Lie quasi-bialgebras with quasi-triangular decomposition

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Abstract. A class of Lie bialgebras and Lie quasi-bialgebras related to a triangular decomposition of the underlying Lie algebras is discussed. New examples are presented.

1. Introduction

A Lie bialgebra is a vector space which is simultaneously a Lie algebra and a Lie coalgebra, both structures connected by a cocycle condition. This fundamental concept was introduced by Drinfeld [5] as the infinitesimal counterpart of the notion of Poisson-Lie group; a Lie group which is a Poisson manifold, both structures related by imposing the multiplication to be a Poisson manifold mapping. Poisson-Lie groups appear naturally in deformation-quantization theory. Their quantizations are the quantum groups. The subsidiary notion of Lie quasi-bialgebra was again introduced by Drinfeld in his approach to the quantization of classical solutions of the quantum Yang-Baxter equations [7]. Being more flexible that Lie bialgebras, the context of Lie quasi-bialgebras allows to use twistings, a technical tool that became very useful.

In this article, we present a unified way to endow Lie algebras with additional data (a so-called “triangular decomposition” or “quasi-triangular decomposition”, see Definition 3.1), a Lie bialgebra or Lie quasi-bialgebra cobracket. Then we provide a systematic iterative way of constructing Lie algebras with quasi-triangular decomposition, analogous to a construction of Witt [14].

The paper is organized as follows: in §2, we recall the necessary definitions and results, mostly due to Drinfeld. In §3, we introduce the notion of Lie algebra with quasi-triangular decomposition and show that a Lie algebra with (quasi-)triangular decomposition is a factorizable Lie (quasi)-bialgebra. Examples of Lie algebras with triangular decomposition are given in §4: some of them were known, as Kac-Moody Lie algebras [6], extended Heisenberg algebras [4]; some of them are new, e. g. motion Lie algebras with respect to the adjoint representation.

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As a byproduct, we provide new examples of classical $r$-matrices. In Section §5, we discuss more examples arising form the analogue of Witt’s construction; in particular, we endow many generalized Heisenberg algebras with Lie bialgebra structures. These examples are also new, see however [12].

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2. Preliminaries

For simplicity of the exposition, we shall work over $\mathbb{C}$. We collect in this section the necessary definitions and theorems, due mostly to Drinfeld [7, [5]; see [3] for further properties of Lie quasi-bialgebras. By abuse of notation, $ad$ will mean a representation which is tensor product of copies of the adjoint representation.

**Definition 2.1.** A Lie quasi-bialgebra is a triple $(g, \delta, \phi)$, where $g$ is a Lie algebra, $\delta : g \to \Lambda^2 g \subset g \otimes g$ is a 1-cocycle and $\phi \in \Lambda^3 g \subset g \otimes g \otimes g$ such that the following axioms hold:

$$
\text{Alt}(\delta \otimes \text{id})\delta(x) = ad\, x(\phi), \quad x \in g; \tag{1}
$$

$$
\text{Alt}(\delta \otimes \text{id} \otimes \text{id})(\phi) = 0, \tag{2}
$$

where Alt is the alternation map and “1-cocycle” means that $\delta$ is linear and $\delta([x, y]) = \text{ad}\, x(\delta(y)) - \text{ad}\, y(\delta(x))$. Examples of 1-cocycles are the 1-coboundaries: if $r \in g \otimes g$, then $\partial r : g \to g \otimes g$, the map given by $\partial r(x) := \text{ad}\, x(r) - r$, is called the coboundary of $r$. Furthermore, if $\phi = 0$ we say that $(g, \delta)$ is a Lie bialgebra.

So that equation (1) becomes $\text{Alt}(\delta \otimes \text{id})\delta(x) = 0$ and equation (2) is identically satisfied. The equality $\text{Alt}(\delta \otimes \text{id})\delta(x) = 0$ is called the co-Jacobi identity.

**Definition 2.2.** Let $(g, \delta)$ be a Lie bialgebra. A Lie subalgebra $h \subset g$ is a Lie subbialgebra if $\delta(h) \subset h \otimes h$.

**Definition 2.3.** A Manin pair is a data $(p, p_1, p_2)$, where $p$ is a Lie algebra provided with a $p$-invariant, symmetric, non degenerate bilinear form $\langle \cdot, \cdot \rangle$: $p \times p \to \mathbb{C}$, $p_1$ is an isotropic Lie subalgebra of $p$ and $p_2$ is an isotropic subspace of $p$ complementary to $p_1$. That is, $p_1 \perp p_2 = p$, $\langle p_i|p_1 \rangle = 0$, $i = 1, 2$. If $p_2$ is a Lie subalgebra of $p$, we say that $(p, p_1, p_2)$ is a Manin triple.

The terminology “Manin pair” is justified as follows: to a Manin pair $(p, p_1, p_2)$ corresponds a Lie quasi-bialgebra structure on $p_1$; and changing the complementary subspace $p_2$ amounts only to a twisting of this structure, so that up to twisting only the pair $(p, p_1)$ counts. See below for details. We remark that Drinfeld does not fixes the isotropic complement $p_2$ since he is interested in the notion of Lie quasi-bialgebra up to twisting.

We now recall the relation between Manin pairs and Lie quasi-bialgebras. Let $(p, p_1, p_2)$ be a Manin pair such that $p$ is finite dimensional. Then there is a Lie quasi-bialgebra structure on $p_1$. Indeed, the restriction of the bracket to $p_2 \otimes p_2 \to p = p_1 \oplus p_2$ has two components: $[\cdot, \cdot]_2 : p_2 \otimes p_2 \to p_2$ and
\[ [\cdot, \cdot] : p_2 \otimes p_2 \to p_1. \] Since \( p_1 \simeq p_2^* \) and \( p_1 \otimes p_1 \simeq (p_2 \otimes p_2)^* \), the first defines by transposition a cobracket \( \delta : p_1 \to p_1 \otimes p_1; \) that is, \( < \delta(x)|u \otimes v >= < x|[u,v] > \), \( x \in p_1, \ u, v \in p_2. \) Similarly, the second defines an element \( \psi \in p_1 \otimes p_1 \otimes p_1 \). Put \( \phi = -\psi \). Then \((p_1, \delta, \phi)\) is a Lie quasi-bialgebra.

Conversely, let \((\mathfrak{g}, \delta, \phi)\) be a finite dimensional Lie quasi-bialgebra. Put \( p = g \oplus g^* \), \( p_1 = g, \ p_2 = g^* \) and endow \( p \) with the canonical scalar product. Let \( \delta^* : g^* \otimes g^* \to g^* \) be the transpose of the bracket and let \( \theta : g^* \otimes g^* \to g \) be induced by \( \phi. \) Take \( \delta^* - \theta \) as the commutator in \( g^* \); it takes values in \( g \oplus g^* \). Then \([x, l]\) can be uniquely defined for \( x \in g, \ l \in g^* \) so that \( p \) is a Lie algebra and the scalar product in \( p \) is invariant. Explicitly, if \( \{ x_i \} \) is a basis of \( g, \ \{ x^i \} \) is the dual basis in \( g^* \) and

\[ [x_i, x_j] = c_{ij}^k x_k, \quad \delta(x_i) = d_{ik}^j x_j \otimes x_k, \quad \text{and} \quad \phi = \phi^i_{jl} x_i \otimes x_j \otimes x_l \] (3)

(here and below, summation is assumed for repeated indices), then \( \{ x_i \} \cup \{ x^i \} \) is a basis of \( p \) and

\[ < x_i | x^j > = \delta_i^j, \quad < x_i | x_j > = 0, \quad < x^i | x^j > = 0, \] \[ [x_i, x^j] = d_{ik}^j x^k - \phi^j_{il} x_i, \quad [x_i, x^j] = d_{ik}^j x^k + c_{ik}^j x^j. \] (4)

It is clear from the preceding discussion that \( p_2 \) is a Lie subalgebra if and only if \( \phi = 0 \).

**Example 2.4.** We recall that, for a Lie algebra \( g \) and a \( g \)-module \( V \), the motion Lie algebra \( g \ltimes V \) is the vector space \( g \oplus V \) with the bracket

\[ [(x, u), (y, v)] = ([x, y], x.v - y.u), \quad x, y \in g, u, v \in V. \]

If \( (g, 0, 0) \) is the trivial Lie bialgebra with underlying Lie algebra \( g \) then \( p = g \oplus g^{**} \) is the motion Lie algebra with respect to the coadjoint representation.

**Remark 2.5.** If \( (g, \delta) \) is a finite dimensional Lie bialgebra and \((p, p_1, p_2)\) is the corresponding Manin triple, there is a one-to-one correspondence between subbialgebras \( q \subset g \) and subalgebras \( q \) of \( p_1 = g \) such that \( q^{**} \cap p_2 \) is an ideal of \( p_2 \).

Let \((g, \delta)\) be a finite dimensional Lie bialgebra and let \((p, p_1, p_2)\) be the corresponding Manin triple. The double of \( g \) is the Lie bialgebra \( \mathcal{D}(g) \) whose underlying Lie algebra is \( p \) and whose Lie cobracket is \( \delta r \), where \( r \) is the image of the canonical element of \( g \otimes g^* \) under the embedding \( g \otimes g^* \hookrightarrow \mathcal{D}(g) \otimes \mathcal{D}(g) \) (the canonical element is \( e_i \otimes e^i \), where \( e_i \) is a basis of \( g \) and \( e^i \) is the dual basis in \( g^* \)). Let \((q, q_1, q_2)\) be the Manin triple corresponding to the Lie bialgebra \( \mathcal{D}(g) \) and identify \( q_2 \) with \( p \) by means of the bilinear form \( \langle \cdot | \cdot \rangle \); the Lie bracket in \( q_2 \), denoted \([\cdot, \cdot]\), is

\[ [u, v]_* = [v_1, u_1] + [u_2, v_2], \]

where \( u_i, v_i \) belongs to \( p_i \) and the bracket in the right hand side is that of \( p \). Indeed, \( \langle \delta(x)|u \otimes v \rangle = \sum_i \langle [x, e_i]|u\rangle \langle e^i|v \rangle + \langle e_i|u \rangle \langle [x, e^i]|v \rangle \) = \([x, \sum_i \langle e^i|v \rangle e_i]u\) + \([x, \sum_i \langle e_i|u\rangle e^i|v \rangle = \langle [x, v_1]|u \rangle + \langle [x, u_2]|v \rangle = \langle x[v_1, u] \rangle + \langle x[u_2, v] \rangle \). Here, \( x \in q_1, u, v \in q_2. \)
Remark 2.6. Let now $(p, p_1, p_2)$ be a Manin pair with $p$ not necessarily finite dimensional. We decompose again $[x, y] = [x, y]_1 + [x, y]_2$ for $x, y \in p_2$ and let $\delta : (p_2)^* \rightarrow (p_2 \otimes p_2)^*$ be the transpose of $[1, 2]$. Identifying $p_1$ with a subspace of $(p_2)^*$, the space of those $x \in p_1$ such that $\delta(x) \in p_1 \otimes p_1$ is a Lie subalgebra of $p_1$. Let $\{x_i : i \in I\}$ be a basis of $p_1$ and assume that there exists a family $\{x^i : i \in I\}$ in $p_2$ such that $\langle x_i, x^j \rangle = \delta_i^j$. If the support of the family $(\delta_i^j)_{i,j \in I}$ is finite for each $k$, then $\delta(p_1) \subset p_1 \otimes p_1$.

Let $\phi_{ij}^k$ be given by $[x^i, x_j]_1 = -\phi_{ij}^k x_i$. If the support of the family $(\phi_{ij}^k)_{i,j,k \in I}$ is finite then it defines $\phi \in p_1 \otimes p_1 \otimes p_1$ and $(p_1, \delta, \phi)$ is a Lie quasi-bialgebra. In fact, a weak version would be that $ad x(\phi_{ij}^k) \in p_1 \otimes p_1 \otimes p_1$ for any $x \in p_1$.

We now recall the notion of twisting of Lie quasi-bialgebras [7]. If $r$ is an element of $g \otimes g$, then set

$$\hat{r} := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}];$$

the identity $\hat{r} = 0$ is the classical Yang-Baxter equation (CYBE). If $r = r_{ij} x_i \otimes x_j$, then $\hat{r} = \delta_{ijk} x_i \otimes x_j \otimes x_k$, where, keeping the notation from (3),

$$\phi_{ijk} = r_{ij} \epsilon_{ik}^j + r_{ik} \epsilon_{jk}^i + r_{jk} \epsilon_{ji}^k.$$  

Let $(g, \delta, \phi)$ be a Lie quasi-bialgebra and let $r \in \wedge^2 g$. Put

$$\delta_r = \delta(x) + ad x r, \quad \phi_r = \phi + Alt(\delta \otimes id)r - \hat{r}.$$  

Then $(g, \delta_r, \phi_r)$ is also a Lie quasi-bialgebra; we shall say that it is obtained from $(g, \delta, \phi)$ by twisting via $r$. If $(g, \delta)$ is a Lie bialgebra, $\hat{r} = 0$ and $Alt(\delta \otimes id)r = 0$ then $(g, \delta_r)$ is a Lie bialgebra. These hypotheses hold if $r \in \wedge^2 g_0$ where $g_0$ is an abelian subalgebra of $g$ such that $\delta(g_0) = 0$.

Lemma 2.8 below is stated in [7] without proof; we include one for completeness. We need the following elementary linear algebra facts:

Remark 2.7. (a). Let $W$ be a vector subspace of a vector space $V$. Fix a complement $U$ of $W$ in $V$, i.e., $V = W \oplus U$. There is a bijection between the set of all complements of $W$ in $V$ and $\text{hom}(U, W)$. Explicitly, if $Z$ is such a complement and $x \in U$ then write $x = x_W + x_Z$, with $x_W \in W, x_Z \in Z$ and define $\varphi_Z(x) := x_W$. Conversely, if $\varphi \in \text{hom}(U, W)$ then $Z := \text{im} \varphi$, where $\Phi(x) := x - \varphi(x)$, is a complement of $W$.

(b). If in addition $V$ is provided with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$, $W$ is isotropic and admits an isotropic complement $U$ then there is a bijection between the set of all isotropic complements of $W$ in $V$ and $\{\varphi \in \text{hom}(U, W) : (\varphi(x), y) = -(x, \varphi(y))\}$.

(c). If in addition $V$ is finite dimensional, there is a bijection between the set of all isotropic complements of $W$ in $V$ and $\wedge^2 W$. Explicitly, let $\{x_i\}$ be a basis of $W$ and let $x^j \in U$ such that $(x_i, x^j) = \delta_i^j$. If $r = r_{ij} x_i \otimes x_j \in \wedge^2 W$ then the subspace $V := \langle x^i + r_{ij} x^j \rangle$ of $V$ is an isotropic complement of $W$.

Lemma 2.8. Let $(g, \delta, \phi)$ be a finite dimensional Lie quasi-bialgebra and let $(p, p_1, p_2)$ be the corresponding Manin pair. Then changing the isotropic complement of $p_1$ amounts to twisting the corresponding Lie quasi-bialgebra. Explicitly,
if \( r \in \wedge^2 \mathfrak{g} \) and \( u = p_{-r} \) corresponds to \(-r\) as described in Remark 2.7 (c), then the Manin pair \((p, p_1, u)\) corresponds to the Lie quasi-bialgebra \((\mathfrak{g}, \delta, \phi_r)\).

**Proof.** Let \( x_1, \ldots, x_n \) be a basis of \( p_1 \) and \( x^1, \ldots, x^n \) be the dual basis in \( p_2 \); so equations (3) and (4) hold. Let \( u_1, \ldots, u_n \) be the basis in \( u \) given by \( u^i := x^i + r^{jk} x_j \). Let \( \hat{r} \) have the same meaning as in (6). Then

\[
[u^i, u^j] = [r^{ij} x_i + x_i, r^{jk} x_j + x_j] = [x_i, x_j] + r^{jk} [x_i, x_j] + r^{ij} r^{jk} [x_i, x_k] + r^{ij} r^{jk} [x_i, x_k] = d^{ij}_{sk} x_s - \phi^{ij}_{rs} x_r + r^{jk} c^{i}_{sl} x_s + r^{ki} c^{s}_{jl} x_s + r^{ij} r^{jk} c^{s}_{kl} x_s + r^{ik} c^{j}_{sl} x_s - \phi^{ij}_{rs} x_r \]

(8)

It follows that \( \phi_r = \phi + \text{Alt}(\delta \otimes \text{id}) r - \hat{r} \) and

\[
\delta_r(x_i) = d^{ij}_{sk} x_i \otimes x_j + r^{jk} c^{i}_{sl} x_i \otimes x_j + r^{ij} r^{jk} c^{s}_{kl} x_s + r^{ik} c^{j}_{sl} x_s - \phi^{ij}_{rs} x_r \]

\[
= \delta(x_i) + \text{ad} x_k(r^{jk} x_k) \otimes x_j + r^{ik} x_i \otimes \text{ad} x_l(x_k) = \delta(x_i) + \text{ad} x_l(r).
\]

**Definition 2.9.** A Lie quasi-bialgebra \((\mathfrak{g}, \delta, \phi)\) is quasitriangular if there exists \( r \in \mathfrak{g} \otimes \mathfrak{g} \), such that:

1. the coboundary of \( r \) is the cobracket of \( \mathfrak{g} \), i.e. \( \partial r = \delta \), and
2. \( \hat{r} = \phi \). (The definition of \( \hat{r} \) is given in formula (5).)

So that if \( \mathfrak{g} \) is Lie bialgebra (i.e. \( \phi = 0 \)), then it is quasitriangular if and only if \( \partial r = \delta \) and \( r \) satisfies the classical Yang-Baxter equation. The following result is also stated in [7] without proof; a proof appears in [3].

**Lemma 2.10.** Let \((\mathfrak{g}, \delta, \phi)\) be a finite dimensional Lie quasi-bialgebra and let \((p, p_1, p_2)\) be the corresponding Manin pair. Let \( u \) be a subspace of \( p \) such that \( p = p_1 \oplus u \), and let \( r \in \mathfrak{g} \otimes \mathfrak{g} \) be the tensor associated to \( u \) (i.e. \( p = u \)). Then

(a) \([p_1, u] \subseteq u\) if and only if the coboundary of \( r \) is the cobracket of \( \mathfrak{g} \).

(b) \([p, u] \subseteq u\) if and only if the coboundary of \( r \) is the cobracket of \( \mathfrak{g} \) and \( \hat{r} = \phi \). In other words, \((\mathfrak{g}, \delta, \phi)\) is quasitriangular if and only if \( p_1 \) admits a complementary ideal in \( p \).

**Proof.** (a) Let \( x_1, \ldots, x_n \) be a basis of \( \mathfrak{g} = p_1 \) and \( x^1, \ldots, x^n \) be a dual basis in \( \mathfrak{g}^* = p_2 \). Let \( u^i = r^{ji} x_j + x_i \); then \([x_i, u^j] = (r^{ji} c^{i}_{kl} + d^{ji}_{kl}) x_k + c^{i}_{kl} x^k\). Thus \([x_i, u^j] \in u\) if and only if \((r^{ji} c^{i}_{kl} + d^{ji}_{kl}) x_k + c^{i}_{kl} x^k = \alpha^{i}_{kl}(r^{ji} c^{i}_{kl} + r^{ji} x_j)\), for some scalars \( \alpha^{i}_{kl} \). This happens if and only if \( c^{i}_{kl} = \alpha^{i}_{kl} x^k \) and \( r^{ji} c^{i}_{kl} + d^{ji}_{kl} = c^{i}_{kl} r^{ji} x_j \), or \( d^{ji}_{kl} = r^{ji} c^{i}_{kl} + r^{ji} c^{i}_{kl} \), for all \( j, k, l \). On the other hand,

\[
\partial r(x_i) = (r^{ik} c^{i}_{kl} + r^{jk} c^{i}_{kl}) x_k \otimes x_i = -(r^{lk} c^{i}_{kl} + r^{li} c^{i}_{kl}) x_k \otimes x_i.
\]

(9)

That is, \([p_1, u] \subseteq u\) if and only if \( d^{ji}_{kl} = r^{ji} c^{i}_{kl} + r^{jk} c^{i}_{kl} \) for all \( j, k, l \) if and only if \( \partial r(x_i) = d^{ji}_{kl} x_k \otimes x_i = \delta(x_i) \).
For (b), we can assume that $\partial r = \delta$ by (a). We have $[x^l, u^k] = (r^{sk}d^l_s - \phi^{jkt})x^j + (r^{sk}c^l_{sp} + d^l_p)x^p$. Thus $[x^l, u^k] \in \mathfrak{u}$ if and only if $[x^l, u^k] = (r^{sk}c^l_{sp} + d^l_p)(r^{w^p}x^j + x^p)$, i.e., $r^{sk}d^l_s - \phi^{jkt} = r^{sk}c^l_{sw}r^{tw} + d^l_s r^{fs}$. Thus

$$r^{sk}r^{vl} c^l_{sw} + r^{sk}r^{tw} c^l_{sw} + r^{sk}r^{tw} c^l_{us} + r^{wl} r^{s} c^k_{sw} + r^{kw} r^{s} c^l_{sw} = \phi^{jkt}.$$  

The second and the third term cancel because of the antisymmetry of the bracket. Performing some permutations in the other terms, we have

$$r^{sk}r^{vl} c^l_{sw} + r^{ts} r^{vl} c^k_{sw} + r^{ts} r^{kw} c^l_{sw} = \phi^{jkt}.$$  

That is, $\mathfrak{u}$ is an ideal if and only if $\hat{\gamma} = \phi$ and $\partial r = \delta$.

\textbf{Definition 2.11.} [13]. A quasi-triangular Lie quasi-bialgebra $(\mathfrak{g}, r)$ is factorizable if the map $\mathfrak{g}^\ast \to \mathfrak{g}$, $\alpha \mapsto (\alpha \otimes \text{id}, r + \tau(r))$, is a bijection, where $\tau$ is the usual transposition.

3. Lie algebras with quasi-triangular decomposition

In this section we introduce the notions of “Lie algebra with quasi-triangular decomposition” and “Lie algebra with triangular decomposition”; these definitions are inspired by [6, Ex. 3.2] and are related to but not the same as the notion discussed in [11]. We show that such a Lie algebra has a canonical structure of quasi-triangular Lie quasi-bialgebra. We give two proofs of this fact; the second one uses the double and suggests a method of constructing Lie algebras with triangular decomposition.

\textbf{Definition 3.1.} Let $\mathfrak{g}$ be a Lie algebra. We shall say that the collection $(\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-, (\{\} ))$ is a quasi-triangular decomposition (QTD) of $\mathfrak{g}$ if $\mathfrak{g}_0$ is a subspace of $\mathfrak{g}$, $\mathfrak{g}_-, \mathfrak{g}_+$ are subspaces of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, and $(\{\} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a $\mathfrak{g}$-invariant, non degenerate, symmetric, bilinear form such that

$$0 = (\mathfrak{g}_+|\mathfrak{g}_+) = (\mathfrak{g}_-|\mathfrak{g}_-) = (\mathfrak{g}_+|\mathfrak{g}_0) = (\mathfrak{g}_0|\mathfrak{g}_-).$$

Furthermore, we shall say that $(\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-, (\{\} ))$ is a triangular decomposition (TD) if $\mathfrak{g}_0$ is abelian, $\mathfrak{g}_-, \mathfrak{g}_+$ are subalgebras of $\mathfrak{g}$ and $[\mathfrak{g}_\pm, \mathfrak{g}_0] \subset \mathfrak{g}_\pm$.

In what follows, we shall simply say “$\mathfrak{g}$ is a Lie algebra with quasi-triangular decomposition or triangular decomposition”, without mentioning the data defining it. We shall use the notation $x = x_+ + x_0 + x_-$ for $x \in \mathfrak{g}$, if $x_j \in \mathfrak{g}_j$, $j \in \{+, 0, -\}$.

\textbf{Theorem 3.1.} Let $\mathfrak{g}$ be a finite dimensional Lie algebra with QTD (respectively TD). Then $\mathfrak{g}$ admits a canonical structure of Lie quasi-bialgebra (resp. Lie bialgebra), which is quasi-triangular. If $\mathfrak{g}$ has another structure of QTD with the same non-degenerate invariant form then the corresponding structures of Lie quasi-bialgebra are related by a twist.

\textbf{Proof.} Let $\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}$ with the product Lie algebra structure, $\mathfrak{p}_1 = \{(a, a) : a \in \mathfrak{g}\}$ and $\mathfrak{p}_2 = \{(a_+ + a_0, a_+ - a_0) : a_+ \in \mathfrak{g}_+, a_0 \in \mathfrak{g}_0\}$. Then $\mathfrak{p}_1$ is a Lie subalgebra of $\mathfrak{p}$ and $\mathfrak{p}_2 \subset \mathfrak{p}$ is a subspace complementary to $\mathfrak{p}_1$. Let $\langle \cdot, \cdot \rangle : \mathfrak{p} \times \mathfrak{p} \to \mathbb{C}$ be
Let \( \mathbb{C} \) be the bilinear form defined by \( \langle (x,y)|(u,v) \rangle = (x|u) - (y|v) \). Then \( \langle \cdot, \cdot \rangle \) is \( p \)-invariant, non degenerate and \( \langle p_1,p_2 \rangle \). If \( (x,y) = (x_+ + x_0, x_+ - x_0) \) and \( (u,v) = (u_+ + u_0, u_+ - u_0) \) belong to \( p_2 \), then

\[
\langle (x,y)|(u,v) \rangle = (x_+ + x_0, x_+ - x_0) - (y_+ + y_0, y_+ - y_0) = 0;
\]

that is \( \langle p_2,p_2 \rangle = 0 \). Hence \( (p, p_1, p_2) \) is a Manin pair and \( p_1 \simeq p \) has a structure of Lie quasi-bialgebra. If \( p \) has a TD, then \( p_2 \) is a Lie subalgebra of \( p \) and \( p \) has a structure of Lie bialgebra. Since \( u = \{(0,x) : x \in g\} \) is an ideal complementary to \( p_1 \), \( g \) is quasi-triangular by Lemma 2.10.

Let \( (g'_0, g'_+, g'_-, (\cdot)) \) be another QTD of \( g \); its Manin pair is \( (p,p_1,p'_2) \), where \( p'_2 = \{(a_+ + a_0, a_+ - a_0) : a_+ \in g'_+, a_0 \in g'_+, a'_0 \in g'_0\} \). The corresponding bilinear form is again \( \langle (x,y)|(u,v) \rangle = (x_+ + x_0, x_+ - x_0) - (y_+ + y_0, y_+ - y_0) \); we have two Manin pairs that only differ in the complement of \( p_1 \), so Lemma 2.8 applies.

**Remark 3.2.** \( \delta = 0 \) if and only if \( [g_+, g_+] = [g_-, g_-] = [g, g_0] = 0 \).

Let \( g = g_+ \oplus g_0 \oplus g_- \) be a finite dimensional Lie algebra with TD, and consider on \( g \) the structure of Lie bialgebra provided by Theorem 3.1.

**Lemma 3.3.**

(a) \( b_+ = g_0 \oplus g_+ \) and \( b_- = g_0 \oplus g_- \) are Lie subbialgebras of \( g \).

As Lie algebras, \( b^*_+ \simeq b_+ \).

(b) \( \partial(b_+) \) is isomorphic as a Lie algebra to the direct product \( g \times g_0 \).

**Proof.** (a). Keep the notation of the proof of Theorem 3.1. The subspace orthogonal to \( b_+ \) (resp., \( b_- \)) in \( p_2 \) is \( 0 \times g_+ \) (resp., \( g_- \times 0 \)) which is clearly an ideal of \( p_2 \), and clearly \( p_2/(0 \times g_+) \) (resp., \( p_2/g_- \times 0 \)) is isomorphic to \( b_- \) (resp., \( b_+ \)), as Lie algebras. Notice that the pairing \( \langle \cdot, \cdot \rangle \) between \( b_+ \) and \( b_- \) is

\[
\langle x, y \rangle = (x_0 | y_0) + (x | y).
\]

(b). Let \( \Upsilon : \partial(b_+) \to g \times g_0 \) be the linear isomorphism \( \Upsilon(x_0 + x_0, y_0 + y_0) = (x_0 + x_0 + y_0 + y_0, y_0 - y_0) \). We want to show that \( \Upsilon([u,v]) = [\Upsilon(u) \Upsilon(v)] \); it suffices to consider \( u = x \in b_+, v = y \in b_- \). Let us write \( [x,y] = [x,y]_1 + [x,y]_2 \), where \( [x,y]_1 \in b_+, [x,y]_2 \in b_- \). We deduce easily from (10) that

\[
[x,y]_1 = [x,y]_1 + \frac{1}{2} [x,y]_0, \quad [x,y]_2 = [x,y]_2 - \frac{1}{2} [x,y]_0.
\]

Indeed, if \( u \in b_- \), \( \langle [x,y]_1, u \rangle = \langle [x,y]_1, u \rangle = (x_0 | y_0, u) = (x_0 | y_0) + \frac{1}{2} [x,y]_0, u \). Now (11) implies our claim.

Let \( \{x_j : j \in J\} \) be a basis of \( g_+ \), \( \{y_j\} \) be its dual basis in \( g_- \), \( \{h_i : i \in I\} \) is an orthonormal basis of \( g_0 \). Then the dual basis of \( B = \{x_j\} \cup \{y_j\} \cup \{h_i\} \) in \( p_2 \) is constituted by the vectors

\[
x^*_j = (y_j, 0), \quad y^*_j = (0, -x_j) \quad \text{and} \quad h^*_i = \frac{1}{2}(h_i, -h_i), \quad j \in J, i \in I.
\]

**Corollary 3.4.** Let \( g \) be a finite dimensional Lie algebra with TD. Then the Lie cobracket on \( g \) provided by Theorem 3.1 is \( \partial r_0 \), where, in the notation above,

\[
r_0 = \sum_{j \in J} x_j \otimes y_j + \frac{1}{2} \sum_{i \in I} h_i \otimes h_i.
\]
This gives a new proof of the quasitriangularity of $\mathfrak{g}$.

**Proof.** Preserve the notation of the preceding proof. The orthogonal of the ideal $\mathcal{Y}^{-1}(0 \times \mathfrak{g}_0)$ is $\{(u, v) \in \mathfrak{d}(\mathfrak{b}_+) : u_0 = v_0\}$, clearly a Lie subalgebra of the dual of $\mathfrak{d}(\mathfrak{b}_+)$. Then $\mathfrak{d}(\mathfrak{b}_+)/\mathcal{Y}^{-1}(0 \times \mathfrak{g}_0) \simeq \mathfrak{g}$ inherits a Lie bialgebra structure and the canonical projection is a morphism of Lie bialgebras. We claim that this Lie bialgebra structure coincides with the structure defined in Theorem 3.1. Let $\{(u, v) \in \mathfrak{d}(\mathfrak{b}_+) : u_0 = v_0\} \rightarrow \mathfrak{p}_2$ be the application $(u, v) \mapsto (v, -u)$; it is easy to check that it is an isomorphism of Lie algebras. Since the introduced isomorphisms preserve the corresponding dualities, the claim follows. Let $r$ be the canonical element of $\mathfrak{d}(\mathfrak{b}_+)$. It is easy to see that the image of $r$ under the above projection is $r_0$; the latter satisfies CYBE because the former does. 

**Corollary 3.5.** $\delta(\mathfrak{g}_0) = 0$.

**Proof.** If $H \in \mathfrak{g}_0$, then write $[H, x_j] = \sum_i \varphi_{ji}(H)x_i$. It follows from the invariance of the bilinear form that $[H, y_j] = -\sum_i \varphi_{ij}(H)y_i$. Hence $\delta(H) = \sum_{j \in \mathfrak{f}} [H, x_j] \otimes y_j + \sum_{j \in \mathfrak{f}} x_j \otimes [H, y_j] = 0$.

**Corollary 3.6.** A finite dimensional Lie bialgebra with TD is factorizable.

Lemma 3.3 and Corollary 3.4 suggest the following method of constructing Lie algebras with TD.

**Theorem 3.2.** Let $\mathfrak{b}$ be a finite dimensional Lie bialgebra. Consider $\mathfrak{b} \subset \mathfrak{d}(\mathfrak{b})$ the double of $\mathfrak{b}$. Assume that

(a) there exists an abelian subalgebra $\mathfrak{h}$ such that, as vector spaces, $\mathfrak{b} = \mathfrak{h} \oplus [\mathfrak{b}, \mathfrak{b}]$;

(b) $\mathfrak{h}^\perp = [\mathfrak{b}^*, \mathfrak{b}^*]$; there exists an abelian subalgebra $\widetilde{\mathfrak{h}}$ such that, as vector spaces, $\mathfrak{b}^* = \widetilde{\mathfrak{h}} \oplus [\mathfrak{b}^*, \mathfrak{b}^*]$, and $\widetilde{\mathfrak{h}}^\perp = [\mathfrak{b}, \mathfrak{b}]$;

(c) for any $x \in \mathfrak{h}$, there exists a unique $\tilde{x} \in \widetilde{\mathfrak{h}}$ such that $\text{ad} \tilde{x}$ coincides with $\text{ad} x$ on $\mathfrak{b}^* \subset \mathfrak{d}(\mathfrak{b})$.

Given $h$ in $\mathfrak{h}$, let $\tilde{h}$ be the unique element of $\widetilde{\mathfrak{h}}$ such that $\langle x | \tilde{h} \rangle = \langle \tilde{x} | h \rangle$, for all $x$ in $\mathfrak{h}$.

Let $\mathfrak{g}_+ = [\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{g}_- = [\mathfrak{b}^*, \mathfrak{b}^*]$, $\mathfrak{g}_0 = \{h + \tilde{h} : h \in \mathfrak{h}\}$. Then

$\mathfrak{g} =: \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$

is a Lie subalgebra of $\mathfrak{d}(\mathfrak{b})$ with TD. The non-degenerate invariant bilinear form is the one inherited from $\mathfrak{d}(\mathfrak{b})$.

**Proof.** First, we remark that if $x \in \mathfrak{h}$, $u \in \mathfrak{b}$ and $w \in \mathfrak{b}^*$, then

$\langle \text{ad}(x)u | w \rangle = \langle [x, u] | w \rangle = -\langle u | [x, w] \rangle = -\langle u | \text{ad}(x)w \rangle$.

By (c) we know that $\text{ad}(x)w = \text{ad}(\tilde{x})w$, hence

$-\langle u | \text{ad}(x)w \rangle = -\langle u | \text{ad}(\tilde{x})w \rangle = -\langle u | [\tilde{x}, w] \rangle = \langle \text{ad}(\tilde{x})u | w \rangle$.

Thus, $\text{ad}(\tilde{x})$ coincides with $\text{ad}(x)$ on $\mathfrak{b}$. 

Let \( \tau = \{ x - \tilde{x} : x \in \mathfrak{h} \} \), then \( \tau \) is an ideal of \( \mathfrak{d}(\mathfrak{b}) \). In fact if \( u \in \mathfrak{b} \), then
\[
[u, x - \tilde{x}] = [u, x] - [u, \tilde{x}] = 0 \text{ (see above). In an analogous way, for (c), we obtain that if } w \in \mathfrak{b}^* \text{, then } [w, x - \tilde{x}] = 0. \text{ So, } [\tau, \mathfrak{d}(\mathfrak{b})] = 0 \text{ and clearly } \tau \text{ is an ideal.}
\]

Let \( z \in \mathfrak{r}^* \), so \( z = u + w \), \( u \in \mathfrak{b} \), \( w \in \mathfrak{b}^* \) and \( < u|x-\tilde{x}> + < w|x-\tilde{x}> = 0 \) for all \( x \in \mathfrak{h} \). For (a) and (b) we have \( u = h_1 + u_1 \) and \( w = s_1 + w_1 \), with \( h_1 \in \mathfrak{h} \), \( u_1 \in [\mathfrak{b}, \mathfrak{b}] \), \( s_1 \in \mathfrak{h} \) and \( w_1 \in [\mathfrak{b}^*, \mathfrak{b}^*] \). So, \( 0 = < z|x-\tilde{x}> = < h_1 + u_1 | x-\tilde{x} > + < s_1 + w_1 | x > = < h_1 | x - \tilde{x} > + < s_1 + w_1 | x > \). Thus, \( < h_1 | x - \tilde{x} > = < s_1 | x > \) for all \( x \in \mathfrak{h} \), then by hypothesis \( s_1 = \tilde{h_1} \). This implies that
\[
\mathfrak{r}^* = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ = \mathfrak{g}.
\]
As \( \tau \) is an ideal, \( \mathfrak{g} \) is also an ideal. In particular, \( \mathfrak{g} \) is a subalgebra of \( \mathfrak{d}(\mathfrak{b}) \).

4. Examples.

**Example 4.1.** A Lie algebra \( \mathfrak{g} \) with TD such that \( \mathfrak{g}_0 = 0 \) is equivalent to a Manin triple \( (\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-) \).

**Example 4.2.** Let \( \mathfrak{g} \) be a Lie bialgebra with TD. If we twist via \( r \in \wedge^2 \mathfrak{g}_0 \) then \( (\mathfrak{g}, \delta_r) \) is a Lie bialgebra (use Corollary 3.5).

**Example 4.3.** Let \( \mathfrak{g} \) be a Lie algebra and let \( < | > \) be a non degenerate invariant bilinear form on \( \mathfrak{g} \). Then \( (\mathfrak{g}, 0, 0, < | > ) \) is a QTD. It is a TD if and only if \( \mathfrak{g} \) is abelian.

**Example 4.4.** Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the complexification of a decomposition of a real simple Lie algebra. It is known that the representation of \( \mathfrak{k} \) on \( \mathfrak{p} \) is either irreducible or a direct sum of two irreducible components; in the latter case, the corresponding symmetric space is hermitian. See [9]. Assume we are in the hermitian case, i.e. that \( \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \) as \( \mathfrak{k} \)-modules, with \( \mathfrak{p}_1 \) and \( \mathfrak{p}_2 \) irreducible. Then \( (\mathfrak{k}, \mathfrak{p}_1, \mathfrak{p}_2, (|) \) is the Killing form, is a QTD. It is seldom a TD; only when \( \mathfrak{g} = sl(2, \mathbb{C}) \).

**Example 4.5.** Let \( A \) be a symmetrizable complex matrix of size \( n \times n \) and let \( D = (d_1, \ldots, d_n) \) be an invertible diagonal matrix such that \( DA = A'D \). Let \( \hat{\mathfrak{g}} = \mathfrak{g}(A) \) be the Lie algebra defined in [10, §1.2] and let \( \mathfrak{g} = \mathfrak{g}(A) \) be the corresponding contragradient Lie algebra [10, Ch. 1]. We preserve the notation \( \hat{\mathfrak{h}}, \hat{\mathfrak{n}}^\pm, e_i, f_i, \alpha_i^\vee \), etc. from loc cit. Let \( h_i = d_i \alpha_i^\vee \). Let \( \tau \) be the unique maximal ideal among the ideals intersecting \( \mathfrak{h} \) trivially; then \( \mathfrak{g} \cong \hat{\mathfrak{g}}/\tau \). Then \( \mathfrak{g} \) has a well-known triangular decomposition, cf [10, 1.2, 2.2], which gives rise to a Lie bialgebra structure by the method of Proposition 3.1. It is well-known [6] that the corresponding cobracket is given by
\[
\delta(h_i) = 0, \quad \delta(e_i) = \frac{1}{2} (e_i \otimes h_i - h_i \otimes e_i) \quad \text{and} \quad \delta(f_i) = \frac{1}{2} (f_i \otimes h_i - h_i \otimes f_i).
\]

(14)

Alternatively, it is not difficult to see that formula (14) determines a Lie bialgebra structure on \( \hat{\mathfrak{g}} \) or \( \mathfrak{g} \). Indeed, [2, Ch. II §2 Prop. 8] allows to define the 1-cocycle on \( \hat{\mathfrak{g}} \) or \( \mathfrak{g} \). The co-Jacobi identity is also easy to check; it suffices to verify it on generators.
Example 4.6. Let \( \mathfrak{g} \) be a Lie algebra and \((\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-, (|))\) a triangular decomposition of \( \mathfrak{g} \). Let \( \mathfrak{l} = \mathfrak{g} \times \mathfrak{g} \) be the motion Lie algebra with respect to the adjoint representation, cf. Example 2.4. Take

\[
\mathfrak{l}_0 = \mathfrak{h} \times \mathfrak{h}, \quad \mathfrak{l}_+ = \mathfrak{n}_+ \times \mathfrak{n}_+, \quad \text{and} \quad \mathfrak{l}_- = \mathfrak{n}_- \times \mathfrak{n}_-.
\]

Thus \( \mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_- \). Let \( k_l(|) : \mathfrak{l} \times \mathfrak{l} \to \mathbb{C} \) be defined by

\[
k_l((x,y)|(u,v)) = (x|u) + (y|u) + (x|v).
\]

Then \((\mathfrak{l}_0, \mathfrak{l}_+, \mathfrak{l}_-, k_l(|))\) is a TD of \( \mathfrak{l} \).

Now, we assume that \( \mathfrak{g} \) is a simple Lie algebra and \( \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \) is the usual decomposition, where \( \mathfrak{h} \) is a Cartan subalgebra, \( \mathfrak{n}_\pm \) is the span of the positive, resp. negative, root vectors. In this context, \((|)\) will be the Killing form. Let \( A \) be the Cartan matrix of \( \mathfrak{g} \), \( \Phi \) be the root system of \( \mathfrak{g} \), \( \Phi^+ \) be the set of positive roots and \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) be the set of simple roots. We choose \( a_\alpha \in \mathfrak{g}_a - \{0\}, \alpha \in \Phi \), \( (\mathfrak{g}_a \) is the root space) and \( H_i, m_i \in \mathfrak{h} \) such that:

\[
(H_i|H) = \alpha_i(H), \quad \forall H \in \mathfrak{h}, \quad [a_\alpha, a_{-\alpha}] = H_i, \quad (a_\alpha|a_{-\alpha}) = 1, \quad (H_i|m_j) = \delta_{ij}.
\]

Let us consider the following elements of \( \mathfrak{l} \):

\[
x_\alpha = (a_\alpha, 0), \quad y_\alpha = (0, a_{-\alpha}), \quad u_\alpha = (0, a_\alpha), \quad v_\alpha = (a_{-\alpha}, -a_\alpha), \quad \alpha \in \Phi,
\]

\[
h_i = (H_i, 0), \quad l_i = (m_i, 0), \quad r_i = (0, H_i), \quad s_i = (m_i, -m_i) \quad i = 1, \ldots, n.
\]

Then it is clear that \( \{x_\alpha, u_\alpha\}_{\alpha \in \Phi^+} \) (resp. \( \{h_i, r_i\}_{1 \leq i \leq n} \)) is a basis of \( \mathfrak{l}_+ \) (resp. \( \mathfrak{l}_0 \)), whose dual basis is \( \{y_\alpha, v_\alpha\}_{\alpha \in \Phi^+} \) (resp. \( \{l_i, s_i\}_{1 \leq i \leq n} \)).

Applying (4) and (12), we obtain the cobracket \( \delta \):

\[
\delta(x_{\pm \alpha}) = \frac{1}{2} x_{\pm \alpha} \wedge r_i + \frac{1}{2} u_{\pm \alpha} \wedge (h_i - r_i) \quad \delta(u_{\pm \alpha}) = \frac{1}{2} u_{\pm \alpha} \wedge r_i \quad \delta(h_i) = \delta(r_i) = 0.
\]

The corresponding \( r \)-matrix is given by

\[
r_0 = \sum_{\alpha \in \Phi^+} x_\alpha \otimes y_\alpha + u_\alpha \otimes v_\alpha + \frac{1}{2} \sum_{i,t} (m_i|m_i)(h_i \otimes h_t - r_i \otimes r_t) \quad (15)
\]

Note that (15) is a new example of a classical \( r \)-matrix.

Remark 4.7. Let \( V \) be a \( \mathfrak{g} \)-module and consider the motion Lie algebra \( \mathfrak{g} \oplus V \), i.e. with the Lie bracket given by \( [[x,y],(u,v)] = ([x,u],xv-uy) \). Suppose that \( \mathfrak{g} \oplus V \) admits a non degenerate invariant bilinear form \((|)\). Then \((V|\mathfrak{g}V) = 0\). If \( V \) is irreducible and non trivial, \((V|V) = 0\) and we obtain a monomorphism of \( \mathfrak{g} \)-modules \( V \to \mathfrak{g}^* \). Assume that \( \mathfrak{g} \) is simple: then \( V \simeq \mathfrak{g}^* \). In addition \( \mathfrak{g} \) is finite dimensional, identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \) via the Killing form. Then any invariant non-degenerate bilinear form on \( \mathfrak{g} \oplus V \) is \( a(x|u) + b(y|u) + b(x|v) \), for some scalars \( a, b \). Let \( c \) be a scalar and let \( T_c \) be the Lie algebra automorphism of \( \mathfrak{g} \oplus V \), \( T_c((x,v)) = (x,cv) \). By using an appropriate \( T_c \), we may assume that an invariant non-degenerate bilinear form on \( \mathfrak{g} \oplus V \) is a multiple of the one considered in Example 4.6.
Example 4.8. The extended Heisenberg algebras have a quasitriangular Lie bialgebra structure considered in [4] as well as their quantizations. It is easy to see that the Lie bialgebra structure arises from a TD; see Example 5.6 below.

Example 4.9. Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in $t$. Recall that the residue of a Laurent polynomial $P$ is defined by $\text{Res} P = -\text{coefficient of } P \text{ at degree } -1$. Let $\phi : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ be defined by $\phi(P, Q) = \text{Res} \left. \frac{dP}{dt} Q \right|_{t=1}$. Then

$$\phi(P, Q) = -\phi(Q, P), \quad (16)$$

$$\phi(PQ, R) + \phi(QR, P) + \phi(RP, Q) = 0 \quad (P, Q, R \in \mathcal{L}). \quad (17)$$

Let $\mathfrak{g}$ be a Lie algebra with QTD. As in [10], consider the loop algebra $\mathcal{L}(\mathfrak{g}) := \mathcal{L} \otimes \mathfrak{g}$, with the bracket $[\cdot, \cdot]_0$ given by $[P \otimes x, Q \otimes y]_0 = PQ \otimes [x, y]$, $P, Q \in \mathcal{L}$, $x, y \in \mathfrak{g}$. Let $\psi : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathbb{C}$, $\psi(P \otimes x, Q \otimes y) = (xy)\phi(P, Q)$. It is easy to check, using (16), (17) and the symmetry and invariance of ($\cdot$), that $\psi$ is a 2-cocycle on $\mathcal{L}(\mathfrak{g})$:

$$\psi(a, b) = -\psi(b, a),$$

$$\psi([a, b]_0, c) + \psi([b, c]_0, a) + \psi([c, a]_0, b) = 0, \quad a, b, c \in \mathcal{L}(\mathfrak{g}).$$

Denote by $\tilde{\mathcal{L}}(\mathfrak{g})$ the extension of the Lie algebra $\mathcal{L}(\mathfrak{g})$ by a 1-dimensional center, associated to the cocycle $\psi$. Explicitly, $\tilde{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}K$ and the bracket is given by

$$[a + \lambda_1 K, b + \lambda_2 K] = [a, b]_0 + \psi(a, b) K, \quad a, b \in \mathcal{L}(\mathfrak{g}); \lambda_1, \lambda_2 \in \mathbb{C}.$$}

The derivation $t \frac{d}{dt} : \mathcal{L} \to \mathcal{L}$ extends to a derivation of $\mathcal{L}(\mathfrak{g})$ by $t \frac{d}{dt}(x \otimes P) = x \otimes t \frac{d}{dt} P$. Let $\tilde{\mathcal{L}}(\mathfrak{g})$ be the Lie algebra obtained by adjoining to $\tilde{\mathcal{L}}(\mathfrak{g})$ a derivation $D$ which acts on $\mathcal{L}(\mathfrak{g})$ as $t \frac{d}{dt}$ and which kills $K$. In other words, $\tilde{\mathcal{L}}(\mathfrak{g})$ is $\tilde{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}D$ with the bracket

$$[a + \lambda_1 K + \mu_1 D, b + \lambda_2 K + \mu_2 D] = [a, b]_0 + \psi(a, b) K + \mu_1 D(b) - \mu_2 D(a),$$

$a, b \in \mathcal{L}(\mathfrak{g}), \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. Even more explicitly

$$[x \otimes t^n + \lambda_1 K + \mu_1 D, y \otimes t^n + \lambda_2 K + \mu_2 D] =$$

$$[x, y] \otimes t^{n+s} + m \delta_m, -n(x|y)K + \mu_1 ny \otimes t^n - \mu_2 mx \otimes t^n,$$

$x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. We extend the form ($\cdot$) to a form ($\cdot$)$_t$ on $\mathcal{L}(\mathfrak{g})$ by:

$$\left. (P \otimes x|Q \otimes y)_t \right| = \text{Res} \left. \left( t^{-1} PQ \right|_{t=1} \right| = \text{Res}(t^{-1} PQ |_{t=1} x|y).$$

Then we extend further ($\cdot$)$_t$ to a bilinear symmetric form ($\cdot$) on $\tilde{\mathcal{L}}(\mathfrak{g})$ imposing ($K|D) = 1$, ($K|\mathcal{L}(\mathfrak{g}) \oplus K) = 0$ and ($D|\mathcal{L}(\mathfrak{g}) \oplus D) = 0$. It is easy to see that ($\cdot$) is non degenerate and $\tilde{\mathcal{L}}(\mathfrak{g})$-invariant (see [10, p. 102]).
We see, with all these coventions, that here are two QTD of \( \mathcal{L}(g) \), namely \((G_0, G_+, G_-, (|))\) and \((G_0, L_+, L_-, (|))\), where
\[
\begin{align*}
G_+ &= g_+ \otimes \mathbb{C}[t, t^{-1}] \oplus g_0 \otimes t\mathbb{C}[t], \\
G_- &= g_- \otimes \mathbb{C}[t, t^{-1}] \oplus g_0 \otimes t^{-1}\mathbb{C}[t^{-1}], \\
G_0 &= g_0 \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}D, \\
L_+ &= (g_- + g_0) \otimes t\mathbb{C}[t] \oplus g_+ \otimes \mathbb{C}[t], \\
L_- &= (g_+ + g_0) \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus g_- \otimes \mathbb{C}[t^{-1}].
\end{align*}
\]
If \( g \) is a Lie algebra with TD then these are TD of \( \mathcal{L}(g) \). It can be shown that these two QTD give rise to Lie quasi-bialgebra structures which are topological twistings of each other.

5. A variation of Witt’s construction

We now discuss a family of examples arising from a construction due to Witt [14], see also [8]. Let \( g \) be a Lie algebra, let \( V \) be a vector space and \( V^* \) be the dual vector space of \( V \). Let \( \pi : g \to \text{End}(V) \) and \( \rho : g \to \text{End}(V^*) \) be representations of \( g \) and denote \( \tilde{g} = V^* \oplus g \oplus V \).

Lemma 5.1. Let \( \beta : V \times V^* \to g \) be a bilinear form. Then the bracket
\[
[(\lambda, x, v), (\lambda', x', v')] = (\rho(x)\lambda' - \rho(x')\lambda, [x, x'] + \beta(v, \lambda') - \beta(v', \lambda), \pi(x)v' - \pi(x')v),
\]
\[\lambda, \lambda' \in V^*, \; x, x' \in g, \; \text{and} \; v, v' \in V, \]
defines a Lie algebra structure on \( \tilde{g} \) if and only if for all \( x \in g, \; v, v' \in V \) and \( \lambda, \lambda' \in V^* \)
\[
[x, \beta(v, \lambda)] = \beta(\pi(x)v, \lambda) + \beta(v, \rho(x)\lambda),
\]
\[\pi(\beta(v, \lambda))v' = \pi(\beta(v', \lambda))v \quad \text{and} \quad \rho(\beta(v, \lambda))\lambda' = \rho(\beta(v', \lambda))\lambda. \tag{20}\]

Proof. The antisymmetry of the bracket (18) is evident. A straightforward computation shows that the Jacobi identity is equivalent to (19), (20).

Lemma 5.2. Suppose that \( g \) is provided with a \( g \)-invariant nondegenerate symmetric bilinear form \( < | > \). Let \( < | > : V \times V^* \to \mathbb{C} \) be the canonical bilinear form. Extend these forms to \( \tilde{g} \) in the following way
\[
< \lambda + x + v|\lambda' + x' + v'> = < v'|\lambda > + < x|x' > + < v|\lambda' >, \tag{21}
\]
x, x' \in g, \; v, v' \in V, \; \lambda, \lambda' \in V^*.

Suppose that \( \rho = \pi^* \) with respect to the form on \( V \times V^* \) and define \( \beta : V \times V^* \to g \) by
\[
< \beta(v, \lambda)|x> = < \pi(x)v|\lambda >, \quad (x \in g, \; v, \lambda \in V^*). \tag{22}
\]
Then the bracket (18) defines a Lie algebra structure on \( \tilde{g} \) if and only if the equations (20) hold. In such case, the form \( < | > \) on \( \tilde{g} \) is \( g \)-invariant.
Proof. It is clear that if \( v \in V \) and \( \lambda \in V^* \), then \( [v, \lambda] \in \mathfrak{g} \) is the unique element such that \( <x|v, \lambda>] = <\pi(x) v|\lambda> \) for all \( x \in \mathfrak{g} \). Hence, \((19)\) implies \((22)\). Thus the bracket \((18)\) define a Lie algebra structure if and only if \((20)\) hold. Let us check that the \( \widetilde{\mathfrak{g}} \)-invariance of \((21)\): let \( x \in \mathfrak{g}, \ v \in V \) and \( \lambda \in V^* \), then
\[
<x|[v, \lambda]> = <\beta(v, \lambda)|x> = <\pi(x)|v\lambda> = <[x, v]||\lambda> \\
\]
and
\[
<v|[x, \lambda]> = <v|\rho(x)\lambda> = - <\beta(v, \lambda)|x> = - <\pi(x)|v\lambda> = <[v, x]|\lambda>.
\]
We can deduce the other cases from the definition of \( <\ | \ > \) and the invariance of the form on \( \mathfrak{g} \).

Remark 5.3. Instead of defining \( \beta \) by \((22)\), we could define \( \pi \) by the formula \((22)\); then we should check that \( \pi \) is a representation of \( \mathfrak{g} \).

Corollary 5.4. Let \( \mathfrak{g} \) be a Lie algebra with QTD (respectively, with TD). Let \( \pi : \mathfrak{g} \to V \) be a representation and let \( \rho, \beta \) be as in Lemma 5.2. Then \((\mathfrak{g}_0, \mathfrak{g}_+ \oplus V^*, \mathfrak{g}_- \oplus V, <\ | \ >)\) is a QTD (respectively, TD) of \( \widetilde{\mathfrak{g}} \) if and only if the equations \((20)\) hold. In such case, the motion Lie algebra \( \mathfrak{g} \oplus V \) is a subalgebra of \( \widetilde{\mathfrak{g}} \).

Proof. We leave the first part to the reader. Let \((p_1, p_1, p_2)\) be the Manin triple associated to \((\widetilde{\mathfrak{g}}, \delta)\) as in Theorem 3.1. Clearly, \( \mathfrak{q} = \{(t, t) : t \in \mathfrak{g} \oplus V\} \) is a subalgebra of \( p_1 \) and \( \mathfrak{q}^+ \cap p_2 = \{(v, 0) : v \in V\} \) is an ideal of \( p_2 \), so \( \mathfrak{g} \oplus V \) is a subalgebra of \( \widetilde{\mathfrak{g}} \).

Example 5.5. We preserve the notation above. We assume that \( \mathfrak{g} \) is a finite dimensional semisimple Lie algebra, the invariant bilinear form is the Killing form and \( V \) a finite dimensional representation of \( \mathfrak{g} \). Let \( C_\mathfrak{g} \) be the value of the action of the Casimir element on the adjoint representation and assume that the action of the Casimir element on \( V \) has a single eigenvalue \( C_V \). Then equations \((20)\) hold whenever
\[
\frac{2 \dim V}{\dim \mathfrak{g}} + \frac{C_\mathfrak{g}}{C_V} = 2.
\]
Indeed, let \( M = V \oplus V^* \) and let \( \psi \) be the symmetric bilinear form on \( M \) which restricted to \( V \times V^* \) is the usual evaluation and such that \( V \) and \( V^* \) are isotropic. It is clearly \( \mathfrak{g} \)-invariant. On the other hand, it is clear that the Casimir element acts on \( V^* \) and \textit{a fortiori} on \( M \) with a single eigenvalue \( C_V \). The claim then follows from [8, Th. 12.1].

Example 5.6. We now consider the opposite situation to the example above. Let \( \mathfrak{g} \) be a finite dimensional Lie algebra and let \( \pi : \mathfrak{g} \to \text{End}(V) \) be a finite dimensional representation. Let \( \mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}^* \) be the motion Lie algebra corresponding to the codjoint representation. We extend \( \pi \) to a representation of \( \mathfrak{l} \) of the same name by letting \( \mathfrak{g}^* \) act by \( 0 \). The bilinear form on \( \mathfrak{l} \) given by evaluation between \( \mathfrak{g} \) and \( \mathfrak{g}^* \), and such that \( \mathfrak{g} \) and \( \mathfrak{g}^* \) are isotropic, is invariant (e. g. by Example 2.4). Let \( \beta : V \times V^* \to \mathfrak{g}^* \) be the bilinear map given by \( <\beta(v, \lambda)|x> = <\pi(x)|v\lambda> \), \( v \in V, \lambda \in V^* \), \( x \in \mathfrak{g} \). Then equations \((20)\) hold because \( \mathfrak{g}^* \subset \ker \pi \); therefore \( \widetilde{\mathfrak{l}} := V^* \oplus \mathfrak{l} \oplus V \) has a Lie algebra structure by Lemma 5.2. Furthermore, if \( \mathfrak{g} \) has a TD then \( \mathfrak{l} \) also does by Example 4.6 (note that the bilinear form considered in Example 4.6 is not the same as the one coming from Example 2.4; however \((22)\) holds for both). By Corollary 5.4, \( \widetilde{\mathfrak{l}} \) also has a TD.
Lemma 5.7. The Lie subalgebra $\mathfrak{h} = V^* \oplus \mathfrak{g}^* \oplus V$; it is a of $\widetilde{I}$ is a Lie subbialgebra.

Proof. Let $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be the Manin triple corresponding to $\widetilde{I}$ as constructed in Theorem 3.1. By Remark 2.5, it is enough to show that $\mathfrak{h} \cap \mathfrak{p}_2$ is an ideal of $\mathfrak{p}_2$. This is not difficult to see using the definitions.

Notice that $\mathfrak{h}$ is a two-step nilpotent Lie algebra, or Heisenberg-type Lie algebra since Heisenberg Lie algebras correspond to the case $\dim \mathfrak{g} = 1$. Hence the procedure just described allows to obtain many new Lie bialgebras with underlying Lie algebra of Heisenberg-type and to provide many new examples of factorizable Lie bialgebras.

References


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