A NUMBER THEORETIC PROBLEM ON THE DISTRIBUTION OF POLYNOMIALS WITH BOUNDED ROOTS

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Abstract
Let \( \mathcal{E}_d^{(s)} \) denote the set of coefficient vectors \((a_1, \ldots, a_d) \in \mathbb{R}^d\) of contractive polynomials \(x^d + a_1 x^{d-1} + \cdots + a_d \in \mathbb{R}[x]\) that have exactly \(s\) pairs of complex conjugate roots and let \(v_d^{(s)} = \lambda_d(\mathcal{E}_d^{(s)})\) be its \((d\text{-dimensional})\) Lebesgue measure. We settle the instance \(s = 1\) of a conjecture by Akiyama and Pethő, stating that the ratio \(v_d^{(s)}/v_d^{(0)}\) is an integer for all \(d \geq 2s\). Moreover we establish the surprisingly simple formula \(v_d^{(1)}/v_d^{(0)} = (P_d(3) - 2d - 1)/4\), where \(P_d(x)\) are the Legendre polynomials.

- Dedicated to Prof. Dominique Foata on the occasion of his 80\(^{th}\) birthday.

1. Introduction
Let \( \mathcal{E}_d \) denote the set of all coefficient vectors \((a_1, \ldots, a_d) \in \mathbb{R}^d\) of polynomials \(x^d + a_1 x^{d-1} + \cdots + a_d \in \mathbb{R}[x]\) with coefficients in \(\mathbb{R}\) and all roots having absolute value less than 1, and let \(\mathcal{E}_d^{(s)}\) denote the subset of the coefficient vectors of those polynomials in \(\mathcal{E}_d\) that have exactly \(s\) pairs of complex conjugate roots. Let furthermore \(v_d =...\)

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\( \lambda_d(\mathcal{E}_d) \) and \( v_d^{(s)} = \lambda_d(\mathcal{E}_d^{(s)}) \) denote the \( d \)-dimensional Lebesgue measures of the referring sets.

The sets \( \mathcal{E}_d \) have been studied by several authors in different context, compare e.g. Schur [11], Fam and Meditch [4] or Fam [3]. More recently, the regions \( \mathcal{E}_d \) have become of interest in the study of “shift radix systems”, since the regions where those systems have a certain periodicity property are in close connection with the regions \( \mathcal{E}_d \) (compare e.g. Kirschenhofer et al. [7]). Fam [3] established the formula

\[
v_d = \begin{cases} 
2^{2m^2} \prod_{j=1}^{m} \frac{(j-1)!^4}{(2j-1)!^2} & \text{if } d = 2m, \\
2^{2m^2+2m+1} \prod_{j=1}^{m} \frac{j^2(j-1)!^2}{(2j-1)!^2} & \text{if } d = 2m + 1.
\end{cases} (1.1)
\]

In [1] Akiyama and Pethô gave a number of results on the quantities \( v_d^{(s)} \), including an integral representation for general \( s \) from which they derived an explicit formula in the instance \( s = 0 \) as well as a somewhat involved expression for \( s = 1 \) reading

\[
v_d^{(0)} = \frac{2^{d(d+1)/2}}{d!} S_d(1, 1, 1/2), \quad v_d^{(1)} = 2^{(d-1)(d-2)/2} - \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^d \cdot k \cdot 2^{d-2k-j}}{j! k! (d - 2 - j - k)!} B_{d-2}(d - 2 - k, d - 2 - k - j) \int_{z=0}^{1} \int_{y=-2\sqrt{\pi}}^{2\sqrt{\pi}} y^j (y + z + 1)^k \, dy \, dz
\]

(1.2)

for \( d \geq 2 \) and \( 0 \leq k \leq j \leq d \) where

\[
S_d(1, 1, 1/2) := \frac{1}{\prod_{i=0}^{d-1} \left( \frac{2i+1}{i} \right)} \quad (1.3)
\]

is a special instance of the Selberg integral \( S_n(\alpha, \beta, \gamma) \) and where

\[
B_d(j, k) := \left( \prod_{i=1}^{k} \frac{2 + (d - i - 1)/2}{3 + (2d - i - 1)/2} \prod_{i=1}^{j} (1 + (d - i)/2) \prod_{i=1}^{k} (1 + (d - i)/2) \right) \frac{1}{\prod_{i=1}^{j+k} (2 + (2d - i - 1)/2)} S_d(1, 1, 1/2).
\]

(1.4)

is a special instance of Aomoto’s generalization of the Selberg integral (compare Andrews et al. [2, Section 8] for Selberg’s and Aomoto’s integrals).

Furthermore, Akiyama and Pethô in [1] proved that the ratios \( v_d^{(s)}/v_d^{(0)} \) are rational, and, motivated by extensive numerical evidence, stated the following

**Conjecture 1.1.** [1, Conjecture 5.1] The quotient

\[
v_d^{(s)}/v_d^{(0)}
\]

is an integer for all non-negative integers \( d, s \) with \( d \geq 2s \).
In Section 2 of this paper we will prove this conjecture for the instance \( s = 1 \) and in addition give a surprisingly simple explicit formula for the quotient in this case involving the Legendre polynomials evaluated at \( x = 3 \). In the proof we will combine several transformations of binomial sums, one of them corresponding to a special instance of Pfaff’s reflection law for hypergeometric functions. We refer the reader in particular to the standard reference [6, Section 5] for the techniques that we will apply.

In Section 3 we will use our main theorem to establish a linear recurrence for the sequence \( \left( v_d^{(1)}/v_d^{(0)} \right)_{d \geq 0} \), and from its generating function will derive its asymptotic behaviour for \( d \to \infty \). Combined with a result from [1], this also gives information on the asymptotic behaviour of the probability \( p_d^{(1)} = v_d^{(1)}/v_d \) of a contractive polynomial of degree \( d \) to have exactly one pair of complex conjugate roots.

In the final section we discuss possible generalizations of our results.

2. Main Result

**Theorem 2.1.** The quotient \( v_d^{(1)}/v_d^{(0)} \) is an integer for each \( d \geq 2 \). Furthermore we have

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{P_d(3) - 2d - 1}{4},
\]

where

\[
P_d(x) := 2^{-d} \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k \binom{d-k}{k} \binom{2d-2k}{d-k} x^{d-2k} = \sum_{k=0}^{d} \binom{d+k}{2k} \binom{2k}{k} \left( x - \frac{1}{2} \right)^k
\]

are the Legendre polynomials (cf. [10, p. 66]).

**Proof.** In a first step we solve the double integral in identity (1.2) for \( v_1^{(1)} \). Let \( j \geq 0, k \geq 0 \). Then

\[
\int_{z=0}^{2\sqrt{2}} \int_{y=-2\sqrt{2}}^{2\sqrt{2}} y^j(y+z+1)^k \ dy \ dz = \int_{y=-2}^{2} \int_{z=y^2/4}^{1} y^j(y+z+1)^k \ dz \ dy
\]

\[
= \frac{1}{k+1} \left( \int_{-2}^{2} y^j(y+2)^{k+1} \ dy - \int_{-2}^{2} y^j(y/2+1)^{2k+2} \ dy \right)
\]

\[
= \frac{1}{k+1} \left( 2^{j+k+2} \int_{-1}^{1} y^j(y+1)^{k+1} \ dy - 2^{j+1} \int_{-1}^{1} y^j(y+1)^{2k+2} \ dy \right)
\]

where we performed the substitution \( y/2 \to y \) in the last step. By iterated partial
integration we gain now from the last expression that

\[
\int_{z=0}^{2\sqrt{\pi}} \int_{y=-2\sqrt{\pi}}^{2\sqrt{\pi}} y^j (y + z + 1)^k \, dy \, dz = \frac{2^{j+2k+4}}{k+1} \left( \sum_{r=1}^{j+1} (-2)^{r-1} \binom{j}{r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1} \binom{j}{r}}{(2k+r+2)_r} \right)
\]

(2.1)

with \((x)_j := \prod_{i=0}^{j-1} (x - i)\).

In the following we insert (2.1) in formula (1.2) and perform stepwise a first evaluation of \(v_d^{(1)}/v_d^{(0)}\) mainly as a sum of products of factorials.

\[
v_d^{(1)}/v_d^{(0)} = \left( \frac{2}{d-1}(d-2) \right)^{-\frac{d-2-j}{2}} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^d \frac{(2d-2k-j)\binom{j}{k}}{(k+1)!}}{j!(d-j-k-2)!} \prod_{i=1}^{d-k-j} \frac{2 + \frac{d-2-i-1}{2}}{2d-i-1} \prod_{i=1}^{d-2-k} \frac{1}{(2d-k-2)_j}
\]

\[
\times \left( \sum_{r=1}^{j+1} \frac{(-2)^{r-1} \binom{j}{r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1} \binom{j}{r}}{(2k+r+2)_r}}{(k+r+1)_r} \right) / \left( \prod_{i=1}^{d-l} \frac{2^{d-1} \binom{d}{i}}{(2i+1)} \right)
\]

\[
= \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} \frac{(-1)^d \frac{(2d-k-2j)\binom{j}{k}}{(k+1)!}}{j!(d-j-k-2)!} \prod_{i=1}^{d-k-j} \frac{2 + \frac{d-2-i-1}{2}}{2d-i-1} \prod_{i=1}^{d-2-k} \frac{1}{(2d-k-2)_j}
\]

\[
\times \left( \sum_{r=1}^{j+1} \frac{(-2)^{r-1} \binom{j}{r} - \sum_{r=1}^{j+1} \frac{(-2)^{r-1} \binom{j}{r}}{(2k+r+2)_r}}{(k+r+1)_r} \right) / \left( \prod_{i=1}^{d-l} \frac{2^{d-1} \binom{d}{i}}{(2i+1)} \right)
\]

In the next step we rewrite the last expression as a sum over products of binomial
coefficients.

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{j=0}^{d-2} \sum_{k=0}^{d-j-2} (-1)^{d+k+1} \binom{d}{j+k+2} \binom{d+j+k+2}{j+k+2} \frac{j+k+2}{j+2k+3} \\
\left( \sum_{r=1}^{j+1} (-2)^{r-2} \binom{j+2k+3}{2k+r+2} \binom{2k+r+2}{k+1} - \sum_{r=1}^{j+1} (-2)^{r-2} \binom{j+2k+3}{2k+r+2} \binom{2k+2}{k+1} \right).
\]

Using the substitution \( j+k+2 \rightarrow a, k+1 \rightarrow b \) the latter expression reads

\[
\sum_{a=2}^{d} \sum_{b=0}^{d-a-2} (-1)^{d+b} (-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{d} \binom{a+b}{2b+r} \binom{2b+r}{b} - \\
\sum_{a=2}^{d} \sum_{b=0}^{d-a-2} (-1)^{d+b} (-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{d} \binom{a+b}{2b+r} \binom{2b+r}{b}
\]

so that

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d} (-1)^{d} a \binom{d}{a} \binom{d+a}{d} \sum_{r=1}^{a} (-2)^{r-2} \\
\left( \sum_{b=0}^{a-r} (-1)^{b} \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} - \sum_{b=0}^{a-r} (-1)^{b} \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} \right).
\]

(2.2)

In the following we will simplify the two innermost sums.

We start with the first sum. If \( r = a \) the sum trivially equals \( \frac{1}{a} \). Let us assume \( 1 \leq r \leq a-1 \) now. Then we have

\[
\sum_{b=0}^{a-r} (-1)^{b} \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} = \frac{1}{a-r} \sum_{b=0}^{a-r} (-1)^{b} \binom{a-r}{b} \binom{a+b-1}{b+r} \\
= \frac{(-1)^{r}}{a-r} \sum_{b=0}^{a-r} \binom{a-r}{b} \binom{r-a}{b+r} = \frac{(-1)^{r}}{a-r} \binom{0}{a} = 0,
\]

where we used

\[
(-1)^{k} \binom{k-n-1}{k} = \binom{n}{k} \quad (n \in \mathbb{Z}, k \geq 0)
\]

for the second identity, and Vandermonde’s identity

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{s}{k+t} = \sum_{k=0}^{n} \binom{n}{k} \binom{s}{n+t-k} = \binom{n+s}{n+t} \quad (s \in \mathbb{Z}, n,t \geq 0)
\]
for the third one. Altogether we have established

$$
\sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} = \frac{1}{a} \delta_{r,a} \quad (1 \leq r \leq a),
$$

(2.3)

where \( \delta_{r,a} \) denotes the Kronecker symbol.

Now we turn to the second sum in question. Since this is a sum reminiscent of a sum treated in [6, Section 5.2, Problem 7] we first try to adopt the strategy followed there and use [6, Section 5.1, identity 5.26]

$$
\binom{l + q + 1}{m + n + 1} = \sum_{0 \leq k \leq l} \binom{l-k}{m} \binom{q+k}{n} \quad (l, m \geq 0, n \geq q \geq 0).
$$

(2.4)

With \( l = a + b - 1, q = 0, m = 2b, n = r - 1 \) and \( k = s \) we get

$$
\sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} = \sum_{b=0}^{a-r} \sum_{s=0}^{a+b-1} (-1)^b \frac{1}{a+b} \binom{a+b-s-1}{2b} \binom{s}{r-1} \binom{2b}{b}
$$

which by a change of summations yields

$$
= \sum_{s=r-1}^{2a-r-1} \binom{s}{r-1} \sum_{b=0}^{a-s-1} (-1)^b \frac{1}{a+b} \binom{a+b-s-1}{2b} \binom{2b}{b}
$$

$$
= \sum_{s=r-1}^{a-1} \binom{s}{r-1} \sum_{b=0}^{a-s-1} (-1)^b \frac{1}{a+b} \binom{a+b-s-1}{2b} \binom{2b}{b}.
$$

(2.5)

Now we are ready to apply sum \( S_m \) from [6, Section 5.2, Problem 8]

$$
S_m = \sum_{k=0}^{n} (-1)^k \frac{1}{k + m + 1} \binom{n+k}{2k} \binom{2k}{k} = (-1)^n \frac{m! n!}{(m + n + 1)!} \binom{m}{n} \quad (m, n \geq 0).
$$

(2.6)

With \( m = a - 1, n = a - s - 1 \) and \( k = b \) we find that (2.5) from above equals

$$
\sum_{s=r-1}^{a-1} \binom{s}{r-1} \frac{(-1)^{a+s+1}(a-1)!(a-s-1)!}{(2a-s-1)!} \frac{a-1}{a-s-1}
$$

$$
= \frac{(-1)^{a+1}(a-1)!(a-1)!}{(2a-1)!} \frac{2a-1}{r-1} \sum_{s=r-1}^{a-1} (-1)^s \binom{2a-r}{s-r+1}
$$

$$
= \frac{(-1)^{a+r}(a-1)!(a-1)!}{(2a-1)!} \frac{2a-1}{r-1} \sum_{s=0}^{a-r} (-1)^s \binom{2a-r}{s}
$$

(2.7)
where we applied the substitution \( s - r + 1 \rightarrow s \) in the last step. Using the basic identity

\[
\sum_{j=0}^{k} (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k} \quad (n, k \geq 0)
\]

to evaluate the last sum in (2.7) we finally get

\[
\sum_{k=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b} \binom{2b}{b} = \frac{(a-1)!(a-1)!}{(2a-1)!} \binom{2a-1}{r} \binom{2a-r-1}{a-r}
\]

\[
= \frac{1}{2a-r} \binom{a-1}{a-r}.
\]

Now we go on plugging the results (2.3) and (2.8) from above in (2.2) and find

\[
\frac{v_{d}^{(1)}}{v_{d}^{(0)}} = \sum_{a=2}^{d} (-1)^{d-a} a \binom{d}{a} \binom{d}{a} \left( \frac{-2}{a} \right) ^{r-2} \left( \sum_{r=1}^{a} \frac{1}{2a-r} \binom{a-1}{a-r} \right)
\]

\[
= \sum_{a=2}^{d} (-1)^{a+1} a \binom{d}{a} \binom{d}{a} \left( \sum_{r=0}^{a} (-2)^{r-1} \frac{1}{2a-r-1} \binom{a-1}{r} \right) - (-2)^{d-2} \frac{1}{a}.
\]

In order to get rid of the inner sum we use an identity that may be proved as an application of the classical reflection law

\[
\frac{1}{(1-z)^{a}} F \left( a, b \left| c \right. \right) = F \left( a, c+b \left| c \right. \right)
\]

(2.10)

for hypergeometric functions by J.F. Pfaff [8], namely

\[
\sum_{k=0}^{m} (-2)^{k} \frac{1}{2m-k+1} \binom{m}{k} = (-1)^{m} 2^{2m} \left( \frac{m}{2m} \right) \quad (m \geq 0),
\]

(2.11)

cf. [6, identity (5.104)]. In this way we find

\[
\frac{v_{d}^{(1)}}{v_{d}^{(0)}} = \sum_{a=2}^{d} (-1)^{d+a+1} a \binom{d}{a} \binom{d+a}{d} \left( \frac{2^{2a-3}}{2a-1} \frac{1}{2a-2} \binom{2a-2}{2a-2} \frac{1}{2a} \right)
\]

\[
= \sum_{a=2}^{d} (-1)^{d+a} a \binom{d}{a} \binom{d+a}{d} \left( \frac{2^{2a-2}}{2a-2} \frac{1}{2a} \right),
\]

i.e.

\[
\frac{v_{d}^{(1)}}{v_{d}^{(0)}} = \sum_{a=2}^{d} (-1)^{d+a} 2^{a-2} \binom{d+a}{2a} \left( \binom{2a}{a} - 2^{a} \right),
\]

(2.12)
so that we have proved that the ratio \( \frac{v_d^{(1)}}{v_d^{(0)}} \) is an integer.

In the last step of the proof we establish the explicit formula for the ratios. Recall the Legendre polynomials \( P_d(x) \), as defined in the theorem, and let

\[
\rho_d(x) := \sum_{k=0}^{d} \binom{d+k}{d-k} x^k
\]

(2.13)
denote the associated Legendre polynomials (cf. [10, p. 66]). Then (2) yields

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = (-1)^d P_d(-3) - \rho_d(-4) \quad (2.14)
\]

Now (cf. [9, p. 158])

\[
P_d(-x) = (-1)^d P_d(x).
\]

(2.15)
Furthermore \( \rho_d \) satisfies the recursive formula

\[
\rho_d(x) = (x+2)\rho_{d-1}(x) - \rho_{d-2}(x)
\]

(2.16)
\( \rho_0(x) = 0, \rho_1(x) = x + 1 \)

(cf. [10, p. 66]) so that \((-1)^d \rho_d(-4) = 2d+1\), which completes the proof of (2.1).

\[\Box\]

3. Recurrence, Asymptotic Behaviour, and Probabilities

In this section we apply Theorem 2.1 in order to establish a recurrence for the quotients \( \frac{v_d^{(1)}}{v_d^{(0)}} \) as well as to establish the asymptotic behaviour of this sequence for \( d \to \infty \) and its consequence on the probabilities \( \frac{v_d^{(1)}}{v_d} \).

Since the Legendre polynomials satisfy the recursive formula

\[
dP_d(x) - (2d-1)xP_{d-1}(x) + (d-1)P_{d-2}(x) = 0 \quad (d \geq 2),
\]

\[
P_0(x) = 1, \ P_1(x) = x
\]

(3.1)
(cf. [9, p. 160]) we get the following second order linear recurrence for \( \frac{v_d^{(1)}}{v_d^{(0)}} \).

\textbf{Corollary 3.1.} We have

\[
d\frac{v_d^{(1)}}{v_d^{(0)}} - 3(2d-1)\frac{v_{d-1}^{(1)}}{v_{d-1}^{(0)}} + (d-1)\frac{v_{d-2}^{(1)}}{v_{d-2}^{(0)}} = 2d(d-1) \text{ for } d \geq 2, \ \frac{v_0^{(1)}}{v_0^{(0)}} = \frac{v_1^{(1)}}{v_1^{(0)}} = 0.
\]

We turn our attention now to the asymptotic behaviour of the ratios for \( d \to \infty \) and start by their generating function. The generating function of the Legendre polynomials is given by ([10, p. 78])

\[
\sum_{d \geq 0} P_d(x)z^d = \frac{1}{\sqrt{1 - 2xz + z^2}}.
\]

(3.2)
so that the generating function of our ratios reads

**Corollary 3.2.** We have

\[ V_1(z) := \sum_{d \geq 0} \frac{v_d^{(1)}}{v_d^{(0)}} z^d = \frac{1}{4} \left( \frac{1}{\sqrt{1 - 6z + z^2}} - \frac{1 + z}{(1 - z)^2} \right). \]

Performing singularity analysis the latter result allows to establish the asymptotic behaviour of the ratios for \( d \to \infty \) as follows.

**Proposition 3.3.** For \( d \to \infty \)

\[ \frac{v_d^{(1)}}{v_d^{(0)}} = \frac{1}{8 \sqrt{2} \sqrt{3 - 2 \sqrt{2}}} (3 + 2 \sqrt{2})^{d + \frac{1}{2}} \left( 1 + O \left( \frac{1}{d} \right) \right). \]

**Proof.** We adopt the usual technique of singularity analysis of generating functions, compare e.g. [5, Chapter IV] or [12, Chapter 8]. The dominating singularity of the generating function \( V_1(z) \) is given by the zero \( 3 - 2 \sqrt{2} \) of \( 1 - 6z + z^2 \) closest to the origin, whereas the other zero of \( 1 - 6z + z^2 \) as well as the term \( \frac{1 + z}{(1 - z)^2} \) will give a contribution that is exponentially smaller than the contribution of the main term.

The local expansion of \( V_1(z) \) about the dominating singularity reads

\[ V_1(z) = \frac{1}{8 \sqrt{2} \sqrt{3 - 2 \sqrt{2}}} \left( 1 - \frac{z}{3 - 2 \sqrt{2}} \right)^{-1/2} \left( 1 + O \left( 1 - \frac{z}{3 - 2 \sqrt{2}} \right) \right) \]

for \( z \to 3 - 2 \sqrt{2} \), from which the asymptotics is immediate. \( \square \)

In [1] Akiyama and Pethő also discussed the probabilities

\[ p_d^{(s)} := \frac{v_d^{(s)}}{v_d} \]  \hspace{1cm} (3.3)

for a contractive normed polynomial of degree \( d \) in \( \mathbb{R}[x] \) to have \( s \) pairs of complex conjugate roots. In particular they derived (cf. [1, Theorem 6.1])

\[ \log p_d^{(0)} = -\frac{\log 2}{2} d^2 + \frac{1}{8} \log d + O(1), \quad \text{for } d \to \infty, \]  \hspace{1cm} (3.4)

for the probability of totally real polynomials and, by numerical evidence for \( d \leq 100 \), conjectured that

\[ \log p_d^{(1)} \leq -\frac{\log 2}{2} d^2 + d \log q \]  \hspace{1cm} (3.5)

for some constant \( q \). Now, obviously, \( p_d^{(1)} = \frac{v_d^{(1)}}{v_d^{(0)}} p_d^{(0)} \), so that from (3.4) and our Proposition 3.3 we gain
Corollary 3.4. The probability $p_d^{(1)}$ for a contractive normed polynomial of degree $d$ in $\mathbb{R}[x]$ to have exactly one pair of complex conjugate roots fulfills

$$ \log p_d^{(1)} = -\frac{\log 2}{2} d^2 + d \log(3 + 2\sqrt{2}) + O(\log d) \quad \text{for } d \to \infty. $$

4. Concluding Remarks

In this paper we were able to settle the instance $s = 1$ of Conjecture 1.1. The question arises, whether our methods could be used to prove the conjecture for additional instances of $s \geq 2$ or even for general $s \geq 1$. A crucial point for a possible application of our method would be to establish a generalization of the Selberg-Aomoto integral for integrands that will occur with the evaluation of $v_d^{(s)}$ for $s \geq 2$ similar to formula (1.2) in the instance $s = 1$. Work is in progress on this question, but even the explicit evaluation of the integrals that appear in instance $s = 2$ seems to be very hard.

References


