Fitting decomposition of Casimir operators
of quadratic Lie superalgebras.

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Abstract. Semisimple Lie superalgebras are Lie superalgebras with non-degenerate Killing forms [1]. In this paper we show that a quadratic Lie superalgebra \((\mathfrak{g}, B)\) is semisimple if and only if its Casimir operator, \(\Omega_B\), associated to \(B\) is invertible. This result enables us to characterize quadratic Lie superalgebras with nilpotent Casimir operators: \(\Omega_B\) is nilpotent if and only if \(\mathfrak{g}\) is a Lie superalgebra without any nonzero semisimple ideal.

1. Introduction

Lie superalgebras considered in this paper are finite dimensional over a commutative field \(\mathbb{K}\) with characteristic zero.

In our paper [1] we have choose a definition of semisimplicity for Lie superalgebras: a Lie superalgebra \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) is semisimple if and only if its Killing form \(\mathcal{K}\) is non-degenerate. Next, we have study relation between semisimplicity and invertibility of the Casimir operator in the case of a quadratic Lie superalgebra \((\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, B)\). In particular, we have shown that if the representation of the Lie algebra \(\mathfrak{g}_0\) on \(\mathfrak{g}_1\) is completely reducible then semisimplicity is equivalent to the invertibility of the Casimir operator. The first main result of the present paper gives equivalence between invertibility of the Casimir operators and semisimplicity of an arbitrary quadratic Lie superalgebra. Recall that if \((\mathfrak{g}, B)\) is a quadratic Lie algebra, the previous result has been obtained by the second author [3]. The second main result gives the characterization of quadratic Lie superalgebras with nilpotent Casimir operators. Finally, we obtain the Fitting decomposition of Casimir operators of quadratic Lie superalgebras.
2. Semisimplicity and Casimir Operators

Let \((g, B)\) be a quadratic Lie superalgebra: in the sense that \(B\) is a supersymmetric even non-degenerate and \(g\)-invariant bilinear form. Let \((e_i)_{1 \leq i \leq n}\) and \((e'_i)_{1 \leq i \leq n}\) two basis of the Lie algebra \(g\) such that

\[
B(e_i, e'_j) = \delta_{ij},
\]

where \(\delta_{ij}\) is the Kronecker’s symbol. Let \((f_i)_{1 \leq i \leq m}\) and \((f'_i)_{1 \leq i \leq m}\) two basis of \(g\) such that

\[
B(f_i, f'_j) = \delta_{ij}.
\]

Let \(\omega_B\) be the element of the enveloping algebra \(U(g)\) of \(g\) defined by

\[
\omega_B = \sum_{i=1}^{n} e_i e'_i - \sum_{i=1}^{m} f_i f'_i.
\]

The element \(\omega_B\) is called the Casimir element of \((g, B)\); it is a central element in \(U(g)\). The Casimir element \(\omega_B\) does not depend on the choice of the basis [6].

Let \(T\) denote the extension of the adjoint representation of \(g\) to \(U(g)\). The element \(T(\omega_B)\) of \(End(g)\) is called the Casimir operator of \((g, B)\) associated with \(B\). We write \(\Omega_B = T(\omega_B)\).

Recall that if \(g\) is a simple Lie superalgebra then the following theorem holds

**Theorem 2.1.** ([5]). Let \((g, B)\) be a simple quadratic Lie superalgebra and \(\Omega_B\) its Casimir operator associated to \(B\). Then \(g\) is semisimple if and only if \(\Omega_B\) is invertible.

Let us now give some definitions.

**Definition 2.1** A graded ideal \(I\) of a quadratic Lie superalgebra \((g, B)\) is called non-degenerate (resp. degenerate) if the restriction of \(B\) to \(I \times I\) is a non-degenerate (resp. degenerate) bilinear form.

**Definition 2.2** A quadratic Lie superalgebra \((g, B)\) is said to be \(B\)-irreducible if \(g\) does not contain any non trivial non-degenerate graded ideal.

The following theorem gives the relation between invertibility of Casimir operators and semisimplicity of quadratic Lie superalgebras.

**Theorem 2.2.** Let \((g, B)\) be a quadratic Lie superalgebra and \(\Omega_B\) its Casimir operator associated to \(B\). Then \(\Omega_B\) is invertible if and only if \(g\) is semisimple.

**Proof.** **First case:** We suppose that \((g, B)\) is \(B\)-irreducible. We show the theorem for this particular case.

1. Suppose that \(\Omega_B\) is invertible. We claim that this implies that \(g\) is a simple Lie superalgebra. Indeed:
Suppose that $\mathfrak{g}$ is not simple, then there exists $I$ a minimal graded non-trivial ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is $B$-irreducible and $I^\perp$ is a graded ideal of $\mathfrak{g}$, by minimality of $I$ one has $I \cap I^\perp = I$. Now by invariance of $B$, we have

$$B([I, I^\perp], \mathfrak{g}) = B(I, [I^\perp, \mathfrak{g}]) = \{0\},$$

so the nondegeneracy of $B$ implies that $[I, I^\perp] = \{0\}$ and, in particular, $I$ is abelian. Denote by $V$ the supplementary subspace of $I^\perp$ in $\mathfrak{g}$ such that $\mathfrak{g} = I^\perp \oplus V$. Note that the fact that $B$ is non-degenerate and $I \neq \{0\}$ implies that $V \neq \{0\}$. Let $W = I \oplus V$. Then $W$ is a subspace of $\mathfrak{g}$ such that $B_{|W} \oplus V$ is non-degenerate. More precisely, if $x \neq 0$ an element of $I$ (resp. $V$), then there exists an element $y$ of $V$ (resp. $I$) such that $B(x, y) \neq 0$. Consequently, the map $\phi : I \to V^*$ (resp. $\phi : V \to I^*$) defined by $\phi(x) = B(x, .)$ (resp. $\phi(x) = B(x, .)$) for all $x \in I$ (resp. $V$) is an injective linear map. Hence, $\dim I = \dim V$. Because $B_{|W} \oplus V$ is non-degenerate, then $\mathfrak{g} = I \oplus W^\perp \oplus V$ and $B_{|W} \oplus V$ is non-degenerate.

Let $(e_i)_{1 \leq i \leq n}$ be a basis of $\mathfrak{g}$ consisting of homogeneous elements such that $(e_i)_{1 \leq i \leq n}$ be a basis of $I$, $(e_i)_{p+1 \leq i \leq 2p}$ be a basis of $V$ and $(e_i)_{2p+1 \leq i \leq n}$ be a basis of $W^\perp$. Let $e_i$ be the degree of $e_i$. We introduce a second basis $(f_i)_{1 \leq i \leq n}$ of $\mathfrak{g}$ such that

$$B(f_j, e_i) = \delta_{ij},$$

that $(f_i)_{p+1 \leq i \leq 2p}$ is a basis of $I$, $(f_i)_{1 \leq i \leq n}$ is a basis of $V$ and that $(f_i)_{2p+1 \leq i \leq n}$ is a basis of $W^\perp$. The Casimir operator associated to $B$ is

$$\Omega_B = \sum_{i=1}^{n} (-1)^{e_i} \text{ad} f_i \circ \text{ad} e_i.$$

Then for all $x \neq 0$ in $I$, one has $\Omega_B(x) = 0$ because $[I, I^\perp] = \{0\}$, $I^\perp = I \oplus W^\perp$ and $I$ is a graded ideal of $\mathfrak{g}$, which contradicts the fact that $\Omega_B$ is invertible.

So $(\mathfrak{g}, B)$ is a simple Lie superalgebra such that its Casimir operator is invertible, by Theorem 2.1, we conclude that $(\mathfrak{g}, B)$ is semisimple.

2. Conversely: If the quadratic $B$-irreducible Lie superalgebra $(\mathfrak{g}, B)$ is semisimple then $\mathfrak{g}$ is a direct sum of its simple graded ideals whose Killing forms are non-degenerate [6]. Let $I$ be a non-zero simple graded ideal, then, by invariance and nondegeneracy of $B$, $I \cap I^\perp$ is an abelian graded ideal of $I$. Consequently, $I \cap I^\perp = \{0\}$, so $I$ is a non-degenerate graded ideal of $\mathfrak{g}$, it follows that $\mathfrak{g} = I$ because $\mathfrak{g}$ is a $B$-irreducible Lie superalgebra. We conclude that $\mathfrak{g}$ is simple. Hence $\Omega_B$ is invertible by Theorem 2.1.

Second case: Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. By Proposition 2.6 of [2] one has

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i,$$

such that:

(i) $\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$, for all $1 \leq i \leq n$;
(ii) $\mathfrak{g}_i$ is $B_i$-irreducible, where $B_i$ is the restriction of $B$ to $\mathfrak{g}_i \times \mathfrak{g}_i$, for all $1 \leq i \leq n$;
(iii) for all $i \neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal.
Let $\omega^i_{B_i}$ be the Casimir element of $(\mathfrak{g}_i, B_i)$. Let $T_i$ be the representation of $\mathfrak{A}(\mathfrak{g}_i)$, the unique extension of the adjoint representation of $\mathfrak{g}_i$. Denote by $\Omega^i_{B_i} = T_i(\omega^i_{B_i})$ the Casimir operator of $(\mathfrak{g}_i, B_i)$. Now $\omega_B = \omega^1_{B_1} + \cdots + \omega^r_{B_r}$.

Then it is easy to see that

- $T(\omega^i_{B_i})(\mathfrak{g}_j) = 0$ if $i \neq j$,
- $T(\omega^i_{B_i}|_{\mathfrak{g}_i}) = T_i(\omega^i_{B_i})$,
- $T(\omega^i_{B_i}) \cdot T(\omega^j_{B_j}) = T(\omega^i_{B_i}) \circ T(\omega^j_{B_j})$,
- $\Omega_B = T(\omega_B) = \sum_{i=1}^r T(\omega^i_{B_i})$.

1. Suppose that the Casimir operator $\Omega_B$ of $(\mathfrak{g}, B)$ is invertible. Then, for all $i \in \{1, \cdots, r\}$, $\Omega^i_{B_i}$ is invertible. The first case implies that $\mathfrak{g}_i$ is simple and semisimple. We conclude that $\mathfrak{g}$ is semisimple.

2. Conversely: If $\mathfrak{g}$ is a semisimple Lie superalgebra, then for all $i \in \{1, 2, \cdots, r\}$, $\mathfrak{g}_i$ is a simple semisimple Lie superalgebra [6]. Now by Theorem 2.1, for all $i \in \{1, 2, \cdots, r\}$, the Casimir operator $\Omega^i_{B_i}$ of $(\mathfrak{g}_i, B_i)$ is invertible. So the Casimir operator $\Omega_B$ of $(\mathfrak{g}, B)$ is invertible.

**Remark.** It is well known that the invertibility of the Casimir operators is very useful in the cohomological theory of semisimple Lie superalgebras. Theorem 2.2 shows that, in the case of quadratic Lie superalgebras which are not semisimple, others methods would be necessary for the cohomological theory.

3. **Fitting decomposition of the Casimir operator $\Omega_B$.**

**Lemma 3.1.** Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra and $\Omega_B$ its Casimir operator. Then

1. For all $X \in \mathfrak{g}$, $\Omega_B \circ \text{ad}_\mathfrak{g}(X) = \text{ad}_\mathfrak{g}(X) \circ \Omega_B$.
2. For all $X, Y \in \mathfrak{g}$, $B(\Omega_B(X), Y) = B(X, \Omega_B(Y))$.
3. For all integer $n \neq 1$, $B(\ker(\Omega_B)^n, \text{Im}(\Omega_B)^n) = \{0\}$.
4. For all integer $n \neq 1$, $\ker(\Omega_B)^n$ is an ideal of $\mathfrak{g}$.

**Proof.** The proof is an easy computation.

**Proposition 3.1.** If $(\mathfrak{g}, B)$ is a quadratic, $B$-irreducible Lie superalgebra and $\Omega_B$ its Casimir operator, then $\Omega_B$ is nilpotent or invertible.

**Proof.** There exists an integer $n \neq 1$ such that

$$\mathfrak{g} = \ker(\Omega_B)^n \oplus \text{Im}(\Omega_B)^n.$$  

By Lemma 3.1, we deduce that $\ker(\Omega_B)^n$ is a non-degenerate graded ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is $B$-irreducible, then $\ker(\Omega_B)^n = \mathfrak{g}$ or $\ker(\Omega_B)^n = \{0\}$. So $\Omega_B$ is nilpotent or invertible.

Now we are in position to state the main result of this section. This result gives the characterization of quadratic Lie superalgebras with nilpotent Casimir operators.
Theorem 3.1. Let \((g, B)\) be a quadratic Lie superalgebra and \(\Omega_B\) its Casimir operator associated to \(B\). Then \(\Omega_B\) is nilpotent if and only if \(g\) is a Lie superalgebra without nonzero semisimple graded ideal.

Proof. By Proposition 2.6 of [2],

\[ g = \bigoplus_{i=1}^{r} g_i \]

such that:

(i) \(g_i\) is a non-degenerate graded ideal of \(g\), for all \(1 \leq i \leq n\);
(ii) \(g_i\) is \(B_i\)-irreducible, where \(B_i\) is the restriction of \(B\) to \(g_i \times g_i\), for all \(1 \leq i \leq n\);
(iii) for all \(i \neq j\), \(g_i\) and \(g_j\) are orthogonal.

1. Suppose that \(\Omega_B\) is nilpotent. Then there exists a non-negative integer \(n\) such that \(\Omega_B^n = 0\). Then for all \(1 \leq i \leq r\), the Casimir operator \(\Omega'_{B_i}\) of \((g_i, B_i)\) is nilpotent. By Theorem 2.2, for all \(1 \leq i \leq r\), \(g_i\) are not semisimple. Let \(I\) be a semisimple ideal of \(g\). By Proposition 2.6 of [1] we have \(I = [I, I]\), consequently \(I = [I, g]\). So

\[ I = [I, g_1] \oplus \cdots \oplus [I, g_r], \text{ and } I \cap g_i = [I, g_i] \text{ for all } i = 1, 2, \ldots, r. \]

Let \(i\) be in \(\{1, 2, \ldots, r\}\). Proposition 2.14 of [1] implies that \(I_i = I \cap g_i\) is a semisimple ideal of \(g_i\). By the invariance and the nondegeneracy of \(B_i\), \(I_i \cap (I_i)^{\perp}\) is a graded abelian of \(I_i\), consequently \(I_i \cap (I_i)^{\perp} = \{0\}\). Hence, \(I_i\) is a non-degenerate ideal of \(g_i\). Now, since \((g_i, B_i)\) is \(B_i\)-irreducible and \(g_i\) is not semisimple, we deduce that for all \(i\), \(I_i = \{0\}\). Thus \(I = \{0\}\).

2. Conversely: If \((g, B)\) is a quadratic Lie superalgebra without nonzero semisimple graded ideal, then, for all \(1 \leq i \leq r\), \(g_i\) is not semisimple. By Proposition 3.1, \(\Omega'_{B_i}\) is nilpotent. Using the fact that:

- \(T(\omega^i_{B_j})(g_j) = 0\) if \(i \neq j\),
- \(T(\omega^i_{B_i})|_{g_i} = T_i(\omega^i_{B_i})\),
- \(T(\omega^i_{B_k}) \circ T(\omega^j_{B_j}) = T(\omega^i_{B_k}) \circ T(\omega^j_{B_j})\),
- \(\Omega_B = T(\omega_B) = \sum_{i=1}^{r} T(\omega^i_{B_i})\),

we then deduce that \(\Omega_B\) is nilpotent.

\[ \square \]

Remark. This theorem generalizes Theorem 1 of [4].

We close this paper by given the Fitting decomposition of Casimir operators of quadratic Lie superalgebras.

Corollary 3.1. Let \((g, B)\) be a quadratic Lie superalgebra and \(\Omega_B\) its Casimir operator associated to \(B\). Let \(S\) be a largest semisimple ideal of \(g\). Then there exists an ideal \(J\) of \(g\) such that \(g = I \oplus J\), this decomposition is the Fitting decomposition of \(g\) relative to \(\Omega_B\).

Proof. By Corollary 2.21 of [1], \(g\) contains a largest graded semisimple ideal \(S\). Since \(B\) is non-degenerate and invariant, then \(S \cap S^{\perp}\) is an abelian graded
ideal of \( \mathfrak{g} \), it follows that \( S \cap S^\perp = \{0\} \) because \( S \) is a graded semisimple ideal. Consequently, \( S \) is a non-degenerate ideal of \( \mathfrak{g} \), so \( \mathfrak{g} = S \oplus J \) where \( J = S^\perp \) which is a non-degenerate graded ideal of \( \mathfrak{g} \). Hence \((S, B_{S \times S})\) and \((J, B_{J \times J})\) are quadratic Lie superalgebras. Consequently,

\[
(\Omega_B)|_S = \Omega(B_{S \times S}) \quad \text{and} \quad a(\Omega_B)|_J = \Omega(B_{J \times J}).
\]

By Theorem 2.2, \((\Omega_B)|_S\) is invertible. Now, if we consider \( I \) a graded semisimple ideal of \( J \), then \( I \) is a semisimple graded ideal of \( \mathfrak{g} \) because \([S, J] = \{0\}\), it follows that \( I \subseteq S \), so \( I = \{0\} \). Hence, \( J \) is a Lie superalgebra without nonzero semisimple graded ideal, it follows, from Theorem 3.1, that \((\Omega_B)|_J\) is nilpotent. We conclude that \( \mathfrak{g} = S \oplus J \) is the Fitting decomposition of \( \mathfrak{g} \).

\section*{References}


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