MONOCHROMATIC SOLUTIONS OF EXPONENTIAL EQUATIONS

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Abstract
We show that for every 2-coloring of \( \mathbb{N} \) and every \( k \in \mathbb{N} \), there are infinitely many monochromatic solutions of the system of \( k^2 \) equations \( z_{ij} = x_i^y_j \), \( 1 \leq i, j \leq k \), where \( x_1, \ldots, x_k, y_1, \ldots, y_k \) are distinct positive integers greater than 1. We give similar, but somewhat weaker, results for more than two colors.

– Dedicated to the memory of Paul Erdős.

1. Introduction

Using ultrafilters and results from [3, 4], Alessandro Sisto [5] showed that every 2-coloring of \( \mathbb{N} \) gives infinitely many monochromatic sets of the form \( \{a, b, a^k\} \), where \( a, b > 1, a \neq b \), and he raised the question of whether there is an elementary proof of this fact.

We use van der Waerden’s Theorem on arithmetic progressions to give an elementary proof of a generalization of Sisto’s result. We show that for any 2-coloring of \( \mathbb{N} \) and any \( k \in \mathbb{N} \), there are infinitely many monochromatic sets of the form

\[
\{a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k\} \cup \{a_i^j : 1 \leq i, j \leq k\},
\]

where \( a_1, \ldots, a_k, e_1, \ldots, e_k \) are distinct positive integers greater than 1.

We also show that for any 3-coloring of \( \mathbb{N} \) and any \( k \in \mathbb{N} \), either there are monochromatic sets as just mentioned, or there are monochromatic sets of the form

\[
\{c^{a_1}, c^{a_2}, \ldots, c^{a_k}, e_1^{c^{a_1}}, e_2^{c^{a_2}}, \ldots, c^{a_k}\} \cup \{c^{a_i^j} : 1 \leq i, j \leq k\},
\]

where \( c \) is a power of 3.

Analogous results hold for more than 3 colors. For example, for any 4-coloring of \( \mathbb{N} \) and any \( k \in \mathbb{N} \), either there are monochromatic sets of one of the previous two types, or there are monochromatic sets of the form

\[
\{b^{a_1}, b^{a_2}, \ldots, b^{a_k}, e_1^{b^{a_1}}, e_2^{b^{a_2}}, \ldots, b^{a_k}\} \cup \{b^{a_i^j} : 1 \leq i, j \leq k\},
\]
where \( b, c \) are powers of 3.

In each case, \( a_1, \ldots, a_k, e_1, \ldots, e_k \) are distinct positive integers greater than 1.

A somewhat different result was proved (using non-elementary methods) by Beiglböck et al [1, 2]: For every finite coloring of \( \mathbb{N} \) and \( k \in \mathbb{N} \), there are \( a, b, d \in \mathbb{N} \) such that \( \{ b(a + id)^j : 0 \leq i, j \leq k \} \cup \{ bd^j : 0 \leq j \leq k \} \cup \{ a + id : 0 \leq i \leq k \} \) is monochromatic.

2. Two Colors

**Definition 1.** For \( k \in \mathbb{N} \), an *exponential \( k \)-set* is a set of the form

\[
\{a_1, a_2, \cdots, a_k, e_1, e_2 \cdots e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\},
\]

where \( a_1, \ldots, a_k, e_1, \ldots, e_k \) are distinct positive integers greater than 1.

Thus, an exponential \( k \)-set can be viewed as a non-trivial solution in \( \mathbb{N} \), with distinct \( x_1, \ldots, x_k, y_1, \ldots, y_k \), of the system of equations

\[
z_{ij} = x_i^{y_j}, 1 \leq i, j \leq k.
\]

**Theorem 1.** For every 2-coloring of \( \mathbb{N} \) and \( k \in \mathbb{N} \) there exists a monochromatic exponential \( k \)-set, that is, there exist distinct positive integers \( a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k \) all greater than 1, such that

\[
\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\}
\]

is monochromatic.

**Proof.** Let us first carry out the proof for \( k = 1 \), which illustrates, without the complications which will come later, the basic scheme of the proof.

Let \( f \) be a 2-coloring of \( \mathbb{N} \), using the colors 0 and 1. We seek a monochromatic set \( \{a_1, e_1, a_1^{e_1}\} \) where \( a_1, e_1 > 1 \), \( a_1 \neq e_1 \). We define

\[
g(x) = f(2^{3^x}), x \geq 1.
\]

By van der Waerden's Theorem on arithmetic progressions, there are \( p, d' \in \mathbb{N} \) with \( g \) constant on \( \{p, p + d', p + 2d', \ldots, p + 16d'\} \).

In particular, with \( d = 2d' \), \( g \) is constant on \( \{p, p + d, p + 2d, \ldots, p + 8d\} \), and \( d \geq 2 \). Thus,

\[
\{2^{3^{p+jd}} : 0 \leq j \leq 8\}
\]

is monochromatic with respect to \( f \), say with colour 0, and \( d \geq 2 \). There are now two cases to consider.
Case 1. There exists $x, 1 \leq x \leq 8$, with $f(3^{x_1}) = 0$. Set $a_1 = 2^{3^x}, e_1 = 3^{x_1}$. Then \{1, c_1, c_2\} is monochromatic, and $a_1 \neq e_1$.

Case 2. $f(3^{x_2}) = 1, 1 \leq x \leq 8$. If there exists $x, 2 \leq x \leq 8$, with $f(x) = 1$, then \{3, x, 3x\} is monochromatic and $3x \neq x$, since $x \leq 8 < 3^d$. If no such $x$ exists, then $f(x) = 0, 2 \leq x \leq 8$, and \{2, 3, 8\} is monochromatic.

Now we turn to the general proof for $k > 1$. Let $k$ be fixed, with $k > 1$.

Let $f$ be a 2-coloring of $\mathbb{N}$, using the colors 0 and 1, and define

$$g(x) = f(2^{3^x}), \quad x \geq 1.$$

We require $g$ to be constant on an arithmetic progression with $w+1$ terms, where $w$ is defined as follows.

**Definition 2.** The numbers $t_0, t_1, \ldots, t_{2k-1}$ are defined inductively by setting

$$t_0 = 1, \quad t_{q+1} = (t_q + k)^2(t_q + 2k), \quad 0 \leq q \leq 2k - 2.$$

Then we set

$$w = 2t_{2k-1}.$$

By van der Waerden’s Theorem there are $p, d' \in \mathbb{N}$ so that $g$ is constant on \{\ldots, p + d', p + 2d', \ldots, p + wd'\}, where $e$ is large enough that $3^e \geq w$. Then in particular, with $d = ed'$, $g$ is constant on \{\ldots, p + d, p + 2d, \ldots, p + wd\}, where $3^d \geq 3^e \geq w$. (The inequality $3^d \geq w$ will be used below only in “Subcase 2a.”)

Hence,

$$\{2^{3^p}, 2^{3^{p+d}}, 2^{3^{p+2d}}, \ldots, 2^{3^{p+wd}}\}$$

is monochromatic with respect to $f$, say with color 0, and $3^d \geq w$.

Let

$$T = \{j \in [1, w/2] : f(3^{j_1}) = 0\}.$$

There are now two cases to consider.

**Case 1.** $|T| \geq k$. Let $x_1, \ldots, x_k \in T$. Then $x_j \leq w/2, 1 \leq j \leq k$, and

$$f(3^{x_{j_1}}) = 0, \quad 1 \leq j \leq k.$$

In this case, we take

$$a_i = 2^{3^{x_{i_1}}}d, \quad 1 \leq i \leq k,$$

$$e_j = 3^{x_{j_1}d}, \quad 1 \leq j \leq k.$$

Then

$$a_i e_j = 2^{3^{x_{i_1}+x_{j_1}}}d, \quad \text{and} \quad x_i + x_j \leq 2(w/2) = w,$$
hence
\[
\{a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\}
\]
is monochromatic, with color 0, and clearly \(a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k\) are distinct and greater than 1.

**Case 2.** \(|T| < k\). Thus,
\[
f(3^{md}) = 1, \ x \in [1, w/2] - T, \ \text{and} \ |T| \leq k - 1.
\]

**Subcase 2a.** There exist \(y_1, y_2, \ldots, y_k\), with the following two properties:
\[
\{y_1, y_2, \ldots, y_k\} \cup \{y_i y_j : 1 \leq i, j \leq k\} \subset [1, w/2] - T
\]
and
\[
f(y_i) = 1, \ 1 \leq i \leq k.
\]

Then we have \(f(3^{md}) = 1\) whenever \(x = y_i\) or \(x = y_i y_j\), \(1 \leq i, j \leq k\), and \(f(y_i) = 1, 1 \leq i \leq k\). Hence \(f\) is constant, with color 1, on the set
\[
\{3^{y_1 d}, 3^{y_2 d}, \ldots, 3^{y_k d}, y_1, y_2, \ldots, y_k\} \cup \{(3^{y_i d})^{e_j} : 1 \leq i, j \leq k\}.
\]
We may assume \(y_1 < y_2 < \cdots < y_k\), so that \(3^{y_1 d} < 3^{y_2 d} < \cdots < 3^{y_k d}\). To show that \(3^{y_1 d}, 3^{y_2 d}, \ldots, 3^{y_k d}, y_1, y_2, \ldots, y_k\) are distinct, we simply note that \(y_k < w \leq 3^d \leq 3^{y_k d}\), and hence
\[
y_1 < y_2 < \cdots < y_k < 3^{y_1 d} < 3^{y_2 d} < \cdots < 3^{y_k d}.
\]

**Subcase 2b.** There do not exist numbers \(y_1, y_2, \ldots, y_k\) as in Subcase 2a. This means that for any
\[
\{y_1, y_2, \ldots, y_k\} \cup \{y_i y_j : 1 \leq i, j \leq k\} \subset [1, w/2] - T
\]
there is at least one \(i, 1 \leq i \leq k\), such that \(f(y_i) = 0\).

Now we make explicit use of the numbers \(t_0, \ldots, t_{2k-1}\) defined above.

**Definition.** The sets \(A_q \subset B_q, 1 \leq q \leq 2k-1\), are defined inductively by setting
\[
A_1 = [t_0 + 1, (t_0 + k)^{t_0 + 2k}], \ B_1 = [t_0 + 1, (t_0 + k)^{2(t_0 + 2k)}] = [t_0 + 1, t_1],
\]
\[
A_{q+1} = [t_q + 1, (t_q + k)^{t_q + 2k}], \ B_{q+1} = [t_q + 1, (t_q + k)^{2(t_q + 2k)}] = [t_q + 1, t_{q+1}].
\]
Note that for each $q$, $0 \leq q \leq 2k - 2$, we can write

$$A_q = [t_q + 1, t_q + 2k] \cup [t_q + 2k + 1, (t_q + k)t_q + 2k],$$

so that if we take

$$[t_q + 1, t_q + 2k] = [a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k],$$

then $A_q$ contains the exponential $k$-set

$$\{a_1, \ldots, a_k, e_1, \ldots, e_k\} \cup \{a_i^{e_j} : 1 \leq i,j \leq k\}.$$  

Also, recall that $\max B_{2k-1} = t_{2k-1} = w/2$, hence

$$B_1 \cup \cdots \cup B_{2k-1} \subset [1, w/2].$$

We shall show that one of the sets $A_i$ is monochromatic under $f$, with color 0. We have $2k - 1$ pairwise disjoint subsets $B_1, \ldots, B_{2k-1}$ of $[1, w/2]$, and, since $|T| < k$, at most $k - 1$ of them can meet $T$. Hence, there are $k$ sets $B_1, \ldots, B_k$ (we assume that $i_1 < \cdots < i_k$) such that

$$B_{i_1} \cup \cdots \cup B_{i_k} \subset [1, w/2] - T.$$

(In Lemma 2 below, we consider the corresponding union $A_{i_1} \cup \cdots \cup A_{i_k}.$)

**Lemma 1.** If $i \leq j$ and $y \in A_i$, $z \in A_j$, then $yz \in B_j$.

**Proof.** From the definitions of $A_j, B_j$, we can simplify the notation to write $A_j = [a, b], B_j = [a, b^2]$. Then $y \in A_i, z \in A_j$ implies $2 \leq y \leq b$ and $a \leq z \leq b$, therefore $a \leq z < yz \leq b^2$, hence $yz \in B_j$.  

**Lemma 2.** Let $S = \{y \in A_{i_1} \cup \cdots \cup A_{i_k} : f(y) = 1\}$, and assume that $B_{i_1} \cup \cdots \cup B_{i_k} \subset [1, w/2] - T$. Then $|S| < k$.

**Proof.** Suppose $|S| \geq k$. Let $y_1, \ldots, y_k \in S \subset A_{i_1} \cup \cdots \cup A_{i_k}$, then by Lemma 2.1,

$$\{y_1, y_2, \ldots, y_k\} \cup \{y_i y_j : 1 \leq i, j \leq k\} \subset B_{i_1} \cup \cdots \cup B_{i_k} \subset [1, w/2] - T.$$  

But since we are in Subcase 2b, this immediately implies that $f(y_i) = 0$ for some $i$, a contradiction.

Thus, we now have $|S| < k$ and

if $y \in A_{i_1} \cup \cdots \cup A_{i_k} - S$ then $f(y) = 0$.

Since $S$ can meet at most $k - 1$ of the intervals $A_{i_1}, \cdots, A_{i_k}$, there is some $q, 1 \leq q \leq k$, such that

$$f(y) = 0, \, y \in A_{i_q}.$$  

Since $A_{i_q}$ contains an exponential $k$-set, this finishes the proof of Theorem 1.  

Corollary 1. Given $A, k \in \mathbb{N}$, there exists $M(A, k) \in \mathbb{N}, M(A, k) > A$, such that for every 2-coloring of $[A, M(A, k)]$ there exists a monochromatic exponential $k$-set.

Proof. The exponential $(A + k)$-set

$$\{a_1, \ldots, a_{A+k}, e_1, \ldots, e_{A+k}\} \cup \{a_i^{e_j} : 1 \leq i, j \leq A + k\},$$

where we can assume that $a_1 < \cdots < a_{A+k}$ and $e_1 < \cdots < e_{A+k}$, contains the exponential $k$-set

$$\{a_{A+1}, \ldots, a_{A+k}, e_{A+1}, \ldots, e_{A+k}\} \cup \{a_i^{e_j} : A + 1 \leq i, j \leq A + k\},$$

which is contained in $[A, \infty)$.

Thus, given any 2-coloring $f$ of $[A, \infty)$, extend $f$ arbitrarily to a 2-coloring of $\mathbb{N}$. By Theorem 1, there exists a monochromatic exponential $(A + k)$-set, which contains an exponential $k$-set in $[A, \infty)$. By compactness, the result follows. \qed

Corollary 1 is the basis of the proofs for the results involving more than 2 colors.

Corollary 2. For every 2-coloring of $\mathbb{N}$ and $k \in \mathbb{N}$ there exist infinitely many monochromatic exponential $k$-sets.

Proof. This follows immediately from Corollary 1. \qed

3. Three and Four Colors

Theorem 2. For every 3-coloring of $\mathbb{N}$ and $k \in \mathbb{N}$ there exist distinct $a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k$, all greater than 1, and $c = 3^d > 1$ such that

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k \}$$

or

$$\{c^{a_1}, c^{a_2}, \cdots, c^{a_k}, c^{e_1}, c^{e_2}, \cdots, c^{e_k}\} \cup \{c^{a_i^{e_j}} : 1 \leq i, j \leq k \}$$

is monochromatic.

Proof. Let $k \in \mathbb{N}$ and let $f$ be a 3-coloring of $\mathbb{N}$, using the colors 0, 1, 2. Using the notation of Corollary 1, define $W_q, 1 \leq q \leq k$ by setting

$$W_1 = M(1,k), W_{q+1} = M(W_q,k), 1 \leq q \leq k - 1.$$

We follow closely the first part of the proof of Theorem 1.

By van der Waerden’s Theorem, there are $p, d \in \mathbb{N}$ so that

$$\{2^{3^p}, 2^{3^p+d}, 2^{3^p+2d}, \ldots, 2^{3^p+2W_kd}\}$$
is monochromatic with respect to $f$, say with color 0.

Let

$$T = \{ j \in [1, W_k) : f(3^j) = 0 \},$$

so that

$$f(3^j) \in \{1, 2\}, \forall j \in [1, W_k) - T.$$

There are now two cases to consider.

**Case 1.** $|T| \geq k$. We proceed exactly as in Case 1 of the proof of Theorem 1, to obtain a monochromatic set of colour 0

$$\{a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k\} \cup \{a_i^e : 1 \leq i, j \leq k \}.$$

**Case 2.** $|T| < k$. We make use of Corollary 1. Consider the intervals

$$[1, W_1), [W_1, W_2), \ldots, [W_{k-1}, W_k).$$

The set $T$ can meet at most $k - 1$ of these intervals, so for some $q$ we have

$$[W_q, W_{q+1}) \subset [1, W_k] - T.$$

Thus $g(j) = f(3^j), j \in [W_q, W_{q+1})$ is a 2-coloring of $[W_q, M(W_q, k))$ and by the definition of $M(W_q, k)$ there is an exponential $k$-set

$$\{a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k\} \cup \{a_i^e : 1 \leq i, j \leq k \}$$

which is monochromatic with respect to $g$.

Hence, with $c = 3^d$, we have $g(a_1) = f(c^{a_1}), g(a_2) = f(c^{a_2}), \ldots$, so that

$$\{c^{a_1}, c^{a_2}, \ldots, c^{a_k}, c^{e_1}, c^{e_2}, \ldots, c^{e_k}\} \cup \{c^{a_i^e} : 1 \leq i, j \leq k \}$$

is monochromatic with respect to $f$. \hfill \Box

**Corollary 3.** Given $A, k \in \mathbb{N}$, there exists $M(A, k) \in \mathbb{N}$ such that for every 3-coloring of $[A, M(A, k))$ there are distinct $a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k$, all greater than 1, and $c = 3^d, d \in \mathbb{N}$, such that

$$\{a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k\} \cup \{a_i^e : 1 \leq i, j \leq k \}$$

or

$$\{c^{a_1}, c^{a_2}, \ldots, c^{a_k}, c^{e_1}, c^{e_2}, \ldots, c^{e_k}\} \cup \{c^{a_i^e} : 1 \leq i, j \leq k \}$$

is monochromatic.

**Proof.** The proof is exactly the same as the proof of Corollary 1. \hfill \Box
Theorem 3. For every 4-coloring of \( \mathbb{N} \) and \( k \in \mathbb{N} \) there exist distinct \( a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k \), all greater than 1, \( c = 3^d, b = 3^d, d, d' \in \mathbb{N} \) such that
\[
\{a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\}
\]
or
\[
\{c_1^{a_1}, c_1^{a_2}, \ldots, c_1^{a_k}, c_2^{e_1}, c_2^{e_2}, \ldots, c_2^{e_k}\} \cup \{c_1^{a_i^{e_j}} : 1 \leq i, j \leq k\}
\]
or
\[
\{b_1^{e_1}, b_1^{e_2}, \ldots, b_1^{e_k}, b_2^{e_1}, b_2^{e_2}, \ldots, b_2^{e_k}\} \cup \{b_1^{e_i^{e_j}} : 1 \leq i, j \leq k\}
\]
is monochromatic.

Proof. The proof is virtually the same as the proof of Theorem 2, here using Corollary 3 instead of Corollary 1.

\[\square\]

4. The General Case

Theorem 4. Let \( r, k \in \mathbb{N} \) and let an \( r \)-coloring of \( \mathbb{N} \) be given. If \( r = 2 \) there exist infinitely many monochromatic exponential \( k \)-sets. If \( r > 2 \), either there are infinitely many monochromatic exponential \( k \)-sets, or there exist \( 1 \leq s \leq r - 2 \), and infinitely many monochromatic sets of the form
\[
\{c_1^{a_1}, c_1^{a_2}, \ldots, c_1^{a_k}, c_2^{e_1}, c_2^{e_2}, \ldots, c_2^{e_k}\} \cup \{c_1^{a_i^{e_j}} : 1 \leq i, j \leq k\},
\]
where \( a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k \) are distinct positive integers greater than 1, and the \( c_i, 1 \leq i \leq s \), are (not necessarily distinct) powers of 3.

In fact, given \( A, r \in \mathbb{N} \), there is \( M(A, k, r) \) such that, for every \( r \)-coloring of the interval \( [A, M(A, k, r)] \), there exists a monochromatic set of one of these types.

Proof. The proof is by induction on \( r \), following the methods of the proofs of Theorem 2 and Corollary 1.

We conclude this paper by proposing the following questions.

Questions. Does every \( r \)-coloring of \( \mathbb{N} \) give a monochromatic set \( \{a, b, a^b\} \), with \( a, b > 1, a \neq b \)? Does every 2-coloring of \( \mathbb{N} \) give a monochromatic solution of
\[
w = ax^3?
\]

Let \( h(k) \) denote the smallest \( n \) such that for any 2-coloring of \([1, n]\) there exists a monochromatic exponential \( k \)-set. Let \( W(k) \) denote the smallest \( n \) such that for
any 2-coloring of $[1,n]$ there exists a monochromatic $k$-term arithmetic progression. The proof of Theorem 2.1 shows that $h(1) \leq 2^{3W(16)}$ and that $h(2)$ is bounded above, roughly speaking, by $2^{3W(s-3^s)}$, where $s = 40 \cdot 3^{20 \cdot 3^{10}}$. Perhaps these bounds can be decreased a bit.

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**References**


