1/2-Transitive Graphs of Order $3p$

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Abstract. A graph $X$ is called vertex-transitive, edge-transitive, or arc-transitive, if the automorphism group of $X$ acts transitively on the set of vertices, edges, or arcs of $X$, respectively. $X$ is said to be 1/2-transitive, if it is vertex-transitive, edge-transitive, but not arc-transitive.

In this paper we determine all 1/2-transitive graphs with $3p$ vertices, where $p$ is an odd prime. (See Theorem 3.4.)

Keywords: 1/2-transitive graph, metacirculant, factor graph

1. Introduction

Let $X = (V(X), E(X))$ be a graph (that is, no multiple edges or loops). We call an ordered pair of adjacent vertices an arc of $X$. The set of all arcs associated with a graph $X$ is denoted by $A(X)$. Thus, $|A(X)| = 2|E(X)|$. If $G$ is a subgroup of $\text{Aut}X$ and $G$ acts transitively on the set of vertices, edges, or arcs of $X$, then $X$ is said to be $G$-vertex-transitive, $G$-edge-transitive, or $G$-arc-transitive, respectively. The graph $X$ is said to be vertex-transitive, edge-transitive, or arc-transitive, if it is $\text{Aut}X$-vertex-transitive, $\text{Aut}X$-edge-transitive, or $\text{Aut}X$-arc-transitive, respectively. We call a graph $X$ 1/2-transitive, if it is vertex-transitive, edge-transitive, but not arc-transitive.

D. Marušić, L. Nowitz and the first author of this paper studied 1/2-transitive graphs [1] and found several infinite families of such graphs. In [7], R.J. Wang and the second author gave a classification of arc-transitive graphs of order $3p$, where $p$ is a prime. The purpose of this paper is to determine all 1/2-transitive graphs of order $3p$.

We use standard terminology and notation for the most part and refer the reader to [5, 6, 8] if necessary. For $v \in V(X)$, $X_1(v)$ denotes the neighborhood of $v$ in $X$, that is, the set of vertices adjacent to $v$ in $X$. If $X$ is a graph and $A$ and $B$ are two vertex-disjoint subsets of the vertex-set $V(X)$ of $X$, we let $\langle A \rangle$ and $\langle A, B \rangle$ denote the subgraph induced on $A$ and the bipartite subgraph, with bipartition sets $A$ and $B$, induced on $A \cup B$ by $X$, respectively. We remind the reader that two representations of a group $G$ as transitive permutation groups are said to be equivalent if the pointwise stabilizers of one representation are conjugate in $G$ to the pointwise stabilizers of the other representation.

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Let \( X \) be a 1/2-transitive graph such that \( \text{Aut}X \) acts imprimitively on \( X \). Then

1. \( X \) is 1/2-transitive or arc-transitive;
2. if \( X \) is connected, then so is \( X \); and
3. if \( \langle B_i \rangle \) has an edge, then \( B_i \) is a union of several connected components of \( X \).

Given two graphs \( X \) and \( Y \) the wreath product \( X \wr Y \) is defined as the graph with vertex set \( V(X) \times V(Y) \) such that \( (x, y)(x', y') \) is an edge if and only if either \( xx' \) is an edge of \( X \), or \( x = x' \) and \( yy' \) is an edge of \( Y \).

Informally \( X \wr Y \) is obtained by taking \( |V(X)| \) copies of \( Y \), labelling these copies with the vertices of \( X \), and, whenever \( xx' \) is an edge of \( X \), joining each vertex in the copy of \( Y \) labelled \( x \) to each vertex in the copy of \( Y \) labelled \( x' \). The automorphism group of \( X \wr Y \) contains the wreath product \( \text{Aut}Y \wr \text{Aut}X \) (but may be larger).

The next obvious proposition gives a method of constructing larger 1/2-transitive graphs from smaller ones.

**Proposition 1.2** If \( X \) is a 1/2-transitive graph of order \( n \), then the wreath product \( X \wr mK_1 \) of \( X \) by \( mK_1 \) is a 1/2-transitive graph of order \( nm \).

Next we quote two propositions from [1].

**Proposition 1.3** Every vertex- and edge-transitive Cayley graph on an abelian group is also arc-transitive.

**Proposition 1.4** Every vertex- and edge-transitive graph with \( p \) or \( 2p \) vertices, \( p \) a prime, is also arc-transitive.

Finally we quote a result from [7].

**Proposition 1.5** Let \( X \) be an arc-transitive graph of order \( 3p \) and \( A = \text{Aut}X \). If \( A \) has a block of imprimitivity of length \( p \), then \( X \) is a Cayley graph on a cyclic group \( Z_{3p} \).
2. Reductions

In this section we eliminate some possible types of 1/2-transitive graphs of order 3p. There is no 1/2-transitive graph of order less than 27 as observed in [1]. All suborbits of a primitive group of degree 3p are self-paired [7], which implies there are no vertex-primitive 1/2-transitive graphs of order 3p. So we may assume that \( p \geq 11 \) in what follows, and we only need consider those 1/2-transitive graphs with an imprimitive automorphism group.

Let \( X \) be a 1/2-transitive graph and \( A = \text{Aut}X \). We have two cases: (1) \( A \) has a block of length \( p \), and (2) \( A \) has a block of length 3, but no blocks of length \( p \). We shall show in the next section that the latter case cannot occur.

Assume that \( X = \{ B_i \mid i \in \mathbb{Z}_3 \} \) is a complete block system of \( A \), and that \( K \) is the kernel of the action of \( A \) on \( X \). Set \( \overline{A} = A/K \). We also use \( \overline{X} \) to denote the corresponding factor graph. Then \( \overline{X} \cong K_3 \). Since \( X \) is 1/2-transitive, \( X \) is not isomorphic to \( K_{p,p,p} \).

**Lemma 2.1** The kernel \( K \) acts faithfully on each block \( B_i \).

**Proof:** We use \( K_{B_i} \) to denote the pointwise-stabilizer of \( B_i \) in \( K \). If \( K \) acts unfaithfully on \( B_i \), we have \( K_{B_i} \) is nontrivial for some block \( B_j \) adjacent to \( B_i \). Since \( K_{B_i} \triangleleft K \), \( K_{B_i} \triangleleft K_{B_j} \). Since \( |B_j| = p \), \( K_{B_j} \) is primitive which implies \( K_{B_i} \) is transitive. It follows that the induced subgraph \( (B_i, B_j) \cong K_{p,p} \). The edge-transitivity of \( \overline{X} \) implies that \( X \cong K_{p,p,p} \). The latter is impossible since \( K_{p,p,p} \) is arc-transitive. \( \square \)

Using the same method as above we can prove the following more general result, which will be used in the next section.

**Lemma 2.2** Suppose that \( X \) is a connected 1/2-transitive graph and \( \overline{X} = \{ B_1, B_2, \ldots, B_n \} \) is a complete block system of \( A = \text{Aut}X \). If \( K \), the kernel of the action of \( A \) on \( \overline{X} \), acts on \( B_i \) unfaithfully and \( K_{B_i} \) is primitive, then \( X \) is isomorphic to the wreath product \( \overline{X} \wr mK_1 \), where \( m = |B_i| \).

By virtue of Lemma 2.1, we may assume that \( K \) acts on each \( B_i \) faithfully in what follows. Since \( |B_i| = p \), we have two cases: (1) \( K \) acts on \( B_i \) doubly-transitively, and (2) \( K \) acts on \( B_i \) simply primitively. We shall treat these two subcases next.

**Lemma 2.3** Let \( X \) be a connected vertex-and edge-transitive graph of order 3p, and \( \overline{X} = \{ B_0, B_1, B_2 \} \) be a complete block system of \( A = \text{Aut}X \). If \( K \), the kernel of the action of \( A \) on \( \overline{X} \), acts doubly transitively on a block, then \( X \) is arc-transitive. That is, there are no 1/2-transitive graphs of order 3p for which \( K \) acts doubly transitively on a block.

**Proof:** By the classification of 2-transitive groups (see [4], for example), it is easy to check that every 2-transitive group has at most two non-equivalent 2-transitive representations. So, without loss of generality, we may assume that \( K_{B_0} \) and \( K_{B_1} \) are equivalent. Since \( \overline{X} \) is edge-transitive, all three groups \( K_{B_0}, K_{B_1}, \) and \( K_{B_2} \) are equivalent to each other. Hence, for any vertex \( v_0 \in B_0 \), the stabilizer \( K_{v_0} \) must fix a vertex \( v_1 \) in \( B_1 \), and a vertex
By the transitivity of $K_{v_1}$ on $B_1 - \{v_1\}$, either $v_0$ is adjacent to every vertex in $B_1$, every vertex in $B_1 - \{v_1\}$, or only $v_1$. Thus, the induced bipartite graph $(B_0, B_1)$ is either $K_{p,p}$, $K_{p,p}$ minus a 1-factor, or a 1-factor. By the edge-transitivity of $X$, $X$ is either $K_{p,p,p}$, $K_{p,p,p}$ minus $pK_3$, or of degree 2. In all these cases, $X$ is arc-transitive.

We now consider the case that $K$ acts on $B_i$ simply primitively. Then $K^{B_i} < AGL(1, p)$ is solvable. So $K$ has only one transitive representation of degree $p$, and for any $v \in B_i$, $K_v < Z_{p-1}$ is semiregular on $B_i - \{v\}$. Furthermore, the Sylow $p$-subgroup of $K^{B_i}$ is normal in $K^{B_i}$. Since $K$ acts on $B_i$ faithfully, the Sylow $p$-subgroup $P$ of $K$ is normal, and then is characteristic, in $K$, implying $P \trianglelefteq A$. Since $|P| = p$, $P$ is cyclic. We will use this information later.

In the next proposition, we determine the factor group $\overline{A} = A/K$.

**Proposition 2.4** If $X$ is a 1/2-transitive graph of order $3p$ with three blocks of length $p$, then $\overline{A} \cong Z_3$.

**Proof:** By Proposition 1.4, there is no 1/2-transitive graph of prime order, which implies that $X$ is connected. By Proposition 1.1, $X \cong K_3$ so that $\overline{A} \cong S_3$ or $Z_3$. Assume that $\overline{A} \cong S_3$. Let $P$ be the Sylow $p$-subgroup of $K$. Then $P$ is normal in $A$ and is cyclic of order $p$ by the information above. Put $C = C_A(P)$. We have $A/C$ is isomorphic to a subgroup of $AutP = Z_{p-1}$, so that $A' \leq C$. Since $A/K \cong S_3$, $A'K/K \cong Z_3$. Hence 3 divides $|A'|$, and then 3 divides $|C|$. Assume that $P = \langle g \rangle$. Take $h \in C$ with $o(h) = 3$. Set $H = \langle g, h \rangle$. Since $h \in C = C_A(P)$, $h$ and $g$ commute. Then $H \cong Z_3 \times Z_p \cong Z_{3p}$ is a regular subgroup of $A$. By [3, Lemma 16.3], $X$ is a Cayley graph of $Z_{3p}$. Finally, by Proposition 1.3, $X$ is arc-transitive, a contradiction. 

Now we give an example via the next theorem. We need the concept of metacirculant defined in [2].

Let $n \geq 2$. A permutation on a finite set is said to be $(m, n)$-semiregular if it has $m$ cycles of length $n$ in its disjoint cycle decomposition. We shall be sloppy and refer to the orbits of the group $(\alpha)$ generated by $\alpha$ as the orbits of $\alpha$. A graph $X$ is an $(m, n)$-metacirculant if it has an $(m, n)$-semiregular automorphism $\alpha$ together with another automorphism $\beta$ normalizing $\alpha$ and cyclically permuting the orbits of $\alpha$. Therefore, we may partition the vertex-set of an $(m, n)$-metacirculant into the orbits $B_0, B_1, \ldots, B_{m-1}$ of $\alpha$, where $B_i = B_{i+1}$ for all $i \in Z_m$. We shall refer to the orbits of $\alpha$ as the blocks of the metacirculant graph. It should be pointed out that the blocks of a metacirculant graph need not be blocks of imprimitivity of the automorphism group of the graph.

Recall that a circulant graph is a Cayley graph on a cyclic group. Using additive notation for the underlying cyclic group, the symbol $S$ of a circulant is defined by $S = \{ j : u_0u_j \text{ is an edge of the circulant graph}\}$. If $S_0 \subseteq Z_n \backslash \{0\}$ is the symbol of the subcirculant $(B_0)$ and, for all $i \in Z_m \backslash \{0\}$, $T_i \subseteq Z_n$ is the symbol of the bipartite subgraph $(B_0, B_i)$, then there exists an $r \in Z_n^\times$, where $Z_n^\times$ denotes the multiplicative group of units in $Z_n$, such that for all $j \in Z_m$, the symbol of $(B_j)$ is $r^jS_0$ and the symbol of the bipartite graph $(B_j, B_{j+i})$, $i \in Z_m$, is $r^jT_i$. Moreover, for all $i \in Z_m$, we have $T_{m-i} = r^{m-i}(-T_i)$. Thus, the metacirculant
Theorem 2.5 If \((d, p) \neq (2,7)\) or \((3,19)\), then the graph \(M(d; 3, p)\) is a 1/2-transitive graph of order \(3p\) and of degree \(2d\). This graph is independent of the choice of \(r\). The automorphism group \(A = \text{Aut}M(d; 3, p)\) is isomorphic to a semidirect product of \(Z_p\) and \(Z_{3d}\), and \(A\) acts regularly on the edge set of \(M(d; 3, p)\).

Proof: Checking the vertex-primitive graphs of order \(3p\) listed in [7], we know that \(M(2; 3, 7)\) and \(M(3; 3, 19)\) are the only vertex-primitive \((3, p)\)-metacirculants and both of them are arc-transitive. Suppose now that \(p \geq 11\) and \(d \neq 3\) if \(p = 19\).

Assume that \(B_i = \{x_j \mid j = 0, 1, \ldots, p - 1\}, i = 0, 1, 2\), are the three blocks of \(X = M(d; 3, p)\) as a metacirculant. It is easy to see that the following mappings \(\alpha, \beta\) and \(\gamma\) are automorphisms of \(X\):

\[
\alpha: x_j \mapsto x_{j+1} \\
\beta: x_j \mapsto x_{j+1}^{2d} \\
\gamma: x_j \mapsto x_{ij}.
\]

Assume that \(3^e \mid d\). Then \(o(\alpha) = p, o(\gamma) = d\) and \(o(\beta) = 3^{e+1}\). Set \(P = \langle \alpha \rangle, L = \langle \alpha, \gamma \rangle, M = \langle \beta, \gamma \rangle\) and \(G = \langle \alpha, \gamma, \beta \rangle\). We can see that \(P \triangleleft G, G\) is a semidirect product of \(P\) and \(M\), and the centralizer of \(P\) in \(G\) is \(P\) itself. Thus, \(M \cong G/P\) is isomorphic to a subgroup of \(\text{Aut}P \cong Z_{p-1}\), so that in particular, \(M = \langle \beta \gamma \rangle\) is cyclic. Also it is easy to see that \(X\) is \(G\)-vertex-transitive and \(G\)-edge-transitive.

To prove that \(X\) is not arc-transitive, first we claim that \(A\) has a block of length \(p\). If not, \(A\) is either primitive, or imprimitive but only has blocks of length \(3\) on the vertex set of \(X\). By the reason mentioned at the beginning of the proof, assuming that \(p \geq 11\) and \(d \neq 3\) for \(p = 19\), we have that \(X\) is vertex-imprimitive and \(A\) has only blocks of length \(3\). By a result in the next section, there are no 1/2-transitive graphs having this property, so that \(X\) must be arc-transitive. By a result in [7] the only arc-transitive graphs, which are not vertex-primitive and whose automorphism groups do not have a block of length \(p\), have automorphism groups \(A\), with \(\text{PSL}(2, 2^s) \leq A \leq \text{PGU}(2, 2^s)\), and \(p = 2^s + 1\) being a Fermat prime. In this case \(3\) does not divide \(p - 1 = 2^s\), so this case cannot occur.

We have proved that \(A\) has a block of length \(p\). Since the only blocks of length \(p\) of \(G\), which is a subgroup of \(A\), are \(B_i, i = 0, 1, 2\), they must be blocks of \(A\) too. Let \(K\) be
the kernel of $A$ acting on $\overline{X} = \{B_0, B_1, B_2\}$. By the same argument as in the proof of Lemma 2.3, we know that $K$ is not doubly-transitive on $B_i$. This implies that the Sylow $p$-subgroup of $K$, which is $P$ defined above and generated by $\alpha$, is normal in $A$. Assume that $X$ is arc-transitive. Noting that $X$ is not isomorphic to the multipartite complete graph $K_{p,p,p}$, by Theorem 3 in [7], $X \cong G(3p, d)$ defined in [7]. By Example 3.4 in [7], $A = \text{Aut}G(3p,d) \cong (Z_p \times Z_d) \cdot S_3$, where $G, H$ denotes an extension of $G$ by $H$, and $A$ contains a cyclic subgroup of order $3p$. It follows that the order of a Sylow 3-subgroup of $A$ is $3^{\varepsilon+1}$, where $3^\varepsilon | d$, and that the centralizer of the Sylow $p$-subgroup $P$ contains an element of order 3. Since $o(\beta) = 3^{\varepsilon+1}$, $\langle \beta \rangle$ is the Sylow 3-subgroup of $A$. It follows that $\beta^{3\varepsilon}$ and $\alpha$ commute. However, it is not the case, a contradiction. This shows that $X$ is 1/2-transitive as required.

It is not difficult to show that different choices of $r$ correspond to isomorphic graphs. We leave this as an exercise for the reader.

Now we determine the automorphism group $A = \text{Aut}M(d; 3, p)$. Since $A$ is an extension of the kernel $K$ by $Z_3$, and $K$ is an extension of $P$ by the stabilizer $K_N$ of $v = x_0^0$ in $K$, it is easy to see that $K_N \cong T$. This shows that $K \cong L$ defined before. Note that $\beta$ is not in $K$ and is a 3-element, implying that $A = \langle K, \beta \rangle = \langle L, \beta \rangle = G$, as desired. It follows that $A = G \cong Z_p \times Z_3d$ acts regularly on the edge set of $X$.

(Note that if $3 \not| d$, then $M(d; 3, p)$ is a Cayley graph on $H_{3p}$ with respect to $S = \{\beta \alpha^i \mid i \in T\} \cup \{\beta^2 \alpha^{-u^2} \mid i \in T\}$, while if $3 \mid d, M(d; 3, p)$ is not a Cayley graph.)

Theorem 2.6 Let $X$ be a 1/2-transitive graph of order $3p$. If $\text{Aut}X$ acts imprimitively on $V(X)$ and has a block of length $p$, then $X$ is isomorphic to $M(d; 3, p)$ for some divisor $d$ of $p-1$ where $(d, p) \neq (2, 7)$ or $(3, 19)$.

Proof: Assume that $\overline{X} = \{B_0, B_1, B_2\}$ is a complete block system of $A = \text{Aut}X$ and that $K$ is the kernel of $A$ acting on $\overline{X}$.

(1) We claim that $X$ is connected. If not, every connected component has either $p$ or 3 vertices, and is also 1/2-transitive. But by Proposition 1.4, there are no 1/2-transitive graphs with a prime number of vertices.

(2) It follows from Proposition 1.1 and Proposition 2.4 that there are no edges in any induced subgraph $\langle B_i \rangle$, the factor graph $\overline{X}$ is a triangle, and the factor group $A/K$ is isomorphic to $Z_3$.

(3) By the information preceding Proposition 2.4 we have that the Sylow $p$-subgroup $P$ of $K$ is cyclic and normal in $A$.

(4) We claim that $X$ is a $(3, p)$-metacirculant. Let $P = \langle \alpha \rangle$. Then $\alpha$ is a $(3, p)$-semiregular automorphism of $X$. Since $A/K \cong Z_3$, any element $\beta \in A \setminus K$ permutes $\overline{X} = \{B_0, B_1, B_2\}$ cyclically. Replacing it by its suitable power, we may assume that $\beta$ is a 3-element. Hence, by definition, $X$ is a $(3, p)$-metacirculant. Assume that $\alpha^{3\varepsilon} = \alpha^r$. Then $r$ is a 3-element in $Z_p \cong Z_{p-1}$.

Now we may label the vertices of $X$ as follows: for $i = 0, 1, 2$, let $B_i = \{x_{i0}, x_{i1}, \ldots, x_{i(p-1)}\}$, and we may assume that $x_j^{\alpha} = x_{j+1}^{\beta}$ and $x_j^{\beta} = x_{j+1}^{\alpha}$ for all $i$ and $j$. 

\[ \text{D} \]
(5) Finally, we claim that $X \cong M(d; 3, p)$ for a divisor $d > 1$ of $\frac{p-1}{3}$. Since there are no edges in $(B_i)$ for any $i$, $X$ has a symbol of the form $(r; \emptyset, S)$. Since $X$ has an odd number of vertices, the degree of $X$ is even, say $2d$. Fix a vertex $v = x_0^1$. The neighborhood of $v$ in $X$ is $X_1(v) = X_{B_1}^1(v) \cup X_{B_2}^1(v)$, where $X_{B_i}^1(v) = X_1(v) \cap B_i$.

Consider the stabilizer $A_v$ of $v$ in $A$. Since $A/K \cong Z_3$, $A_v$ fixes $B_i$ setwise for each $i$. So $A_v = K_v$. Since $K$ is solvable, $K$ has only one permutation representation of degree $p$, and $K$ is a Frobenius group or $K = P$. So $K_v$ must fix one vertex in $B_1$ and one vertex in $B_2$. Without loss of generality, we may assume that $K_v$ fixes $v_1 = x_1^1$ in $B_1$ and $v_2 = x_2^2$ in $B_2$. By the edge-transitivity of $X$, $A_v$ has two orbits in $X_1(v)$, which must be $X_{B_1}^1(v) = \{ x_1^j \mid j \in S \}$ and $X_{B_2}^1(v) = \{ x_2^j \mid j \in -r^2S \}$. Since $A_v = K_v$, the action of $A_v$ on $B_1$ is equivalent to the action of $K_v$ on $B_1$, and then to the action of $K_v$ on $B_1$. Since $K_{B_1}$ is a Frobenius group, the subscripts of the vertices in $X_{B_1}^1(v)$, which is an orbit of $K_{v_1}$, is a coset of a subgroup of $Z_{p-1}$ of order $d$, say $aT$, where $T \leq Z_{p-1}$, $[T] = d$ and $a \neq 0$. So we have proved $d$ is a divisor of $p - 1$. If $d = 1$, then $X$ has degree 2, contradicting the fact that $X$ is not arc-transitive. So $d > 1$. If $d$ does not divide $\frac{p-1}{3}$, then $r \in T$. Set $v: x_j^i \mapsto x_{j+1}^i$ for all $i$ and $j$. Then $v \in A$, and $\beta v$ maps $x_j^i$ to $x_j^{i+1}$. Thus $\beta v$ is an automorphism of $X$ of order 3 which commutes with $\alpha$. This implies that $\langle \alpha, \beta v \rangle$ is a regular subgroup of $A$ which is isomorphic to $Z_3$. By Proposition 1.5, $X$ is arc-transitive which is a contradiction. So we have $d | \frac{p-1}{3}$. Finally, noticing that two metacirculants with symbols $(r; \emptyset, T)$ and $(r; \emptyset, aT)$ are isomorphic, we have the desired result. \hfill \Box

3. $A$ has a block of length 3

The results of the previous section leave us with the case where $A$ has a block of length 3 and no blocks of length $p$. We assume that $\overline{X} = \{ B_i \mid i \in Z_p \}$ is a complete block system of $A$, and that $K$ is the kernel of the action of $A$ on $\overline{X}$. Set $\overline{A} = A/K$. We also use $\overline{X}$ to denote the corresponding block graph.

By Lemma 2.2, if $K$ acts on $B_i$ unfaithfully, then $X \cong \overline{X} \setminus 3K_1$, where $\overline{X}$ is an arc-transitive graph of order $p$. Thus, $X$ is also arc-transitive, so that we may assume that $K$ acts on each $B_i$ faithfully. Thus, we have $K \cong S_3$, or $K \cong Z_3$, or $K = 1$. We also know that there are no edges inside any $B_i$.

Lemma 3.1 The group $\overline{A}$ is insolvable, and the graph $\overline{X}$ is isomorphic to $K_p$.

Proof: If $\overline{A}$ is solvable, then $\overline{A}$ has a normal subgroup $\overline{H} = H/K$ of order $p$ since $\overline{A}$ is of degree $p$. Let $P \in \text{Syl}_p(H)$. Then it is easy to check that $P \triangleleft H$, and hence $P \triangleleft A$. So $A$ has a block of length $p$, contradicting our assumption.

Since $\overline{A}$ is insolvable and transitive of degree $p$, the well known theorem of Burnside implies that it is doubly transitive. Since there is no 1/2-transitive graphs of order 3, $\overline{X}$ is connected. Hence, $\overline{X}$ must be isomorphic to $K_p$. \hfill \Box

Lemma 3.2 The group $K = 1$. 


Proof: If \( K \neq 1 \), either \( K \cong S_3 \) or \( K \cong Z_3 \). Hence, \( K \) has a characteristic subgroup \( N \) of order 3 which is normal in \( A \). Put \( C = C_A(N) \). Then \( A/C \) is isomorphic to a subgroup of \( \text{Aut} K \cong Z_2 \). This implies that every element of order \( p \) in \( A \) is contained in \( C \). It follows that \( A \) has a subgroup isomorphic to \( Z_{3p} \), and this subgroup must be regular. Therefore, \( X \) is a Cayley graph on an abelian group. By Proposition 1.3, \( X \) is arc-transitive, which is a contradiction. \( \square \)

We may now assume that \( K = 1 \). In this case, \( A \cong \overline{A} \) as abstract groups. But as permutation groups, \( \overline{A} \) is a group of degree \( p \) and \( A \) is of degree \( 3p \). Since \( \overline{A} \) is insolvable, it is doubly-transitive as observed above. Then \( \overline{A} \) is known by the finite simple group classification.

If \( G \) is a doubly-transitive group of degree \( p \), one necessary condition for \( G \) to be the automorphism group of a 1/2-transitive graph of order \( 3p \) (as abstract groups) is that the point stabilizer \( G_a \) has a subgroup of index 3. A table of 2-transitive groups of degree \( p \) with simple socle is given in [7], and after checking all (insolvable) doubly-transitive groups of degree \( p \) listed there, the only possible groups have socle either \( PSL(3, 2) \), \( p = 7 \), or \( PSL(2, 2^s) \), where \( s > 0 \) and \( p = 2^{2s} + 1 \) is a Fermat prime.

There are no 1/2-transitive graphs with fewer than 27 vertices [1], so we only need to consider the latter case, where the socle is \( PSL(2, 2^s) \) and \( p = 2^{2s} + 1 \) is a Fermat prime. In this case, noting that \( |PGL(2, 2^s) : PSL(2, 2^s)| = 2^s \), the stabilizer of \( \overline{A} \) having a subgroup of index 3 implies that the stabilizer of \( PSL(2, 2^s) \) also has such a subgroup. This is true since \( 2^{2s} - 1 \) is divisible by 3. Hence \( PSL(2, 2^s) \) is vertex-transitive on \( X \).

Lemma 3.3 If \( PSL(2, 2^s) \leq A \leq PGL(2, 2^s) \), \( A \) is not the automorphism group of any 1/2-transitive graph.

Proof: As noted above \( PSL(2, 2^s) \) is vertex-transitive. Then since all orbitals are self-paired (see [7]) it follows that any edge-transitive graph \( X \) admitting the group is arc-transitive. \( \square \)

Summarizing the result of Section 2 and Section 3, we get the main theorem of this paper.

Theorem 3.4 A graph of order \( 3p \) is 1/2-transitive if and only if it is a \((3, p)\)-metacirculant graph of the form \( M(d; 3, p) \), where \( (d, p) \neq (2, 7) \) or \((3, 19)\).

References