Harmonic analysis on $SU(n,n)/SL(n,C) \times \mathbb{R}^+_+$

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Abstract. We find an explicit expression for the spherical functions on the ordered symmetric space $\mathcal{M} = SU(n,n)/SL(n,C) \times \mathbb{R}^+_+$, we formulate and prove a Paley-Wiener theorem for the spherical Laplace transform on $\mathcal{M}$ and we find an inversion formula for the Abel transform on $\mathcal{M}$.

0. Introduction

Let $\mathcal{M} = SU(n,n)/SL(n,C) \times \mathbb{R}^+_+$, let $\mathfrak{a}^-$ be the negative Weyl chamber of a certain Cartan subspace $\mathfrak{a}$ for $\mathcal{M}$, let $\lambda \in \mathfrak{a}^*_+\mathbb{C}$, the complex dual of $\mathfrak{a}$, and let $A^- = \exp \mathfrak{a}^-$. Let $\Phi_\lambda$ denote the Harish-Chandra series on the Riemannian dual $\mathcal{M}^d = SU(n,n)/S(U(n) \times U(n))$ of $\mathcal{M}$. G. Ólafsson proved in [9], §5 an expansion formula (for general ordered symmetric spaces):

$$\varphi_\lambda(a) = \sum_{w \in W_0} c(w,\lambda) \Phi_{w,\lambda}(a), \quad a \in A^-,$$

for the spherical functions $\varphi_\lambda$ on $\mathcal{M}$ (see §3 for a precise definition and construction of $\varphi_\lambda$), where $c(\lambda)$ is the $c$-function for $\mathcal{M}$ and $W_0$ is some Weyl group.

The Berezin-Karpelevič formula for the spherical functions $\psi^d_\lambda$ on $\mathcal{M}^d$ was proved by B. Hoogenboon, see [6], using the Harish-Chandra expansion of $\psi^d_\lambda$ and an explicit expression for $\Phi_\lambda$. We use the expansion formula above to prove a similar (explicit) formula for the spherical functions $\varphi_\lambda$ on $\mathcal{M}$.

The spherical Laplace transform $\mathcal{L}$ on $\mathcal{M}$ is defined in terms of integrating against the spherical functions. We use the explicit formulae for the spherical functions on $\mathcal{M}$ and $\mathcal{M}^d$ to prove a Paley-Wiener Theorem for the spherical Laplace transform, generalizing results in the rank 1 case obtained by G. Ólafsson and the first author, see [1].

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The Abel transform on $\mathcal{M}$ is related to the spherical Laplace transform $\mathcal{L}$ by the classical Laplace transform on the cone $c_{\text{max}} \subset a$. We find an inversion formula for the Abel transform, using an approach similar to the method used by C. Meaney for the inversion formula for the Abel transform on $\mathcal{M}^d$, see [8].

We have tried to keep notations and proofs to a minimum in order to make the presentation as clear as possible, we refer to [3], [5] and [9] for more details on spherical functions and the spherical Laplace and Abel transforms defined on ordered symmetric spaces.

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1. Notation and preliminaries

Let $n$ be a positive integer and let $G^c = SU(n, n)$ denote the connected group of matrices with determinant 1 preserving the hermitian form 

$$(x, y) = x_1 \overline{y}_1 + \cdots + x_n \overline{y}_n - x_{n+1} \overline{y}_{n+1} - \cdots - x_{2n} \overline{y}_{2n}, x, y \in \mathbb{C}^{2n}.$$

The Lie algebra $\mathfrak{g}^c = \mathfrak{su}(n, n)$ is given by $2n \times 2n$-matrices of the form

$$\mathfrak{g}^c = \left\{ \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \left| a = -a^*, b = -b^*, \text{tr}(a + b) = 0 \right. \right\},$$

where $a, b$ and $c$ are $n \times n$-matrices. It is isomorphic (by $c$-duality) to

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{pmatrix} \left| \beta = \beta^*, \gamma = \gamma^*, \exists \text{tr} \alpha = 0 \right. \right\}.$$

We embed $\mathfrak{h} = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R} \cong \left\{ \alpha \in \mathfrak{gl}(n, \mathbb{C}) \mid \exists \text{tr} \alpha = 0 \right\}$ in the diagonal as follows:

$$\alpha \mapsto \begin{pmatrix} \alpha \\ -\alpha^* \end{pmatrix}.$$

Let $G$ and $H$ denote the analytic subgroups of $GL(2n, \mathbb{C})$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The involution $\sigma$ on $\mathfrak{g}$ given by

$$\sigma \left( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix},$$

fixes $\mathfrak{h}$. The $-1$ eigenspace $\mathfrak{q}$ of $\sigma$ is given by:

$$\mathfrak{q} = \left\{ \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \left| \beta = \beta^*, \gamma = \gamma^* \right. \right\}.$$

Let $\mathcal{M} = G/H \cong SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}^*_+$, then $G/H$ is an ordered symmetric space of Cayley type, see [5] or [9], §1.

Let $\theta$ be the classical Cartan involution on $\mathfrak{g}$, i.e. $\theta(X) = -X^*, X \in \mathfrak{g}$, and let $\mathfrak{k}$ and $\mathfrak{p}$ denote the $\pm 1$-eigenspaces of $\theta$. Let $K \cong S(U(n) \times U(n))$
denote the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. Then $G/K$ is isometric to the Riemannian dual $\mathcal{M}^{d}$ of $\mathcal{M}$, see [5] and [9], §1 for details.

We choose a Cartan subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ for $\mathcal{M}$ as follows:

$$\mathfrak{a} = \left\{ X_T = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \mid T = \text{diag}(t_1/2, \ldots, t_n/2), t_1, \ldots, t_n \in \mathbb{R} \right\}.$$ 

We note that $\mathfrak{a}$ also is a Cartan subspace of $\mathfrak{p}$. We identify $\mathfrak{a}$ and $\mathbb{R}^n$ via the map $\mathbb{R}^n \ni t = (t_1, \ldots, t_n) \mapsto T = \text{diag}(t_1/2, \ldots, t_n/2)$. Let $\gamma_i \in \mathfrak{a}^*$ be defined by: $\gamma_i(t) = -t_i$ for $i = 1, \ldots, n$. We identify the complexified dual $\mathfrak{a}_\mathbb{C}^*$ and $\mathbb{C}^n$ by the map:

$$\mathbb{C}^n \ni \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto -\sum_j \lambda_j \gamma_j.$$

The root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is given by $\Delta = \{ \pm \gamma_i \} \cup \left\{ \frac{2\lambda \pm \gamma_i}{2} \right\}$, with multiplicity $m_\alpha = 2$ for the short roots $\alpha = \frac{2\lambda \pm \gamma_i}{2}$ and $m_\alpha = 1$ for the long roots $\alpha = \pm \gamma_i$.

Let $\Delta^+ = \{ \gamma_i \} \cup \left\{ \frac{2\lambda \pm \gamma_j}{2}, i < j \right\}$ be a set of positive roots. Let furthermore $\Delta_0$ denote the root system $\Delta_0 = \left\{ \frac{2\lambda \pm \gamma_j}{2} \right\}$ with positive roots $\Delta_0^+ = \left\{ \frac{2\lambda \pm \gamma_j}{2}, i < j \right\}$. The negative Weyl chamber $\mathfrak{a}^-$ is given by:

$$\mathfrak{a}^- = \{ t \in \mathbb{R}^n \mid 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \}.$$ 

Let $W \cong \{ \pm 1 \}^n \times \mathfrak{S}_n$ and $W_0 \cong \mathfrak{S}_n$ (the permutation group of $n$ elements) denote the Weyl groups of the root systems $\Delta$ and $\Delta_0$ respectively. Let finally $n = \sum_{\alpha \in \Delta^+} 1 + 1 = \sum_{\alpha \in \Delta^+} 1 + 1$, $A = \exp \mathfrak{a}$, $A^- = \exp \mathfrak{a}^-$, $\bar{N} = \exp \bar{N}$, where $\exp$ is the exponential mapping from $\mathfrak{g}$ to $G$.

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We will use the notation $x \geq r$ (or $x > r$) if $x_j \geq r$ (or $x_j > r$) for all $j$. Let $C_{\max}$ be the (unique) closed $H$-invariant cone in $\mathfrak{q}$ defined by $C_{\max} = \mathfrak{a}^- = \{ t \in \mathbb{R}^n \mid 0 \geq t \}$. Let $S = \exp(C_{\max})H$ be the associated semigroup in $G$, and let $S^0$ denote the interior of $S$. Let finally $S^0_\Lambda := S^0 \cap A = \exp e_{\max}$.

Let $\eta : \mathbb{D}(\mathcal{M}) \to \mathbb{D}(\mathcal{M}^d)$ denote the Flenshted-Jensen isomorphism between the commutative algebras of invariant differential operators on $\mathcal{M}$ and $\mathcal{M}^d$ respectively (mapping the Laplace-Beltrami operator $\Delta$ on $\mathcal{M}$ onto the Laplace-Beltrami operator $\Delta^d = \eta(\Delta)$ on $\mathcal{M}^d$). Let $\Pi(D)$ and $\Pi(D^d)$ denote the radial part (on $A^-$) of $D \in \mathbb{D}(\mathcal{M})$ and $D^d \in \mathbb{D}(\mathcal{M}^d)$ respectively.

There exists a unique map $C_c^\infty(H \setminus S^0/H) \ni f \mapsto f^d \in C_c^\infty(K \setminus G/K)$ such that $f|_{A^-} = f^d|_{A^-}$ and $\Pi(D)f = D(\eta(D))f^{d}$, see [5] or [9], §4 for more details.

Let $P_\lambda$ and $Q_\lambda$ denote Legendre functions of the first and second kind.

We note that

$$P_{\lambda - \frac{1}{2}}(\cosh t) = \varphi_{\lambda}(0, -\frac{1}{2})(t) = \varphi_{2i\lambda}(t)/2,$$

and

$$\frac{\Gamma(\lambda + 1)}{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})} Q_{\lambda - \frac{1}{2}}(\cosh t) = \Phi_{\lambda}(0, -\frac{1}{2})(t) = \Phi_{2i\lambda}(t),$$

where $\varphi^{(\alpha, \beta)}$ and $\Phi^{(\alpha, \beta)}$ denote Jacobi functions of the first and second kind. We can furthermore view $P_{\lambda - \frac{1}{2}}(\cosh t)$ and $Q_{\lambda - \frac{1}{2}}(\cosh t)$ as spherical functions on
the Riemannian symmetric space $SO_o(1,2)/SO(2)$, respectively on the ordered symmetric space $SO_o(1,2)/SO_o(1,1)$, of rank 1. From e.g. [7], §2, we get the following estimates on $P_{\lambda - \frac{1}{2}}(\cosh t)$ and $Q_{\lambda - \frac{1}{2}}(\cosh t)$:

$$|P_{\lambda - \frac{1}{2}}(\cosh t)| \leq c e^{(\Re \lambda - \frac{1}{2})|t|},$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, for some constant $c$; and, for any $r > 0$:

$$\left| \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})} \right| |Q_{\lambda - \frac{1}{2}}(\cosh t)| \leq c_r e^{-((\Re \lambda + \frac{1}{2})^r},$$

for $\Re \lambda \geq 0$ and $t \geq r > 0$, where $c_r$ is a constant only depending on $r$.

2. The spherical Fourier transform on $\mathcal{M}^d = SU(n,n)/S(U(n) \times U(n))$

In this section we recall some well-known definitions and results for the spherical Fourier transform on $\mathcal{M}^d$, see e.g. [4], Chapter 4.

Let $\lambda \in \mathbb{C}^n$. The Poisson kernel for $\mathcal{M}^d$ is defined by:

$$N\lambda K \ni n\lambda = x \mapsto a^{\lambda+\rho} =: p^d(x),$$

where $\rho = \sum_{\alpha \in \Delta^+} m_\alpha \alpha$. The spherical functions on $\mathcal{M}^d$ can be written as:

$$\psi^d_\lambda(x) = \int_G p^d(kx)dk,$$

for $x \in G$. The spherical functions are bi-$K$-invariant, $\psi^d_\lambda(\exp 0) = 1$ and $D\psi^d_\lambda = \gamma(D)(\lambda)\psi^d_\lambda$ for all $D \in D(\mathcal{M}^d)$ and all $\lambda \in \mathbb{C}^n$, where $\gamma$ is the Harish-Chandra isomorphism. They are furthermore invariant under the action of the Weyl group $W$, i.e. $\psi^d_{wa} = \psi^d_\lambda$ for all $w \in W$.

Let $\Lambda$ denote the simple roots in $\Delta^+$. The Harish-Chandra series:

$$\Phi^d_\lambda(a) = a^{\rho - \lambda} \sum_{\mu \in (\Lambda \cup \{0\})^\Lambda} a^\mu \Gamma_\mu(\lambda), \quad a \in A^-,$$

is a solution of the differential equation $\Delta^d \Phi^d_\lambda(a) = (\lambda^2 - \rho^2) \Phi^d_\lambda(a)$ for $a \in A^-$, where $\Gamma_0(\lambda) \equiv 1$ and $\Gamma_\mu(\lambda)$, $\mu \in \mathbb{N} \Lambda$ is determined by recursion. The Harish-Chandra expansion formula states that:

$$\psi^d_{-\lambda}(a) = \psi^d_\lambda(a^{-1}) = \sum_{w \in W} c^d(w\lambda)\Phi_{w\lambda}(a), \quad a \in A^-,$$

where the Harish-Chandra c-function $c^d$ for $\mathcal{M}^d$ is given by (modulo constants):

$$c^d(\lambda) := \int_{\mathcal{M}} p^d(\bar{n})d\bar{n} = \prod_{j} \frac{\Gamma(-\lambda_j)}{\Gamma(-\lambda_j + \frac{1}{2}i)} \prod_{i < j} (\lambda_i^2 - \lambda_j^2)^{-1}.$$
The Harish-Chandra series on \( \mathcal{M}^d \) is given by:

\[
\Phi_\lambda(\exp t) = \pi^{-n/2} \prod_i \frac{\Gamma(\lambda_i + \frac{1}{2})}{\Gamma(\lambda_i + \frac{n}{2})} \frac{Q_{\lambda_i - \frac{1}{2}}(\cosh t_i)}{\delta_1(t)},
\]

for \( t > 0 \), where

\[
\delta_1(t) = \prod_{\alpha = \gamma(i+1)} \sinh(-\alpha, t) = 2^{n(n-1)/2} \prod_{i<j} (\cosh t_j - \cosh t_i),
\]

see [6], Theorem 2. Using the Harish-Chandra expansion formula, this yields the Berezin-Karpelević formula for the spherical functions on \( \mathcal{M}^d \):

\[
\psi_\lambda^d(\exp t) = \frac{c}{\prod_{i<j} (\lambda_j^2 - \lambda_i^2)} \frac{\det \left( P_{\lambda_i - \frac{1}{2}}(\cosh t_j) \right)}{\delta_1(t)},
\]

for all \( t \in \mathbb{R}^n \), where \( c \) is a constant, see [6] for more details.

The spherical Fourier transform \( \mathcal{F} \) on \( \mathcal{M}^d \) is defined for any function \( f \in C_c^\infty(K \setminus G/K) \) as:

\[
\mathcal{F}(f)(\lambda) = \int_G f(x) \psi_\lambda^d(x) dx = \int_{\mathbb{A}^n} f(a) \psi_\lambda^d(a) \delta(a) da,
\]

where \( \delta(\exp t) = \prod_{\alpha \in \Delta^+} \sinh^{m_\alpha}(-\alpha, t) = \delta_1(t)^2 \prod_j \sinh t_j \). The inversion formula for \( \mathcal{F} \) reads (after normalizing \( d\lambda \) suitably):

\[
f(x) = \int_{\mathbb{C}^n} \mathcal{F}(f)(\lambda) \psi_\lambda^d(x) |e^d(\lambda)|^{-2} d\lambda,
\]

for all \( f \in C_c^\infty(K \setminus G/K) \) and \( x \in G \).

Let \( R > 0 \). Let \( C^\infty_R(K \setminus G/K) := \{ f \in C^\infty_c(K \setminus G/K) | \text{supp} f \subset \exp B_R \} \), where \( B_R := \{ t \in \mathbb{R}^n | |t| \leq R \} \). Define the Paley-Wiener space \( \mathcal{H}_R(\mathbb{C}^n) \) as the space of \( W \)-invariant holomorphic functions \( g \) on \( \mathbb{C}^n \) of exponential type \( R \), i.e. satisfying the estimate:

\[
\sup_{\lambda \in \mathbb{C}^n} e^{-R|\Re \lambda|}(1 + |\lambda|)^N |g(\lambda)| < \infty,
\]

for all \( N \in \mathbb{N} \). Furthermore denote by \( \mathcal{H}(\mathbb{C}^n) \) the union of the spaces \( \mathcal{H}_R(\mathbb{C}^n) \) for all \( R > 0 \).

**Theorem 1 (The Paley-Wiener Theorem).** The Fourier transform is a bijection of \( C_c^\infty(K \setminus G/K) \) onto \( \mathcal{H}(\mathbb{C}^n) \). More precisely it is a bijection of \( C^\infty_R(K \setminus G/K) \) onto \( \mathcal{H}_R(\mathbb{C}^n) \) for all \( R > 0 \).

### 3. Spherical functions on \( \mathcal{M} = SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}^*_+ \)

We define spherical functions on \( \mathcal{M} \) according to [9], Definition 4.1:
Definition 2. An $H$-biinvariant continuous function $\varphi : S^0 \to \mathbb{C}$ is called a spherical function if there exists a character $\chi$ of $\mathbb{D}(\mathcal{M})$ such that (in the sense of distributions) $D\varphi = \chi(D)\varphi$ for all $D \in \mathbb{D}(\mathcal{M})$.

Define the Poisson kernel for $\mathcal{M}$ (and the open orbit $\mathcal{N}AH$) by:

$$\mathcal{N}AH \ni nah = x \mapsto a^{\theta - \lambda} =: p_\lambda(x),$$

and $p_\lambda \equiv 0$ on $G \setminus \mathcal{N}AH$. We note that $hx \in S \subset \mathcal{N}AH$ for all $h \in H$ and $x \in S$, see [3], Theorem 4.2. We can construct spherical functions $\varphi_\lambda$ as follows:

$$\varphi_\lambda(x) := \int_H p_\lambda(hx)dh,$$

for $x \in S^0$, and $D\varphi_\lambda = \gamma(D)(\lambda)\varphi_\lambda$ for all $D \in \mathbb{D}(\mathcal{M})$, whenever the integral exists, see [3], §5 and [9], Theorem 4.10.

The asymptotic behavior of $\varphi_\lambda$ as $t \to \infty$, $t \in \mathfrak{a}^-$ is given by:

$$\lim_{t \to \infty} e^{(\lambda - \rho)t}\varphi_\lambda(\exp t) = c(\lambda) = c_0(\lambda)c_\Omega(\lambda),$$

see [3], §6 for details, where $c$ is the $c$-function for $\mathcal{M}$ given by:

$$c(\lambda) := \int_{\mathcal{N} \cap \mathcal{N}AH} p_{-\lambda}(\bar{n})d\bar{n},$$

the function $c_\Omega$ is given by (modulo constants):

$$c_\Omega(\lambda) := \int_{K \cap \mathcal{N}AH} p_{-\lambda}(k)dk = \prod_j \frac{\Gamma(\lambda_j + \frac{1}{2})}{\Gamma(\lambda_j + 1)} \prod_{i < j} (\lambda_i + \lambda_j)^{-1},$$

see [2], Corollaire 5.2, and $c_0$ is the $c$-function for a Riemannian symmetric space with root system $\Delta_0$, given by (modulo constants):

$$c_0(\lambda) = \prod_{i < j} (\lambda_j - \lambda_i)^{-1}.$$

We note that $c_\Omega$ is $W_0$-invariant, i.e. $c_\Omega(w\lambda) = c_\Omega(\lambda)$ for $w \in W_0$.

Considering asymptotics of the spherical functions and the correspondence between (the radial parts of) invariant differential operators on $\mathcal{M}$, respectively on $\mathcal{M}^d$, we obtain the following expansion formula for $\varphi_\lambda$:

$$(2) \quad \varphi_\lambda(a) = c_\Omega(\lambda) \sum_{w \in W_0} c_0(w\lambda) \Phi_{w\lambda}(a), \quad a \in A^-,$$

for $\lambda$ in a dense open subset of $\mathbb{C}^n$, see [9], Theorem 5.7. We use this expansion formula to find an explicit expression for $\varphi_\lambda$:...
Theorem 3. The spherical functions on $\mathcal{M}$ are given by:

$$\varphi_{\lambda}(\exp t) = \frac{c}{\prod_{i<j}(\lambda_j^2 - \lambda_i^2)} \det \left( Q_{\lambda_i - \frac{1}{2}}(\cosh t_j) \right),$$

for $\lambda \geq 0$ and $t > 0$, where $c$ is a constant. The map $\lambda \rightarrow \varphi_{\lambda}(\exp t)$ extends (for fixed $t > 0$) to a meromorphic function with simple poles for $\lambda_i \in \mathbb{N} + \frac{1}{2}$, $(i = 1, \ldots, n)$ and $\lambda_i = -\lambda_j (i \neq j)$.

Proof. The expansion formula (2) yields:

$$\varphi_{\lambda}(\exp t) = c_{\Omega}(\lambda)c_0(\lambda) \sum_{w \in W_0} \varepsilon(w)\Phi_{w\lambda}(\exp t) = c(\lambda) \sum_{w \in W_0} \varepsilon(w)\Phi_{w\lambda}(\exp t),$$

since $c_0(w\lambda) = \varepsilon(w)c_0(\lambda)$ for all $w \in W_0 = \mathfrak{S}_n$, where $\varepsilon(w)$ denotes the sign of the permutation $w \in \mathfrak{S}_n$. Inserting the explicit expression (1) of the Harish-Chandra series $\Phi_{\lambda}$ gives the result by definition of the determinant. $
$
We easily get the following estimates of the spherical functions on $\mathcal{M}$:

Lemma 4. Let $r > 0$. There exists a constant $c_r$ such that

$$|\delta_1(t)\varphi_{\lambda}(\exp t)/c(\lambda)| \leq c_r e^{-\min_{w \in W_0} \langle w\Re \lambda, t \rangle} \leq c_r e^{-\langle \Re \lambda, r_t \rangle},$$

for $\Re \lambda \geq 0$ and $t \geq r$, where $t_0 = (1, \ldots, 1)$.

Proof. Let $r > 0$, then:

$$|\delta_1(t)\varphi_{\lambda}(\exp t)/c(\lambda)| = c \left| \frac{\Gamma(\lambda_j + 1)}{\Gamma(\lambda_j + \frac{1}{2})} \det \left( Q_{\lambda_j - \frac{1}{2}}(\cosh t_j) \right) \right|$$

$$\leq c e^{-\min_{w \in W_0} \langle w\Re \lambda, t \rangle},$$

for $\lambda \geq 0$ and $t \geq r$, for some constants $c$. $
$
From the two expansion formulae for the spherical functions we finally obtain the following correspondence between the spherical functions on $\mathcal{M}^d$ and $\mathcal{M}$:

$$\psi^d_{\lambda}(a^{-1}) = \psi^d_{-\lambda}(a) = \sum_{w \in W_0 \setminus W} \frac{c^d(w\lambda)}{c(w\lambda)} \varphi_{w\lambda}(a), \quad a \in A^-,$$

see also [9], Theorem 5.9. We note that the fraction $\frac{c^d(\lambda)}{c(\lambda)}$ is $W_0$-invariant.

4. The spherical Laplace transform on $\mathcal{M}$

We define the normalized spherical Laplace transform $\mathcal{L}^0$ on $\mathcal{M}$ as (cf. [3], §8):

$$\mathcal{L}^0(f)(\lambda) = c_{\Omega}(\lambda)^{-1} \int_{A^-} f(a)\varphi_{\lambda}(a)\delta(a)da,$$
for any $f \in C_c^\infty(H \setminus S^0/H) \cong C_c^\infty(S_A^0)^{W_0}$ (the left-$W_0$-invariant functions in $C_c^\infty(S_A^0)$), whenever the integral converges. From the explicit expression for $\varphi_\lambda$, we see that the function $\lambda \mapsto \mathcal{L}^\alpha(f)(\lambda)$ extends to a meromorphic function on $\mathbb{C}^n$ with at most simple poles for $\lambda_i \in -\mathbb{N} (i = 1, \ldots, n)$.

Let $f \in C_c^\infty(S_A^0)^{W_0}$. We see that $\mathcal{L}^\alpha f$ satisfies the following functional equation:

$$\mathcal{F}(f^d)(\lambda) = \sum_{w \in W_\alpha \setminus W} c_1(w\lambda)\mathcal{L}^\alpha(f)(w\lambda),$$

almost everywhere (and the right hand side extends to an analytic function), where

$$c_1(\lambda) := c^d(\lambda)/c_0(\lambda) = \prod_j \frac{\Gamma(-\lambda_j)}{\Gamma(-\lambda_j + \frac{n}{2})} \prod_{i<j} (\lambda_i - \lambda_j)^{-1}.$$

The inversion formula for the normalized spherical Laplace transform is an easy consequence of (3) and the inversion formula for the spherical Fourier transform, see also [9], Theorem 6.13:

**Theorem 5 (The Inversion Formula).** Let $f \in C_c^\infty(S_A^0)^{W_0}$. Then

$$f(a) = \frac{|W|}{|W_0|} \int_{\mathbb{R}^n} \mathcal{L}^\alpha(f)(\lambda)\psi^d_\lambda(a) \frac{d\lambda}{c_0(\lambda)c^d(-\lambda)},$$

for all $a \in S_A^0$.

Let $R > r > 0$ and define $C_{r,R}^\infty(S_A^0)^{W_0} := \{f \in C_c^\infty(S_A^0)^{W_0} | \text{supp} f \subset \exp(C_r \cap B_R)\}$, where $C_r := \{t \in \mathbb{R}^n | t \geq r\}$. Lemma 4 and (3) suggest the following definition of the Paley-Wiener space, the supposed image space of the normalized spherical Laplace transform acting on $C_c^\infty(S_A^0)^{W_0}$ (or on the subspaces $C_{r,R}^\infty(S_A^0)^{W_0}$):

**Definition 6.** Let $R > r > 0$. We define the Paley-Wiener space $PW_{r,R}(\mathbb{C}^n)$ as the space of $W_0$-invariant meromorphic functions $g$ on $\mathbb{C}^n$, with at most simple poles for $\lambda_i \in -\mathbb{N} (i = 1, \ldots, n)$, such that (i)

$$\sup_{\lambda \geq 0} e^{\mathbb{R}^\alpha(\lambda, rt_\alpha)}(1 + |\lambda|^N |g(\lambda)/c_0(\lambda)| < \infty,$$

for all $N \in \mathbb{N}$, and (ii) the $c_1$-weighted average

$$\mathbb{P}^{\text{av}} g(\lambda) = \sum_{w \in W_\alpha \setminus W} c_1(w\lambda)g(w\lambda)$$

extends to a function in $\mathcal{H}_R(\mathbb{C}^n)$. Furthermore denote by $PW(\mathbb{C}^n)$ the union of the spaces $PW_{r,R}(\mathbb{C}^n)$ over all $R > r > 0$.

It is easily seen that $\mathcal{L}^\alpha$ maps $C_{r,R}^\infty(S_A^0)^{W_0}$ into $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$ (since $\mathcal{L}^\alpha(\Delta f)(\lambda) = (\lambda^2 - \rho^2)\mathcal{L}^\alpha f(\lambda)$ for all $f \in C_c^\infty(S_A^0)^{W_0}$). We remark that $\mathbb{P}^{\text{av}} \mathcal{L}^\alpha$ acts injectively on $C_{r,R}^\infty(S_A^0)^{W_0}$, since $\mathbb{P}^{\text{av}} \mathcal{L}^\alpha(f) = \mathcal{F}(f^d) = 0$ implies $f = f^d = 0$ on $A^-$ for any $f \in C_c^\infty(S_A^0)^{W_0}$ by injectivity of the spherical Fourier transform. The following lemma, due to H. Schlichtkrull in the rank 1 case, see [1], Lemma 7, shows that $\mathbb{P}^{\text{av}}$ is injective on $PW(\mathbb{C}^n)$:
Lemma 7. Let $g$ be meromorphic function on $\mathbb{C}^n$ that satisfies item (i) of Definition 6 (for some $r > 0$). Assume that $P^\gamma g = 0$. Then $g = 0$.

Proof. Let $g_1(\lambda) = g(\lambda)/e^{d(\lambda)c_0(\lambda)}$ and let $W_1 := \{ \pm 1 \}^n \cong W_0 \setminus W$. Then $P^\gamma g_1(\lambda) = |W_1| e^{d(\lambda)c_0(\lambda)} e^{d(\lambda)c_0(\lambda)}g_1(\lambda)$, where

$$\text{avg}_1(\lambda) := \frac{1}{|W_1|} \sum_{w \in W_1} g_1(w\lambda)$$

is the average of $g_1$ over $W_1$. It follows from the assumption $P^\gamma g = 0$ that $\text{avg}_1 = 0$. The function $g_1$ also satisfies item (i) of Definition 6, in particular, $g_1(i \cdot) \in L^1(\mathbb{R}^n)$. Let

$$\gamma(s) = \int_{\mathbb{R}^n} g_1(i\lambda)e^{i(s,\lambda)} d\lambda, \quad s \in \mathbb{R}^n,$$

denote the Euclidean Fourier transform of $g_1(i \cdot)$. The condition (i) implies that $g_1$ is holomorphic in an open set containing $\{ z \in \mathbb{C}^n | Re z \geq 0 \}$, and the standard argument with Cauchy's theorem gives that $\gamma$ is supported on $C_r^-$. On the other hand, the average $\text{avg}_1 \gamma$ of $\gamma$ is the Fourier transform of $\text{avg}_1(i \cdot)$, which vanishes, hence $\text{avg}_1 \gamma$ vanishes as well. Hence $\gamma = 0$ by the support condition. Since the Euclidean Fourier transform is injective on $L^1(\mathbb{R}^n)$, we conclude that $g_1$, and hence also $g$, vanishes.

Theorem 8 (The Paley-Wiener Theorem). The normalized spherical Laplace transform $L^0$ is a bijection of $C^\infty_r(S^0_A)^W_0$ onto $PW(\mathbb{C}^n)$. More precisely it is a bijection of $C^\infty_r(S^0_A)^W_0$ onto $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$.

Proof. It only remains to show that the normalized spherical Laplace transform maps $C^\infty_r(S^0_A)^W_0$ onto $PW_{r,R}(\mathbb{C})$ for all $R > r > 0$.

We define an auxiliary function $\Xi^d_\lambda$ by:

$$\Xi^d_\lambda(\exp t) = \sum_{w \in W_1} c^d(w\lambda) \Phi_{w,\lambda}(\exp t) = c \prod_{i < j} \Gamma_\lambda(\lambda_j + \frac{1}{2}) \prod_j P_{\lambda_j - \frac{1}{2}}(\cosh t_j) \delta_1(t)$$

for $\lambda_i \neq \pm \lambda_j (i \neq j)$ and $t_i \neq t_j (i \neq j)$. Hence $\psi^d_\lambda = \sum_{w \in W_0} \Xi^d_{w,\lambda}$, and we can rewrite the inversion formula as:

$$f(a) = \frac{|W|}{|W_0|} \int_{\mathbb{R}^n} (L^0 f)(\lambda) \psi^d_\lambda(a) \frac{d\lambda}{c_0(\lambda)e^{d(\lambda)c_0(\lambda)}}$$

for all $a \in A^-$, by $W_0$-invariance of the measure $d\lambda$. 

Consider the wave packet \( I g \in C^\infty (S^0_A)^{W_0} \) of \( g \in PW_{r,R}(a^*_c) \) defined by the inversion formula(e) (for \( a \in A^- \)):

\[
I g(a) = \frac{|W|}{|W_0|} \int_{\mathbb{R}^n} g(\lambda) \psi^d_{\lambda}(a) \frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}
= \frac{|W|}{|W_0|} \int_{\mathbb{R}^n} g(\lambda) \Xi^d_{\lambda}(a) \frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}.
\]

Fix \( r > 0 \) and assume that \( t \notin C_r \). There exists \( \lambda_0 > 0 \) such that \( \langle \lambda_0, t - rt_0 \rangle = -\varepsilon < 0 \) (\( t_0 = (1, \ldots, 1) \)). This yields the following estimate:

\[
|\Xi^d_{\lambda + \mu \lambda_0}(\exp t) / e^d(-\lambda - \mu \lambda_0)| \leq c(1 + |\lambda + \mu \lambda_0|)^{n/2} e^{\mu \langle \lambda_0, t_0 \rangle \varepsilon - \mu \varepsilon},
\]

for \( \mu \geq 0 \) and \( \lambda \in i\mathbb{R}^n \), for some constants \( c \) not depending on \( \lambda \). By Cauchy’s theorem and a contour shift we get:

\[
I g(\exp t) = \frac{|W|}{|W_0|} \int_{\mathbb{R}^n} g(\lambda) \Xi^d_{\lambda}(\exp t) \frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}
= \frac{|W|}{|W_0|} \int_{\mathbb{R}^n} g(\lambda + \mu \lambda_0) \Xi^d_{\lambda + \mu \lambda_0}(\exp t) \frac{d\lambda}{c_0(\lambda + \mu \lambda_0)c^d(-\lambda - \mu \lambda_0)}
\to 0 \quad \text{for} \quad \mu \to \infty.
\]

By continuity and \( W_0 \)-invariance this shows that \( I g \) is identically zero on \( S^0_A \setminus \exp C_r \).

An easy calculation shows that (for \( a \in A^- \)):

\[
I g(a) = \frac{|W|}{|W_0|} \int_{\mathbb{R}^n} g(\lambda) \psi^d_{\lambda}(a) \frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}
= \int_{\mathbb{R}^n} \text{P}^{av} g(\lambda) \psi^d_{\lambda}(a) \left| c^d(\lambda) \right|^{-2} d\lambda,
\]

which we recognize as the inverse Fourier transform of \( \text{P}^{av} g \in \mathcal{H}_R(\mathbb{C}) \), whence \( I g(a) = 0 \) for \( a \in S^0_A \setminus \exp B_R \) by the Paley-Wiener theorem for the spherical Fourier transform on \( \mathcal{M}^d \).

Since \( \text{P}^{av} \mathcal{L}^0 f = \mathcal{F} \mathcal{D}^d f \) for all \( f \in C^\infty_c(S^0_A)^{W_0} \), the above also yields:

\[
\text{P}^{av} \mathcal{L}^0 I g = \mathcal{F} (I g)^d = \text{P}^{av} g,
\]

for all \( g \in PW(\mathbb{C}^n) \), hence Lemma 7 implies that \( \mathcal{L}^0 I g = g \) for all \( g \in PW(\mathbb{C}^n) \) and we conclude that \( \mathcal{L}^0 \) maps \( C^\infty_r(S^0_A)^{W_0} \) onto \( PW_{r,R}(\mathbb{C}^n) \) for all \( R > r > 0 \).

5. The Abel transform on \( \mathcal{M} = SU(n, n)/SL(n, \mathbb{C}) \times \mathbb{R}^*_+ \)

The Abel transform \( \mathcal{A} \) of an \( H \)-invariant function \( f \) on the semigroup \( S \) is defined as (cf. [3], §8):

\[
\mathcal{A} f(a) = a^{-\theta} \int_{\mathcal{N}} f(na) dn,
\]
for \( a \in A \), whenever this integral exists (we put \( f(x) \equiv 0 \) for \( x \in NAH \setminus S \)). It has the following connection to the spherical Laplace transform (for \( \lambda \gg 0 \) and otherwise by analytic continuation):

\[
\mathcal{L}f(\lambda) = \int_{\exp c_{\text{max}}} a^{-\lambda} \mathcal{A}f(a) da = \mathcal{L}_A(\mathcal{A}f)(\lambda),
\]

where \( \mathcal{L}_A \) is the Euclidean Laplace transform on \( A \) with respect to the cone \( c_{\text{max}} \), see [3], Proposition 8.5.

Using the explicit expression of the spherical functions from Theorem 3, we get (modulo constants):

\[
\prod_{i<j}(\lambda_j^2 - \lambda_i^2) \mathcal{L}(f)(\lambda) = \int_{t_n > t_{n-1} > \ldots > t_1 > 0} f(\exp t) \det \left( Q_{\lambda_j} - \frac{1}{2} (\cosh t_j) \right) \frac{\delta(t)}{\delta_1(t)} dt
\]

\[
= \sum_{w \in W_0} \int_{t_n > t_{n-1} > \ldots > t_1 > 0} f(\exp t) e(w) \prod_j Q_{\lambda_j} - \frac{1}{2} (\cosh t_j) \delta_1(t) \prod_j \sinh t_j dt
\]

\[
= \sum_{w \in W_0} \int_{t_n > t_{n-1} > \ldots > t_1 > 0} f(\exp t) \prod_j Q_{\lambda_j} - \frac{1}{2} (\cosh t_j) \delta_1(t) \prod_j \sinh t_j dt
\]

\[
= \int_{c_{\text{max}}} f(\exp t) \delta_1(t) \left\{ \prod_j Q_{\lambda_j} - \frac{1}{2} (\cosh t_j) \sinh t_j \right\} dt
\]

\[
= \mathcal{L}_1^\otimes(f(\exp \cdot) \cdot \delta_1)(\lambda) = \mathcal{L}_A \mathcal{A}_1^\otimes(f(\exp \cdot) \cdot \delta_1)(\lambda),
\]

where \( \mathcal{L}_1^\otimes \) is the \( n \)-fold tensor product of the Laplace transform \( \mathcal{L}_1 \) on the ordered symmetric space \( SO_o(1,2)/SO_o(1,1) \) of rank 1:

\[
\mathcal{L}_1 f(\lambda) = \int_0^\infty f(t) Q_{\lambda - \frac{1}{2}}(\cosh t) \sinh t dt,
\]

for \( f \in C^\infty(\mathbb{R}_+) \), and \( \mathcal{A}_1^\otimes \) is the \( n \)-fold tensor product of the Abel transform \( \mathcal{A}_1 \) on \( SO_o(1,2)/SO_o(1,1) \):

\[
\mathcal{A}_1 f(t) = \int_0^t f(\tau)(2 \cosh t - 2 \cosh \tau)^{-1/2} \sinh \tau d\tau,
\]

for \( f \in C^\infty(\mathbb{R}_+) \), see [3], §10 for details (we have identified \( A^- \) in the rank 1 case with \( \mathbb{R}_+ \) via the map \( a_t \mapsto \tau \)).

We furthermore have:

\[
\prod_{i<j}(\lambda_j^2 - \lambda_i^2) \mathcal{L}(f) = \mathcal{L}_A \left( \prod_{i<j}(\partial_j^2 - \partial_i^2) \mathcal{A}(f) \right)(\lambda),
\]

which implies that:

\[
\left( \prod_{i<j}(\partial_j^2 - \partial_i^2) \mathcal{A}(f) \right) = \mathcal{A}_1^\otimes(f(\exp \cdot) \cdot \delta_1),
\]

by injectivity of the Laplace transform \( \mathcal{L}_A \). Finally, inverting one coordinate at a time, we get by [3], §10:
Theorem 9. Let $f \in C_c^\infty(S_A^0)^{W_0}$. Then:

$$f(\exp t) = c \delta(t)^{-1} \prod_j \left( \frac{1}{\sinh t_j} \frac{d}{dt_j} \right) \int_0^{t_0} \cdots \int_0^{t_1} \left( \prod_{k<l} (\partial_k^2 - \partial_l^2)Af \right)(\exp \tau) \times \prod_j \left( \cosh t_j - \cosh \tau_j \right)^{-1/2} \sinh \tau_j \, d\tau_1 \cdots d\tau_n,$$

for $t \in a^-$, for some constant $c$.

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