GENERATING $d$-COMPOSITE SANDWICH NUMBERS

Lenny Jones  
Department of Mathematics, Shippensburg University, Shippensburg, Pennsylvania  
lkjone@ship.edu

Alicia Lamarche  
Department of Mathematics, Shippensburg University, Shippensburg, Pennsylvania  
al5903@ship.edu

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Abstract  
Let $d \in \mathcal{D} = \{1, \ldots, 9\}$, and let $k$ be a positive integer with $\gcd(k, 10d) = 1$. Define a sequence $\{s_n(k, d)\}_{n=1}^{\infty}$ by

$$s_n(k, d) := k \underbrace{d \ldots dk}_n.$$

We say $k$ is a $d$-composite sandwich number if $s_n(k, d)$ is composite for all $n \geq 1$. For a $d$-composite sandwich number $k$, we say $k$ is trivial if $s_n(k, d)$ is divisible by the same prime for all $n \geq 1$, and nontrivial otherwise. In this paper, we develop a simple criterion to determine when a $d$-composite sandwich number is nontrivial, and we use it to establish many results concerning which types of integers can be $d$-composite sandwich numbers. For example, we prove that there exist infinitely many primes that are simultaneously trivial $d$-composite sandwich numbers for all $d \in \mathcal{D}$. We also show that there exist infinitely many positive integers that are simultaneously nontrivial $d$-composite sandwich numbers for all $d \in \mathcal{D}$, where $D \subset \mathcal{D}$ with $|D| = 4$ and $D \neq \{3, 6, 7, 9\}$.

1. Introduction  
In [22], the first author proved that for any given fixed digit $d \in \{1, 3, 5, 7\}$, there exist infinitely many positive integers $k$, such that $\gcd(k, d) = 1$ and every integer in the sequence

$$kd, \ kdd, \ kddd, \ kdddd, \ kddddd, \ldots,$$

is composite. Other authors [16, 23, 24] have handled modifications of this appending-digits problem. An inserting-digits problem was treated in [13]. In this paper, we
investigate the following variation. Let \( d \in D = \{1, \ldots, 9\} \), and let \( k \) be a positive integer with \( \gcd(k, 10d) = 1 \). Define a sequence \( \{s_n(k, d)\}_{n=1}^{\infty} \) by

\[
s_n(k, d) := k dd \ldots dk.\tag{1.1}
\]

We say \( k \) is a d-composite sandwich number if \( s_n(k, d) \) is composite for all \( n \geq 1 \). If there exists a prime \( p \) such that \( s_n(k, d) \equiv 0 \pmod{p} \) for all \( n \geq 1 \), we say \( k \) is trivial, otherwise we say \( k \) is nontrivial. The restriction that \( \gcd(k, 10d) = 1 \) has been imposed to discard obvious trivial situations. If \( d = 0 \), we observe that \( \gcd(k, 10d) = k \), so that \( d = 0 \) is ruled out except possibly in the case of \( k = 1 \). However, since \( s_1(1, 0) = 101 \) is prime, we see that \( k = 1 \) is not a 0-composite sandwich number, and we can exclude the digit \( d = 0 \) from all consideration. Therefore, we assume throughout this article that \( d \in D \) and \( k \) is a positive integer with \( \gcd(k, 10d) = 1 \).

We prove results concerning what types of integers can be d-composite sandwich numbers, and we consider both trivial and nontrivial situations. In particular, we prove the following.

**Theorem 1.1.** For any \( d \in D \), there exist infinitely many primes that are trivial d-composite sandwich numbers.

**Theorem 1.2.** There exist infinitely many primes \( q \) that are simultaneously trivial d-composite sandwich numbers for all \( d \in D \).

**Theorem 1.3.** For any \( d \in D \), there exist infinitely many nontrivial d-composite sandwich numbers.

**Theorem 1.4.** There exist infinitely many positive integers that are simultaneously nontrivial d-composite sandwich numbers for all \( d \in D \), where \( D \subset D \) with \( |D| = 4 \) and \( D \neq \{3, 6, 7, 9\} \).

**Theorem 1.5.** For any \( d \in D \), there are infinitely many sets of 13 consecutive positive integers that are all d-composite sandwich numbers.

**Theorem 1.6.** For any \( d \in D \), there are infinitely many positive integers \( k \) such that \( k^2 \) is simultaneously a d-composite sandwich number, a Sierpiński number and a Riesel number.

**Remark 1.7.** Although Theorem 1.1 follows directly from Theorem 1.2, we nevertheless treat it independently since we find the smallest trivial d-composite sandwich number for each individual digit \( d \in D \).

2. Preliminaries

This section contains some basic definitions and concepts that are useful in this paper. Other preliminary concepts that are needed for the proof of only one theorem are presented in the appropriate section.
Definition 2.1. Let \( a > 1 \) be an integer. A prime divisor \( p \) of \( a^n - 1 \) is called a primitive divisor of \( a^n - 1 \) if \( a^m \not\equiv 1 \pmod{p} \) for all positive integers \( m < n \).

The following theorem concerning the existence of primitive divisors is due to Bang [1].

Theorem 2.2. Let \( a \) and \( n \) be positive integers with \( a \geq 2 \). Then \( a^n - 1 \) has a primitive divisor with the following exceptions:

- \( a = 2 \) and \( n = 6 \)
- \( a + 1 \) is a power of 2 and \( n = 2 \).

In terms of group theory, the prime \( p \) is a primitive divisor of \( a^n - 1 \) if and only if \( n \) is the order of \( a \) in the group of units modulo \( p \). We denote this order as \( \text{ord}_p(a) \).

The following concept, which is due to Erdős [9], plays an essential role in the proofs of many of our results.

Definition 2.3. A (finite) covering system, or simply a covering, of the integers is a system of congruences \( x \equiv a_i \pmod{m_i} \), with \( 1 \leq i \leq t \) such that every integer \( n \) satisfies at least one of the congruences. To avoid a trivial situation, we require \( m_i > 1 \) for all \( i \). We let \( \mathcal{M} = \text{lcm}(m_i) \) for all moduli \( m_i \) in a covering.

Many applications of coverings require an associated set of primes, where each of these primes corresponds in some way to a particular modulus in the covering. It will be convenient throughout this article to represent a covering and the associated set of primes using a set \( \mathcal{C} \) of ordered triples \( (n_i, m_i, p_i) \) (or simply ordered pairs \( (a_i, m_i) \)) if the primes \( p_i \) are too large to display conveniently, where \( x \equiv a_i \pmod{m_i} \) is a congruence in the covering and \( p_i \) is the corresponding prime. When a covering is used for a proof of a theorem in this article, the correspondence between the prime \( p_i \) and the modulus \( m_i \) is that \( p_i \) is either a primitive divisor of \( 10^{m_i} - 1 \) (with the exception of \( p_i = 3 \)), or that \( p_i \) is a primitive divisor of \( 2^{m_i} - 1 \) for all \( i \) in the covering. For certain moduli \( m_i \), the numbers \( 10^{m_i} - 1 \) and \( 2^{m_i} - 1 \) have more than one primitive divisor. In those cases, the corresponding modulus \( m_i \) can be used repeatedly in the covering – once for each primitive divisor. Abusing notation slightly, we refer to \( \mathcal{C} \) as a “covering”.

For a positive integer \( k \), we let \( \ell(k) \) denote the number of digits in the decimal representation of \( k \). In this paper, we are concerned with the sequence \( \{s_n(k, d)\}_{n=1}^{\infty} \) defined in (1.1). It will be convenient to use the following easily-derived formula for \( s_n(k, d) \):

\[
s_n(k, d) = k \left( 10^{\ell(k)+n} + 1 \right) + d \cdot 10^{\ell(k)} \left( \frac{10^n - 1}{9} \right). \tag{2.1}
\]

Computer computations in this paper were performed by the authors using Maple and MAGMA.
3. Trivial Situations

Recall that $k$ is a trivial $d$-composite sandwich number if there exists a prime $p$ such that $s_n(k,d) \equiv 0 \pmod{p}$ for all $n \geq 1$. It may not be immediately apparent that such numbers even exist, but in fact, we shall see that they are quite abundant. In this section we determine necessary and sufficient conditions on $k$, such that $k$ is a trivial $d$-composite sandwich number.

**Theorem 3.1.** Let $d \in \mathcal{D}$ and let $k \geq 1$ be an integer such that $\gcd(k,10d) = 1$. Then $k$ is a trivial $d$-composite sandwich number if and only if $\gcd(9k+d,10^{f(k)}+1) > 1$.

*Proof.* Assume first that $k$ is a trivial $d$-composite sandwich number. Then there exists a prime $p$ such that $s_n(k,d) \equiv 0 \pmod{p}$ for all $n \geq 1$. Note that if $p = 3$, then

$$s_n(k,d) = k \left(10^n + 10^{f(k)} + 1\right) + d \left(10^{n-1} + 10^{n-2} + \cdots + 1\right) 10^{f(k)} \equiv 2k+nd \pmod{3}.$$ 

Hence, $s_n(k,d) \equiv 0 \pmod{3}$ for all $n \geq 1$ if and only if $\gcd(k,d) \equiv 0 \pmod{3}$, which we have excluded from consideration here. Thus, $p \geq 7$, since the condition $\gcd(k,10d) = 1$ also excludes the possibility that $p = 2$ or $p = 5$. Since $s_n(k,d) \equiv 0 \pmod{p}$, we have that

$$0 \equiv 9 \cdot s_n(k,d) \equiv A \cdot 10^n + B \pmod{p},$$

where

$$A \equiv (9k+d)10^{f(k)} \pmod{p} \quad \text{and} \quad B \equiv 9k - d \cdot 10^{f(k)} \pmod{p}. \quad (3.1)$$

Since $s_1(k,d) \equiv s_2(k,d) \equiv 0 \pmod{p}$, we deduce that $90A \equiv 0 \pmod{p}$. Since $p \geq 7$, it follows that $A \equiv B \equiv 0 \pmod{p}$, and so $9k+d \equiv 0 \pmod{p}$. Also, solving the second congruence in (3.1) for $9k$ and substituting into the first congruence in (3.1) gives

$$d \cdot 10^{f(k)} \left(10^{f(k)} + 1\right) \equiv 0 \pmod{p},$$

which implies that $10^{f(k)} + 1 \equiv 0 \pmod{p}$, unless $p = d = 7$. But in this case, since $A \equiv 0 \pmod{7}$, we have that $k \equiv 0 \pmod{7}$, which we have excluded. Thus, $\gcd(9k+d,10^{f(k)}+1) \equiv 0 \pmod{p}$.

Conversely, suppose that $p$ is a prime such that $\gcd(9k+d,10^{f(k)}+1) \equiv 0 \pmod{p}$. Then, in (3.1), we have $A \equiv 0 \pmod{p}$, and since $10^{f(k)} \equiv -1 \pmod{p}$, we also get that

$$B \equiv 9k+d \equiv 0 \pmod{p}.$$ 

Therefore,

$$9 \cdot s_n(k,d) \equiv A \cdot 10^n + B \equiv 0 \pmod{p},$$
and since \( p \geq 7 \), we conclude that \( s_n(k, d) \equiv 0 \pmod{p} \) for all \( n \geq 1 \), and the proof is complete.

**Example 3.2.** Let \( k = 260487394697203 \) and \( d = 2 \). Then \( \ell(k) = 15 \), \( \gcd(k, 10d) = 1 \) and

\[
\gcd(9 \cdot 260487394697203 + 2, 10^{15} + 1) = 211,
\]

so that \( s_n(k, 2) \equiv 0 \pmod{211} \) for all \( n \geq 1 \). Hence, \( k \) is a trivial 2-composite sandwich number.

**Remark 3.3.** We caution the reader that if \( \gcd(9k + d, 10^{\ell(k)} + 1) = 1 \), then it does not follow that \( k \) is a nontrivial \( d \)-composite sandwich number, since not all integers are composite sandwich numbers. However, this condition can be used to detect if a known \( d \)-composite sandwich number is trivial or not. We apply this condition in Section 4.

### 3.1. Theorem 1.1

To prove Theorem 1.1, we first need some preliminary results. The first result, which we state without proof, is a well-known version of the prime number theorem for arithmetic progressions [27]. We use the standard Landau little-o notation.

**Theorem 3.4.** Let \( \gcd(a, m) = 1 \) and let \( \pi(x; m, a) \) be the number of primes \( p \leq x \) such that \( p \equiv a \pmod{m} \). Then

\[
\pi(x; m, a) = \frac{x(1 + o(1))}{\phi(m) \log x}.
\]

The following corollary is immediate from Theorem 3.4.

**Corollary 3.5.** Let \( f(z) > 1 \) and \( g(z) > 1 \) be strictly increasing functions with \( f(z) < g(z) \) for all sufficiently large \( z \). Then the number of primes \( p \) with \( f(z) < p \leq g(z) \) such that \( p \equiv a \pmod{m} \) is

\[
\pi(g(z); m, a) - \pi(f(z); m, a) = \frac{1 + o(1)}{\phi(m)} \left( \frac{g(z)}{\log g(z)} - \frac{f(z)}{\log f(z)} \right).
\]

We need the following lemma.

**Lemma 3.6.** Let \( x \) be a positive integer and let \( p \) be a prime such that \( 10^x + 1 \equiv 0 \pmod{p} \). If \( y \) is an integer with \( 1 \leq y \leq p - 1 \), then there exist nonnegative integers \( N \) and \( z \) such that \( k = pN + y \) is odd with \( \ell(k) = 2z + 1 \).

**Proof.** Note that \( \ell(p) \leq x + 1 \). Let \( z \geq p \) and let \( w = (2z + 1)x - \ell(p) \). Then \( w > \ell(p) \). If \( y \) is odd, let \( N = 10^w \), and if \( y \) is even, let \( N = 10^w + 1 \). Let \( k = pN + y \). Then \( k \) is odd and

\[
\ell(k) = \ell(pN + y) = \ell(p) + w = (2z + 1)x.
\]
3.1.1. Proof of Theorem 1.1

Let \( p \geq 7 \) be a prime such that \( \text{ord}_p(10) \equiv 0 \pmod{2} \). For the sake of brevity of notation, we let \( x = \text{ord}_p(10)/2 \). Then

\[
10^x + 1 \equiv 0 \pmod{p}.
\]

Let \( y \leq p - 1 \) be a positive integer such that \( y \equiv -d/9 \pmod{p} \). By Lemma 3.6, there exists a nonnegative integer \( z \) and an odd positive integer \( k \) such that \( k \equiv y \pmod{p} \) and \( \ell(k) = (2z + 1)x \). Apply Corollary 3.5 with \( a = y \), \( m = p \), \( f(z) = 10^{(2z+1)x-1} \), and \( g(z) = 10^{(2z+1)x} \) to get that

\[
\lim_{z \to \infty} (\pi(g(z); p, y) - \pi(f(z); p, y)) = \infty.
\]

Thus, for any sufficiently large integer \( z \), there is a prime \( q \) such that \( q \equiv y \equiv -d/9 \pmod{p} \) and \( \ell(q) = (2z + 1)x \). Then \( \gcd(9q + d, 10^{\ell(q)} + 1) \equiv 0 \pmod{p} \), and hence, by Theorem 3.1, it follows that \( q \) is a trivial \( d \)-composite sandwich number, and the proof of Theorem 1.1 is complete.

For each \( d \in \mathcal{D} \), we use Theorem 3.1 to give in Table 1 the smallest prime \( q \) that is a trivial \( d \)-composite number with \( \gcd(q, 10d) = 1 \). The primes \( p \) such that \( s_n(q,d) \equiv 0 \pmod{p} \) for all \( n \geq 1 \) are also given.

<table>
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<th>( q )</th>
<th>( p )</th>
</tr>
</thead>
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<td>101</td>
<td>7, 13</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>101</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>101</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>107</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>109</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>89</td>
<td>101</td>
</tr>
<tr>
<td>8</td>
<td>101</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>103</td>
<td>13</td>
</tr>
</tbody>
</table>

Table 1: Smallest Trivial \( d \)-Composite Sandwich Primes \( q \) with \( \gcd(q, 10d) = 1 \)

3.2. Proof of Theorem 1.2

Suppose that \( m \) is a positive integer such that \( 10^m + 1 \) has at least 9 distinct prime factors \( p_1, p_2, \ldots, p_9 \). Using the Chinese remainder theorem, we solve the system of congruences

\[
k \equiv -d/9 \pmod{p_\alpha}.
\]

Thus, for sufficiently large such \( m \), by Lemma 3.6 and Corollary 3.5 (as in the proof of Theorem 1.1), there is a prime \( q \) in the resulting arithmetic progression such that
\( \ell(q) = m. \) Since \( \gcd(q, 10d) = 1 \) and \( \gcd(9q + d, 10^{\ell(q)} + 1) = p_d, \) it follows from Theorem 3.1 that \( q \) is a trivial \( d \)-composite sandwich prime for all \( d \in D. \)

We illustrate the techniques with an example.

**Example 3.7.** The smallest value of \( m \) such that \( 10^m + 1 \) has at least 9 distinct prime factors is \( m = 39. \) These distinct primes are

\[
P = [1058313049, 859, 7, 6397, 157, 388847808493, 13, 11, 216451].
\]

Using the Chinese remainder theorem, we solve the system of congruences

\[
k \equiv -d/9 \pmod{P[d]}, \quad d \in D,
\]

to get

\[
k \equiv 23095145832174487336140994425364380822 \pmod{\prod_d P[d]}.
\]

The smallest prime in this arithmetic progression with 39 digits is

\[
q = 100018222755251410413064071348441303899.
\]

Since \( \gcd(q, 10d) = 1 \) and \( \gcd(9q + d, 10^{\ell(q)} + 1) = P[d], \) it follows from Theorem 3.1 that \( q \) is a trivial \( d \)-composite sandwich number for all \( d \in D. \)

**Remark 3.8.** The particular order of the primes in the list (3.2) gives the smallest prime \( q \) that is simultaneously a trivial \( d \)-composite sandwich number for all \( d \in D. \)

### 4. Nontrivial Situations

Recall that nontrivial \( d \)-composite sandwich numbers are positive integers \( k \) such \( s_n(k, d) \) is composite for all \( n \geq 1, \) but not every term \( s_n(k, d) \) is divisible by the same prime. We use a covering to find such numbers. Given \( d \in D, \) we want to construct a covering \( C = C(d), \) such that for each triple \((a_i, m_i, p_i) \in C, \) we have that \( s_n(k, d) \equiv 0 \pmod{p_i} \) whenever \( n \equiv a_i \pmod{m_i}. \) Here the primes \( p_i, \) with the exception of \( p_i = 3, \) are chosen to be primitive divisors of \( 10^{m_i} - 1 \) (but we still assume that 3 is a divisor of \( (10^{m_i} - 1)/9 \) if \( p_i = 3). \) To find a number \( k \) that satisfies these conditions, we first set \( s_n(k, d) \) congruent to 0 modulo \( p_i \) and solve (2.1) for \( k \) modulo \( p_i \) for each \( i, \) noting that \( 10^n \equiv 10^{m_i} \pmod{p_i}. \) Then we use the Chinese remainder theorem to piece together this information and get an infinite arithmetic progression of such values of \( k \) that simultaneously solve all congruences \( s_n(k, d) \equiv 0 \pmod{p_i}. \) Two additional congruences

\[
k \equiv 1 \pmod{2} \quad \text{and} \quad k \equiv z \pmod{5}, \quad \text{where} \quad z \neq 0 \pmod{5}, \quad (4.1)
\]
can be added to the system, if necessary, to ensure that \( \gcd(k, 10) = 1 \). Note that these two new congruences do not conflict with any congruences in \( \mathcal{C} \) since 2 and 5 are never primitive divisors of numbers of the form \( 10^x - 1 \). Also, it is usually easy to guarantee that \( \gcd(k, d) = 1 \) by adding additional congruences for \( k \) if necessary, or by choosing appropriate residues associated with the primes 3 and 7, if they appear in \( \mathcal{C} \). Once \( k \) is found, we can easily check that \( \gcd(k, 10d) = 1 \), and then use Theorem 3.1 to determine if \( k \) is nontrivial.

Certain criteria must be satisfied to accomplish this task. To begin, we set \( s_n(k, d) = 0 \) and formally solve for \( k \) in (2.1) to get

\[
k = \frac{d \cdot 10^{\ell(k)} (10^n - 1)}{9 \cdot (10^{\ell(k)+n} + 1)}.
\]  

(4.2)

However, modulo \( p_i \) there are two difficulties here: (4.2) makes no sense if \( p_i = 3 \) or if \( 10^{\ell(k)+n} + 1 \equiv 0 \pmod{p_i} \).

The first difficulty is easily overcome with a modest price. If \( p_i = 3 \), then we can expand \( (10^n - 1)/9 \) to get

\[
\frac{10^n - 1}{9} = 10^{n-1} + 10^{n-2} + \cdots + 10 + 1 \equiv n \pmod{3}.
\]

We would like to replace \( n \) here with \( a_i \) since \( n \equiv a_i \pmod{m_i} \) in \( \mathcal{C} \). To do this, we must have that \( a_i \equiv 0 \pmod{3} \) to guarantee that 3 is a divisor of \( (10^{m_i} - 1)/9 \). In other words, if \( p_i = 3 \) in \( \mathcal{C} \), then \( a_i \equiv 0 \pmod{3} \). Hence, since \( 10^{\ell(k)+a_i} + 1 \equiv 2 \not\equiv 0 \pmod{3} \), we can rewrite (4.2) modulo \( p_i \), when \( p_i = 3 \), as

\[
k \equiv \frac{d \cdot 10^{\ell(k)} a_i}{10^{\ell(k)+a_i} + 1} \equiv d a_i \pmod{3}.
\]  

(4.3)

The second difficulty is slightly more annoying. Here we have \( p_i \neq 3 \) and we can reduce (4.2) modulo \( p_i \) to

\[
k \equiv \frac{d \cdot 10^{\ell(k)} (10^{a_i} - 1)}{9 \cdot (10^{\ell(k)+a_i} + 1)} \pmod{p_i},
\]  

(4.4)

which makes sense provided \( 10^{\ell(k)+a_i} + 1 \not\equiv 0 \pmod{p_i} \). Since we are looking for \( k \), we obviously do not know the value of \( \ell(k) \). But, using a suitable specific value \( x \) in place of \( \ell(k) \), we can construct a corresponding value of \( k \). A “suitable” \( x \) is a number that satisfies two conditions. The first condition is that \( 10^{x+a_i} + 1 \) must be invertible modulo \( p_i \) for each \( i \). The second condition we require is that \( x \geq \ell(P) \), where \( P = 10 \prod_{i=1}^{t} p_i \) and \( t \) is the total number of elements in \( \mathcal{C} \). The factor of 10 here arises from the fact that we have added the two additional congruences for \( k \) modulo 2 and 5. This second condition can be achieved since if one value of \( x \) exists that satisfies the first condition, then there exist infinitely many values of \( x \) satisfying the first condition. This follows from the fact that

\[
10^{x+a_i} + 1 \equiv 10^{x+sM+a_i} + 1 \pmod{p_i},
\]
for any positive integer \( s \). Then, since \( x \geq \ell(\mathcal{P}) \), we can “jack-up” the value of \( k \) in the arithmetic progression we get from solving the system of congruences for \( k \) using the Chinese remainder theorem by adding multiples of \( \mathcal{P} \), and hence produce a value of \( k \) with exactly \( \ell(k) = x \). Moreover, if \( k \equiv z \pmod{\mathcal{P}} \), where \( 0 \leq z \leq \mathcal{P} - 1 \), and we choose \( x \geq 2\ell(\mathcal{P}) \), then we can choose \( k = 10^{x-\ell(\mathcal{P})} + z \). Observe then that \( \ell(k) = x \). Hence, we have

\[
k = 10^{\ell(k)-\ell(\mathcal{P})}\mathcal{P} + z. \tag{4.5}
\]

In many situations in this article, it is convenient to restrict our attention to values of \( k \) in the form of (4.5). Note that, in practice, it is not always necessary to choose a value of \( x \geq 2\ell(\mathcal{P}) \) to achieve the form (4.5).

Finally, in order to verify that we have not created a trivial situation, we use Theorem 3.1 and check that \( \gcd(9k + d, 10^{\ell(k)} + 1) = 1 \). We illustrate this method with an example.

**Example 4.1.** Let \( d = 1 \) and let \( \mathcal{C} = \{(2, 3, 3), (1, 3, 37), (0, 6, 7), (3, 6, 13)\} \). Let \( x = 7 \). Then \( x \geq \ell(\mathcal{P}) = \ell(10\prod_{i} p_{i}) = \ell(101010) = 6 \), and it is straightforward to check that \( 10^{x+i_1} + 1 \not\equiv 0 \pmod{p_{i}} \), for all \( i \). Using (4.3) and (4.4), we get that the resulting system of congruences for \( k \) is:

\[
\begin{align*}
k &\equiv 2 \pmod{3} \\
k &\equiv 1 \pmod{37} \\
k &\equiv 0 \pmod{7} \\
k &\equiv 2 \pmod{13} \\
k &\equiv 1 \pmod{2} \\
k &\equiv 1 \pmod{5}.
\end{align*}
\tag{4.6}
\]

Using the Chinese remainder theorem to solve (4.6) gives \( k \equiv 85841 \pmod{101010} \). Then \( k = 10 \cdot 101010 + 85841 = 1095941 \) is in this arithmetic progression and \( \ell(k) = 7 \). Clearly, \( \gcd(k, 10) = 1 \), and using a computer we verify that \( \gcd(9k + 1, 10^{\ell(k)} + 1) = 1 \). Hence, by Theorem 3.1, we conclude that \( k \) is a nontrivial 1-composite sandwich number.

While it is true that the infinitely many values of \( x \) described in the discussion prior to Example 4.1 give rise to infinitely many \( d \)-composite sandwich numbers \( k \), it is not true that \( k \) is necessarily nontrivial. In Example 4.1, we chose to let \( x = 7 \). Since \( \mathcal{M} = 6 \), every value of \( x \equiv 1 \pmod{6} \), with \( x \geq 7 \), will generate 1-composite sandwich numbers \( k \), although \( k \) may be trivial. For example, if \( x = 13 \) and \( k = 10000371 \cdot 101010 + 85841 = 1010137560551 \), then \( \ell(k) = 13 \) and \( k \) is a 1-composite sandwich number. However, \( k \) is trivial by Theorem 3.1 since \( \gcd(k, 10) = 1 \) and \( \gcd(9k + 1, 10^{\ell(k)} + 1) = 859 \). Note that this particular value of \( k \) is not in the form (4.5); and indeed every value of \( k \) in the form (4.5) with \( \ell(k) \equiv 1 \pmod{6} \) is nontrivial. To see this, we let \( k = 10^{\ell(k)-6} \cdot 101010 + 85841 \) and we suppose that \( q \) is
a prime divisor of $\gcd(9k+1, 10^{\ell(k)} + 1)$. Since $10^{\ell(k)} \equiv -1 \pmod{q}$ and $9k+1 \equiv 0 \pmod{q}$, we deduce that $q$ is a prime divisor of

$$9 \cdot 10^6 \cdot 85841 + 10^6 - 9 \cdot 101010 = 772569090910 = 2 \cdot 5 \cdot 77256909091.$$ 

Clearly, $q \not\in \{2, 5\}$, and so $q = 77256909091$. But then $2\ell(k) = \ord_q(10) = 77256909091$ implies that

$$\ell(k) = 77256909090/2 = 38628454545 \equiv 0 \pmod{3},$$

which contradicts the fact that $\ell(k) \equiv 1 \pmod{6}$.

The question still remains as to whether a suitable value of $x$ can always be found to generate $d$-composite sandwich numbers using the method described in this section. We provide conditions on $C$ to guarantee the existence of such an $x$, but first we need a lemma.

**Lemma 4.2.** Let $x$, $a$ and $m$ be positive integers with $m \equiv 0 \pmod{2}$. Let $\hat{m} = m/2$, and let $p$ be a primitive divisor of $10^m - 1$. Then

$$10^{x+a} + 1 \equiv 0 \pmod{p} \iff x \equiv \hat{m} - a \pmod{m}.$$ 

**Proof.** Suppose first that $10^{x+a} + 1 \equiv 0 \pmod{p}$. Then $10^{2(x+a)} \equiv 1 \pmod{p}$ so that $m = \ord_p(10)$ divides $2(x + a)$. Thus, $x + a \equiv 0 \pmod{\hat{m}}$. Note that if $x + a \equiv 0 \pmod{m}$, then $0 \equiv 10^{x+a} + 1 \equiv 2 \pmod{p}$, which is impossible since $p$ is odd. Hence, $x + a \not\equiv 0 \pmod{m}$. Therefore, there is an integer $b$ such that

$$x + a = \frac{(2b + 1)m}{2} = bm + \hat{m} \equiv \hat{m} \pmod{m}.$$ 

Conversely, suppose that $x \equiv \hat{m} - a \pmod{m}$. Then $2(x + a) \equiv 0 \pmod{m}$, which implies that $(10^{x+a} - 1)(10^{x+a} + 1) = 10^{2(x+a)} - 1 \equiv 0 \pmod{p}$. Hence, $10^{x+a} + 1 \equiv 0 \pmod{p}$ since $x + a \not\equiv 0 \pmod{m}$. \hfill $\Box$

**Proposition 4.3.** Let $C = \{(a_i, m_i, p_i)\}$ be a covering with exactly $t$ congruences such that $p_i$ is a primitive divisor of $10^{m_i} - 1$ for each $p_i \neq 3$. Relabel, if necessary, so that $m_1, \ldots, m_s$ are even, where $s \leq t$. Let $\hat{m}_i = m_i/2$ for each $i$ with $1 \leq i \leq s$. If there exists $y \in \mathbb{Z}$ such that

$$\hat{m}_i - a_i \not\equiv y \pmod{m_i} \quad \text{for all } i \text{ with } 1 \leq i \leq s,$$ 

then there exist infinitely many values of $x$ such that $10^{x+a_i} + 1 \not\equiv 0 \pmod{p_i}$ for all $i$.

**Proof.** We first claim that $10^{x+a_i} + 1 \not\equiv 0 \pmod{p_i}$ for each $i$ with $m_i$ odd. If $p_i = 3$, then this fact is obvious, so assume that $p_i \neq 3$. If $10^{x+a_i} + 1 \equiv 0 \pmod{p_i}$, then $10^{2(x+a_i)} \equiv 1 \pmod{p_i}$. Hence, $2(x + a_i) \equiv 0 \pmod{m_i}$, since by the construction
of \( C \) we have that \( \text{ord}_{p_i}(10) = m_i \). Thus \( x + a_i \equiv 0 \pmod{m_i} \) since \( m_i \) is odd. But then, \( 0 \equiv 10^{x + a_i} + 1 \equiv 2 \pmod{p_i} \), which is impossible since clearly \( p_i \neq 2 \). Hence, if \( m_i \) is odd, any value of \( x \) will suffice to ensure that \( 10^{x + a_i} + 1 \not\equiv 0 \pmod{p_i} \), and the claim is established. This also proves the proposition if \( s = 0 \).

Suppose now that \( s \geq 1 \) and consider the list of congruences

\[
x \equiv \hat{m}_1 - a_1 \pmod{m_1} \\
x \equiv \hat{m}_2 - a_2 \pmod{m_2} \\
\vdots \\
x \equiv \hat{m}_s - a_s \pmod{m_s}.
\]

(4.8)

It follows from (4.7) that (4.8) is not a covering, and therefore there exist infinitely many positive integers \( x \) that do not satisfy any of the congruences in (4.8). Hence, for any such value of \( x \), we have by Lemma 4.2 that \( 10^{x + a_i} + 1 \not\equiv 0 \pmod{p_i} \) for all \( i \), which completes the proof of the proposition.

\[\square\]

4.1. Proof of Theorem 1.3

We first show that if there exists a covering \( C \) that can be used to generate a nontrivial \( d \)-composite sandwich number \( k \) (as in Example 4.1) with \( \gcd(\ell(k), M) = 1 \), then there exist infinitely many nontrivial \( d \)-composite sandwich numbers. To complete the proof, we then find a covering \( C \) to construct a nontrivial \( d \)-composite sandwich number \( k \) with \( \gcd(\ell(k), M) = 1 \) for each value of \( d \).

Assume that \( k \) is a nontrivial \( d \)-composite sandwich number in the form (4.5) that has been constructed from \( C = \{(a_i, m_i, p_i)\} \) using the method described at the beginning of Section 4. Suppose also that \( \gcd(\ell(k), M) = 1 \) and \( k \equiv z \pmod{\mathcal{P}} \) with \( 1 \leq z \leq \mathcal{P} - 1 \), where \( \mathcal{P} = 10 \prod_i p_i \) or \( \mathcal{P} = \prod_i p_i \), depending on whether the two additional congruences (4.1) have been added. Then \( k = 10^{\ell(k)} + \ell(\mathcal{P}) + z \), and \( \gcd(9k + d, 10^{\ell(k)} + 1) = 1 \), by Theorem 3.1. Let \( p \) be the largest prime divisor of

\[
\mathcal{Z} = 9 \left( z \cdot 10^{\ell(\mathcal{P})} - \mathcal{P} \right) + d \cdot 10^{\ell(\mathcal{P})}.
\]

Since \( \gcd(\ell(k), M) = 1 \), by Dirichlet’s theorem on primes in an arithmetic progression, there exist infinitely many positive integers \( j \) such that \( \ell(k) + jM \) is a prime with \( \ell(k) + jM > P \). We claim that

\[
k_j := 10^{\ell(k) - \ell(\mathcal{P}) + jM} + z
\]

is a nontrivial \( d \)-composite sandwich number for each such value of \( j \). Note that \( k_j \) is indeed a \( d \)-composite sandwich number since

\[
s_n(k_j, d) \equiv s_n(k, d) \equiv 0 \pmod{p_i},
\]

when \( n \equiv a_i \pmod{m_i} \). Also note that \( \ell(k_j) = \ell(k) + jM \). Let

\[
A = 9k_j + d \quad \text{and} \quad B = 10^{\ell(k) + jM} + 1,
\]
and assume that \( k_j \) is trivial. Then, by Theorem 3.1, we have that \( \gcd(A, B) \equiv 0 \pmod{p} \) for some prime \( p \). Since \( B \equiv 0 \pmod{p} \), we get that \( 10^{\ell(k) - \ell(P) + jM} \equiv -10^{-\ell(P)} \pmod{p} \). Then, since \( A \equiv 0 \pmod{p} \), we have

\[
A \equiv 9 \left( -10^{-\ell(P)}P + z \right) + d \equiv 0 \pmod{p},
\]

which implies that \( Z \equiv 0 \pmod{p} \). In other words, if \( k_j \) is trivial, then \( s_n(k_j, d) \equiv 0 \pmod{p} \) for all \( n \geq 1 \), for some prime divisor \( p \) of the fixed positive integer \( Z \). However, since \( 10^{\ell(k) + jM} + 1 \equiv 0 \pmod{p} \), and \( \ell(k) + jM \) is prime, it follows that

\[
2(\ell(k) + jM) = \operatorname{ord}_p(10) \text{ divides } p - 1,
\]

which is impossible since \( \ell(k) + jM \geq p \). This contradiction proves the existence of infinitely many nontrivial \( d \)-composite sandwich numbers \( k \) if one is known to exist with \( \gcd(\ell(k), M) = 1 \). To complete the proof of the theorem, we construct such a nontrivial \( d \)-composite sandwich number for each value of \( d \). We use the following covering for each \( d \):

\[
\mathcal{C} = \{(0, 3, 37), (3, 4, 101), (2, 5, 41), (0, 5, 271), (2, 6, 13), (2, 10, 9091),
(5, 12, 9901), (1, 15, 31), (28, 30, 211), (19, 30, 241), (13, 30, 2161)\}.
\]

Here \( M = 60 \), and \( \ell(k) = 29 \) for each \( d \), so that \( \gcd(\ell(k), M) = 1 \). Also, it is easily verified using a computer that \( \gcd(9k + d, 10^{\ell(k)} + 1) = 1 \) in each case. We add the two congruences \( k \equiv 1 \pmod{2} \) and \( k \equiv 1 \pmod{5} \) for \( k \). We omit the details and simply provide in Table 2 the value of \( k \) for each \( d \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17447080826852847307281356871</td>
</tr>
<tr>
<td>2</td>
<td>18177573287783379196675646521</td>
</tr>
<tr>
<td>3</td>
<td>1725295798971701638754384961</td>
</tr>
<tr>
<td>4</td>
<td>17983450450642232528148674611</td>
</tr>
<tr>
<td>5</td>
<td>1705883515257055970227413051</td>
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<tr>
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<td>1778932761350108759621702701</td>
</tr>
<tr>
<td>7</td>
<td>16864712315429410301700441141</td>
</tr>
<tr>
<td>8</td>
<td>17595294776359942191094730791</td>
</tr>
<tr>
<td>9</td>
<td>1667058947828264633173469231</td>
</tr>
</tbody>
</table>

Table 2: Nontrivial \( d \)-Composite Sandwich Numbers \( k \) with \( \gcd(\ell(k), M) = 1 \)

5. Proof of Theorem 1.4

In order to prove the existence of infinitely many positive integers that are simultaneously nontrivial \( d \)-composite sandwich numbers for all \( d \in D \subset D \), where \( |D| = 4 \)
and $D \neq \{3, 6, 7, 9\}$, we use an argument similar to the one used in the proof of Theorem 1.3. In this situation, however, we must construct four separate coverings $C_1$, $C_2$, $C_3$ and $C_4$ corresponding to the four digits. As before, in each of these coverings, with the exception of $p_i = 3$, $p_i$ is a primitive divisor of $10^{m_i} - 1$, where $m_i$ is a modulus in the covering. The tricky part is that if we use the same prime $p_i$ in more than one covering, the resulting congruences for $k$ must be the same. For this reason, we have avoided using any prime more than once. The price we pay for this avoidance is that the coverings are more difficult to construct. The four coverings we use are as follows:

$C_1 = \{(1, 4, 101), (0, 6, 13), (3, 8, 73), (7, 8, 137), (8, 12, 9901), (10, 16, 17), (2, 16, 5882353)
\}
\{(16, 18, 19), (14, 24, 9999001), (28, 36, 9999990000001), (22, 48, 999999900000001)
\}
\{(4, 72, 3169), (40, 72, 98641), (46, 72, 3199044596370769), (94, 144, 89299)
\},

$C_2 = \{(1, 3, 3), (0, 3, 37), (5, 6, 7), (2, 18, 52579), (26, 27, 757), (14, 27, 440334654777631),
\}
\{(44, 54, 70541929), (32, 54, 14175966169), (35, 81, 163), (77, 81, 9397), (50, 81, 2462401),
\}
\{(62, 81, 676421558270641), (23, 81, 130654897808007778425046117),
\}
\{(8, 162, 456502382570032651)
\},

$C_3 = \{(1, 5, 41), (4, 5, 271), (7, 10, 9091), (13, 15, 31), (8, 15, 2906161), (15, 20, 3541),
\}
\{(20, 25, 21401), (0, 25, 25601), (5, 25, 182521213001), (3, 30, 211),
\}
\{(18, 30, 241), (22, 30, 2161), (40, 50, 251), (10, 50, 5051), (15, 50, 78875943472201),
\}
\{(32, 60, 61), (12, 60, 4188901), (45, 60, 39526741), (35, 75, 151), (10, 75, 4201)
\},

$C_4 = \{(1, 7, 239), (6, 7, 4649), (7, 9, 333667), (0, 14, 9099091), (5, 21, 43), (17, 21, 1933),
\}
\{(9, 21, 10838689), (11, 28, 29), (18, 28, 281), (7, 28, 121499449),
\}
\{(17, 32, 353), (9, 32, 449), (1, 32, 641), (25, 32, 1409), (2, 42, 127),
\}
\{(24, 42, 2689), (12, 42, 459691), (4, 56, 7841), (32, 56, 1275220010201503761),
\}
\{(31, 63, 10873), (37, 63, 23311), (40, 63, 45613), (19, 63, 45121231),
\}
\{(10, 63, 1921436048294281), (23, 84, 226549), (3, 84, 4458192232302340849),
\}
\{(93, 96, 97), (69, 96, 206209), (45, 96, 66554101249), (21, 96, 75118313082913),
\}
\{(53, 112, 113), (109, 112, 72767555896403138401),
\}
\{(21, 112, 119968369144846370226083377), (58, 126, 5274739),
\}
\{(121, 126, 18977242267323558874485732659), (149, 168, 603812429055411913),
\}
\{(243, 252, 22906246896437231227899575633620139766044690040039603689929),
\}
\{(75, 252, 43266855241), (159, 252, 1009),
\}
\{(77, 336, 14333198271594667890696685637947917991613652942792832495021393),
\}
\{(301, 336, 2070270028985341766616009080161)
\}.

Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$, and let $P$ be the product of all the primes in $C$ multiplied by 10 (adding the primes 2 and 5). Then $\ell(P) = 741$. We give details only for the case of $D = \{1, 2, 3, 4\}$ since the other cases are similar. Here, $C_i$ corresponds to the digit $i$. We create a system of congruences for $k$ using (4.3) and (4.4), together with
the two added congruences \( k \equiv 1 \pmod{2} \) and \( k \equiv 1 \pmod{5} \). Then we search over values of \( x \geq 741 \) (in place of \( \ell(k) \)) to find a solution to the system (via the Chinese remainder theorem), in accordance with (4.5), that simultaneously satisfies the criteria from Proposition 4.3 and Theorem 1.3. The smallest value of \( x \) that satisfies these conditions is \( x = 779 \), and a solution with \( \ell(k) = 779 \) is

\[
\begin{align*}
&k = 354402919536891231666174784482258650911694860681098307631700188746035217 \\
&407222366102321143987082440414733816824588900746485360827171938317666194 \\
&1962697722999232448362746920592650000444434954589845840512762885513871166 \\
&7148130944698360348586790802593012202168051368277728925506156403451950936 \\
&57809847315372321302963019726914499525385017784735089846291075875191298265 \\
&19133083235144744855685269649268693233138949862013450740415771923058 \\
&4123751445781205058216540960465709674609765053940878499898580013583705525 \\
&8947392368038231957687185581251473983275672044111171706569043689583610615 \\
&967835508356441937227174321720691339534757036108747356645837948550243646 \\
&5471117811031822098052647767973565747345134645161034457609214662167122526 \\
&639148429313444597795160981393953097819656099201.
\end{align*}
\]

Indeed, it is straightforward to check that

\[ \gcd(9k + d, 10^\ell(k) + 1) = \gcd(k, 10d) = 1, \]

for each digit \( d \in \{1, 2, 3, 4\} \). In addition, \( M = 453600 \) so that \( \gcd(\ell(k), M) = 1 \), which completes the proof of the theorem.

**Remark 5.1.** Note that there are 4! possible ways the coverings \( C_1, C_2, C_3 \) and \( C_4 \) can be “assigned” to the digits in a set \( D \subset D \) with \( |D| = 4 \). For a particular \( D \), some of these correspondences do not yield a solution that satisfies all of the necessary criteria. However, with the exception of \( D = \{3, 6, 7, 9\} \), at least one correspondence produces a desired solution using these four coverings. The exceptional case \( D = \{3, 6, 7, 9\} \) in Theorem 1.4 is a result of the fact that, regardless of how we assign these four coverings to the digits in \( D \), the congruences arising from the covering \( C_2 \) force either \( k \equiv 0 \pmod{3} \) or \( k \equiv 0 \pmod{7} \). Although \( k \) is still a composite sandwich number in this case, it is trivial.

**6. Proof of Theorem 1.5**

Given any \( d \in D \), to prove that there are infinitely many sets of 13 consecutive positive integers that are all \( d \)-composite sandwich numbers, we utilize the method and coverings that were used in the proof of Theorem 1.4. However, here we use the same \( d \) for all four coverings \( C_1, C_2, C_3 \) and \( C_4 \). In addition, we substitute \( k + 2 \) for \( k \) in the congruences generated from \( C_2 \), \( k + 4 \) for \( k \) in the congruences for \( k \) in the
congruences generated from \(C_3\), and \(k + 6\) for \(k\) in the congruences generated from \(C_4\). We also add the congruences \(k \equiv 1 \pmod{2}\) and \(k \equiv 2 \pmod{5}\) to the system for \(k\). Since the techniques are similar, we give details only in the case of \(d = 1\). The smallest value of \(x \geq 741\) here, in accordance with (4.5), that simultaneously satisfies the criteria from Proposition 4.3 and Theorem 1.3 is \(x = 811\). Using the Chinese remainder theorem, a solution to the system of congruences for \(k\), with \(\ell(k) = 841\), is

\[
k = 354402919536891231666617478448225865090832683647239060200876864650079431 \\
1711603812697324230525538611838646043882766742346069163764391075438928881 \\
3699789159029469059047678298092672792216574482270249499003001497798585976 \\
53443081833646371358102361408628239297159604420693600828207861578411170 \\
66811862621597843416753694935763341807568419205446721718502316694487437 \\
1689757054755673346721553522756962087212178947796157521768786309652796994 \\
468576518557579581179806157246989868578577877932376808715073572119986790 \\
478277025337010376759748042869229543778058998648693949970580537837368249 \\
5758123653025610997354753176939604748433221775845235692325886596409193285 \\
72137430076309602222108680710478868396867354398689532792465149164194750 \\
8378705604292541419937489028854509228736851403542592414929979630436913 \\
87637597.
\]

Note that \(k + z \equiv 0 \pmod{2}\) for all odd integers \(z\) with \(-3 \leq z \leq 9\). Also, \(k + z \equiv 0 \pmod{5}\) for \(z \in \{-2, 3, 8\}\). Hence, in these cases, the integers \(k + z\) are trivial 1-composite sandwich numbers. For the values of \(z\) with \(z \in \{0, 2, 4, 6\}\), we have used the respective coverings \(C_1, C_2, C_3, C_4\), to ensure that \(k + z\) is a 1-composite sandwich number. It is again easy to check that

\[
gcd(9k + 1, 10^{\ell(k)} + 1) = \gcd(k, 10) = 1
\]

and \(\gcd(\ell(k), \mathcal{M}) = 1\). Thus, we can construct infinitely many nontrivial 1-composite sandwich numbers \(k\) such that \(k - 3, k - 2, \ldots, k + 9\) are all 1-composite (not necessarily nontrivial) sandwich numbers. As mentioned, the methods are similar for the other digits.

\section{Theorem 1.6}

**Definition 7.1.** A Sierpiński number \(k\) is an odd positive integer such that \(k \cdot 2^n + 1\) is composite for all integers \(n \geq 1\).

**Definition 7.2.** A Riesel number \(k\) is an odd positive integer such that \(k \cdot 2^n - 1\) is composite for all integers \(n \geq 1\).
In 1956, Riesel [26] proved that there are infinitely many Riesel numbers, and in 1960, Sierpiński [28] proved that there are infinitely many Sierpiński numbers. Since then, other authors have examined extensions and variations of these ideas [2, 4, 5, 6, 7, 8, 10, 12, 14, 17, 18, 19, 20, 21, 22, 23, 24]. Coverings are used quite extensively in these investigations.

7.1. Proof of Theorem 1.6

We provide details only for \( d = 2 \) since the procedure is identical for other values of \( d \in D \).

The approach we use is similar to one used in [23]. We build three coverings \( C_1 \), \( C_2 \) and \( C_3 \) with the following properties:

- \( C_1 \) is used to generate infinitely many 2-composite sandwich numbers that are also perfect squares. The corresponding primes here are primitive divisors of numbers of the form \( 10^m - 1 \), where \( m \) is a modulus in \( C_1 \).

- \( C_2 \) is used to generate infinitely many squares that are also Sierpiński numbers. The corresponding primes here are primitive divisors of numbers of the form \( 2^m - 1 \), where \( m \) is a modulus in \( C_2 \).

- \( C_3 \) is used to generate infinitely many squares that are also Riesel numbers. The corresponding primes here are primitive divisors of numbers of the form \( 2^m - 1 \), where \( m \) is a modulus in \( C_3 \).

- The only corresponding prime that appears in more than one covering is \( p = 3 \).

Even though we use the prime \( p_i = 3 \) in all three coverings, the resulting congruence for \( k^2 \) is the same in all cases. Thus, we can piece together all three coverings and use the Chinese remainder theorem to get squares that are simultaneously 2-composite sandwich, Sierpiński and Riesel.

The first covering is: \( C_1 = \{(2, 3, 3), (1, 3, 37), (3, 6, 7), (0, 6, 13)\} \). Here, we have \( \ell \left( \prod_{i=1}^{4} p_i \right) = \ell(10101) = 5 \), and so we need to choose \( x \) with \( x \geq 5 \). In addition, by Proposition 4.3, we need to choose \( x \) such that \( x \not\equiv 0 \pmod{3} \). We choose \( x = 5 \). Replacing \( k \) with \( k^2 \) in (4.3) and (4.4), we get the following set of congruences for \( k^2 \):

\[
\begin{align*}
k^2 & \equiv 1 \pmod{3} \\
k^2 & \equiv 11 \pmod{37} \\
k^2 & \equiv 1 \pmod{7} \\
k^2 & \equiv 0 \pmod{13}.
\end{align*}
\] (7.1)

Note that each of the residues in (7.1) is a square modulo the corresponding prime.

The second covering is:

\( C_2 = \{(1, 2, 3), (2, 4, 5), (4, 8, 17), (8, 16, 257), (16, 32, 65537), (32, 64, 641), (0, 64, 6700417)\} \).
We use $C_2$ and solve for $k^2$ in each congruence $k^2 \cdot 2^n + 1 \equiv 0 \pmod{p_i}$. The resulting system of congruences is:

$$
\begin{align*}
    k^2 &\equiv 1 \pmod{3} & k^2 &\equiv 1 \pmod{65537} \\
    k^2 &\equiv 1 \pmod{5} & k^2 &\equiv 1 \pmod{641} \\
    k^2 &\equiv 1 \pmod{17} & k^2 &\equiv -1 \pmod{6700417}.
\end{align*}
$$

(7.2)

Again, each of the residues in (7.2) is a square modulo the corresponding prime.

The third covering is:

$$
C_3 = \{ (0, 2, 3), (3, 5, 31), (6, 7, 127), (8, 9, 73), (4, 15, 151), (7, 20, 41), (12, 21, 337),
(9, 24, 241), (11, 25, 601), (21, 25, 1801), (11, 27, 262657), (17, 28, 113), (0, 35, 71),
(15, 35, 122921), (37, 40, 561681), (14, 45, 631), (29, 45, 23311), (37, 56, 15790321),
(39, 60, 1321), (5, 63, 92737), (25, 70, 281), (49, 72, 433), (25, 72, 38737),
(16, 75, 10567201), (1, 75, 100801), (3, 84, 14449), (41, 90, 18837001),
(40, 105, 152041), (100, 105, 29191), (10, 105, 106681), (29, 108, 279073),
(101, 108, 246241), (69, 120, 4562284561), (23, 126, 77158673929), (56, 135, 271),
(101, 135, 49971617830801), (135, 140, 7416361), (31, 150, 1133836730401),
(143, 168, 88959882481), (59, 168, 3361), (116, 175, 39551), (37, 180, 54001),
(110, 189, 207617485544528392970753527), (47, 189, 1560007), (31, 200, 401),
(91, 200, 3173389601), (131, 200, 2787601), (141, 210, 664441), (21, 210, 1564921),
(41, 216, 138991501037953), (185, 216, 33975937), (51, 225, 13861369826299351),
(201, 225, 1348206751), (126, 225, 617401), (167, 270, 15121),
(191, 280, 84179842077657862011867889681), (81, 300, 1201)\}.
$$

Here we solve the congruence $k^2 \cdot 2^n - 1 \equiv 0 \pmod{p_i}$ for $k^2$ in each case to get:
Adding the congruence $k^2 \equiv 1 \pmod{2}$ to the total system comprised of the congruences from $C_1$, $C_2$ and $C_3$, and choosing a square root in each congruence, we solve for $k$. Denote the smallest positive solution arising from this choice of square roots as $k_s$. Then $\ell(k_s^2) \equiv 0 \pmod{6}$. To get a desired value of $k$ with $\ell(k^2) \equiv 5 \pmod{6}$, we simply choose an appropriate value in the arithmetic progression. The smallest such value is

$$
k = k_s + 100 \cdot 2 \left( \prod_{p_i \in C_1 \cup C_2 \cup C_3} p_i \right),$$

where $\ell(k^2) = 755 \equiv 5 \pmod{6}$. Then

$$
k^2 = 1019047736805764081984947455865396072470509397203727838645536786 \\
3814245458582704351902014013046418057986606439796252087699283422 \\
449262089186511473841093149621728527411300462367593616610241532 \\
8833503147456880764322345477551037588779177594977927000285137096 \\
53958430977765851717282427808209963540686171949063146235604344 \\
5355471369350576909107473412666632616920949310511495213282687819 \\
46450436597977898812671503038105142619408620244266209983512079649 \\
0270257386246544934381849990614732027662942914035791594676896536 \\
16888717719317050738539768537653566067446726452080221478290577620 \\
317717900389669162822618774620058507396304452842837394091889306 \\
54643038648628277723975722234454521723228403577353586765723156863 \\
783373786978177733914324199470291788881


is a 2-composite sandwich square that is also a Sierpiński and Riesel number. Since
there are infinitely many such values of $k$ in the arithmetic progression, the theorem is established.

**Remark 7.3.** Since $\gcd(9\kappa^2 + 2, 10^{\ell(\kappa^2)} + 1) = 1$, it follows from Theorem 3.1 that $\kappa^2$ is nontrivial. This is not the case for all such values of $k^2$ produced from the arithmetic progression. For example, the number $\left( k_2 + 10^{2} \cdot 2 \prod_{p_i \in \mathcal{C}_2} c_{p_i} \right)^2$ is a trivial 2-composite sandwich square that is a Sierpiński and Riesel number.

We give in Table 3 a list of coverings to generate $d$-composite sandwich squares. These coverings can be used in conjunction with $\mathcal{C}_2$ and $\mathcal{C}_3$, as in the given case of $d = 2$, to generate $d$-composite sandwich squares that are also Sierpiński and Riesel. The number of digits modulo $\mathcal{M}$ for these squares is also given in each case.

<table>
<thead>
<tr>
<th>$d$</th>
<th>Covering</th>
<th>$\ell \ (k^2) \ (\text{mod} \ \mathcal{M})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${(1, 3, 3), (2, 3, 37), (3, 6, 7), (0, 6, 13)}$</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>$\mathcal{C}_1$</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>${(0, 2, 11), (2, 3, 37), (1, 4, 101), (3, 6, 13), (7, 12, 9901)}$</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>${(1, 3, 3), (2, 3, 37), (3, 6, 7), (0, 6, 13)}$</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>${(2, 3, 3), (1, 3, 37), (0, 6, 7), (3, 6, 13)}$</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>${(0, 2, 11), (0, 3, 37), (1, 6, 13), (5, 9, 333667), (17, 18, 19), (11, 18, 52579)}$</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>${(0, 2, 11), (0, 3, 37), (3, 4, 101), (5, 6, 13), (1, 12, 9901)}$</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>$\mathcal{C}_1$</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>${(0, 2, 11), (1, 3, 37), (3, 6, 13), (5, 9, 333667), (17, 18, 19), (11, 18, 52579)}$</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3: Coverings Used to Generate $d$-Composite Sandwich Squares

### 8. Final Comments

In Theorem 1.1 and Theorem 1.2, we found infinitely many primes that are trivial $d$-composite sandwich numbers. However, in Theorem 1.3, all nontrivial $d$-composite numbers are composite. The reason for this is that using a covering argument to produce such numbers inherently produces composite numbers. To see this, note that any covering $\mathcal{C}$ must have at least one zero residue. Suppose that $(0, m_i, p_i) \in \mathcal{C}$. Then, by (4.3) and (4.4), we have that $k \equiv 0 \text{ (mod } p_i)$ so that $k$ is composite. Thus, a covering argument cannot be used to find nontrivial $d$-composite sandwich primes. Of course, this raises two natural questions: Do any such primes exist; and if so, how do we find them? A computer search has found candidates for such primes. For example, 7 “seems” to be a 1-composite sandwich number, but there is no apparent obstruction to encountering a prime in
the sequence \( \{s_n(7, 1)\}_{n=1}^{\infty} \). A partial covering
\[
\{(1, 3, 3), (2, 6, 11), (5, 6, 13), (0, 6, 7)\}
\]
shows that there is indeed a pattern to the prime divisors of \( s_n(7, 1) \) for these congruence classes modulo \( n \). However, the uncovered class of \( n \equiv 3 \pmod{6} \) remains somewhat of a mystery; and if a prime occurs in \( \{s_n(7, 1)\}_{n=1}^{\infty} \), it must occur in this class. Further splitting of this class reveals that there are still patterns to the prime divisors, but ultimately no covering has been found to explain the entire situation. Similar phenomena have been observed before in [11, 18, 22] where the sequence in question defies a covering argument; but, in fact, the sequence is composite. In these situations, the “bad” class yields to a factorization rather than the same prime divisor. It is conjectured that the smallest prime divisor of these “bad” terms in the sequence is unbounded as \( n \) approaches infinity. It appears, however, in the case of nontrivial \( d \)-composite sandwich numbers, that such a factorization is unlikely, although we have not been able to confirm this belief.

Recently, Bob Hough [17] has given a negative answer to a famous question in covering systems known as the minimum modulus problem. This problem, originally posed by Erdős [9], asks if the minimum modulus in a covering with distinct moduli can be arbitrarily large. Previously, the best known result was due to Nielsen [25], who constructed a covering with distinct moduli and minimum modulus 40. To prove Theorem 1.4 and Theorem 1.5, we constructed four coverings with no repeated associated prime. Admittedly, our situation is not as restrictive as the minimum modulus problem since we can use repeated moduli. So, it is conceivable that we might be able to extend our results somewhat by constructing more than four coverings with no repeated corresponding prime, but it seems computationally infeasible, and perhaps impossible, in light of Hough’s theorem, to extend this process to nine coverings.

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