$, $U^r_\infty(D)$ describes precisely the set of linear differential operators with polynomial coefficients. Now $\mathcal{D}(\mathbb{R})$ being dense in $\mathcal{S}(\mathbb{R})$ for the topology of $\mathcal{S}(\mathbb{R})$, the closure of any operator $U^r_\infty(D)$ is equal to the closure of its restriction to $\mathcal{D}(\mathbb{R})$. In particular, $U^r_\infty(D)$ and its restriction to $\mathcal{D}(\mathbb{R})$ are simultaneously essentially self-adjoint. This self-adjointness will henceforth be described by saying that $U^r(D)$ is essentially self-adjoint.

Let us now consider three examples:

**Example 2.1.** $D = -X^2 + Y^4$. Then $U^r(D) = -(\frac{d}{dt})^2 + r^4 t^4$. It is well known that this operator is essentially self-adjoint for all $r \in \mathbb{R}$ (see [4] XIII.6.15 or [13] X.28). Thus, by Theorem 1.7, $D$, with domain $\mathcal{D}(G)$, is essentially self-adjoint in $L^2(G)$.

**Example 2.2.** (Harmonic oscillator). $D = -X^2 - Y^2$. Then $U^r(D) = -(\frac{d}{dt})^2 + r^2 t^2$. Here again, $U^r(D)$ is essentially self-adjoint for all $r \in \mathbb{R}$, and so $D$ is essentially self-adjoint.

We mention this example because of the (rather farfetched) possibility to draw the inverse conclusion. For instance, if $Z$ being central, $D$ commutes with the elliptic operator $X^2 + Y^2 + Z^2$, and so $D$ is essentially self-adjoint by the theorem of Nelson and Stinespring ([12] 2.4). By Theorem 1.7 it follows that $U^r(D)$ is essentially self-adjoint for almost all $r$. The fact that there are no exceptions can be seen directly as follows: For $r \neq 0$, let $T_r$ be the unitary operator in $L^2(\mathbb{R})$ defined by $[T_r f](t) = |r|^\frac{1}{2} f(|r|^\frac{1}{2} t)$. Then we have:

$$U^r(D) = |r| T_r U^1(D) T_r^{-1}$$

i.e. up to a factor, the various operators $U^r(D)$ are unitarily equivalent. Thus, if one is essentially self-adjoint, so are the others.

**Remark** The operator $D = -X^2 - Y^2$ has an absolutely continuous spectrum, in spite of the fact that for each $r \in \mathbb{R}_+$, $U^r(D)$ has a purely discrete spectrum.

Although this is probably known a proof is included to keep this paper self-contained (an alternative suggested by the referee is to use [13] Thm XIII.85 and 86 and the method of [11]).

First note that, $U^r(D)$ being strictly positive, its spectral measure, which we denote $F_r$, is concentrated on $\mathbb{R}_+^+ = (0, +\infty)$. Therefore, by Propositions 1.4 and 1.9, the spectral measure $E$ of $D$ is also concentrated on $\mathbb{R}_+^+$ (this can of course also be seen by checking that $D = X^*X + Y^*Y$ is strictly positive, i.e. positive and injective). Now let $\Delta$ be a subset of $\mathbb{R}_+^+$ which has Lebesque measure equal to 0. Then, to show that $E(\Delta) = 0$, it is sufficient, by proposition 1.4 and 1.9, to show that the set $S$ of all $r \in \mathbb{R}_+$ such that $F_r(\Delta) \neq 0$ is negligible with respect to the Plancherel measure, i.e. a set of Lebesque measure zero. But by (18) we have

$$F_r(\Delta) = T_r F_1 \left( \frac{1}{|r|} \Delta \right) T_r^{-1}$$

and so $S = \{ r \in \mathbb{R}_+ : F_1 \left( \frac{1}{|r|} \Delta \right) \neq 0 \}$. Now let $\mu$ be a positive bounded measure on $\mathbb{R}_+^+$ having the same sets of measure zero as $F_1$, and let $\rho$ be a strictly positive
function on $\mathbb{R}^+$, regarded as group, integrable with respect to the Haar measure $\frac{d\rho}{r}$. Then the convolution product $\rho \ast \mu$ on $\mathbb{R}^+$ is absolutely continuous, and so we have $0 = \rho \ast \mu(\Delta) = \int \rho(r) \mu\left(\frac{1}{r} \Delta\right) \frac{d\rho}{r}$. Hence $\mu\left(\frac{1}{r} \Delta\right) = 0$ almost everywhere on $\mathbb{R}^+$. Consequently, $S$ is negligible as asserted.

A similar remark and argument applies to the operator in example 2.1.

**Example 2.3.** $D = -X^2 - Y^4$. Then $U^r(D) = -(\frac{d}{dr})^2 - r^4 t^4$. The operator $U^r(D)$ is not essentially self-adjoint for any $r \neq 0$. In fact ‘both’ solutions of the equation $U^r(D)f = if$ belong to $L^2(\mathbb{R})$.

This is a consequence of Wintner’s theorem ([13] X.9 or [4] XIII.6.20) and of Kodaira’s theorem relating the defect indices of the operator on $[0, +\infty), (-\infty, 0]$ and $(-\infty, +\infty)$ (see [4] XIII.2.26). Thus, by Theorem 1.7, the operator $D$, with domain $\mathcal{D}(G)$, is not essentially self-adjoint in $L^2(G)$. With respect to the coordinates $(x, y, z)$, $-D$ has the expression:

$$
\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z}\right)^4
$$

(20)

By exchanging $X$ and $Y$, or on the line $\frac{d}{dr}$ and $i \tau t$, one obtains a slightly simpler example of a left invariant differential operator on the Heisenberg group, which is not essentially self-adjoint, namely:

$$
\left(\frac{\partial}{\partial x}\right)^4 + \left(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z}\right)^2
$$

(21)

**References**


E. G. F. Thomas
Universiteit Groningen
Mathematisch Instituut
Postbus 800, 9700 AV
The Netherlands
E.G.F.Thomas@math.rug.nl

Received December 20, 2000
and in final form May 17, 2001