THE NUMBER OF REPRESENTATIONS OF A NUMBER AS SUMS OF VARIOUS POLYGONAL NUMBERS

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Received: 9/22/11, Revised: 8/9/12, Accepted: 9/19/12, Published: 10/8/12

Abstract
In this paper, we present twenty-five analogues of Jacobi’s two-square theorem which involve squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers.

1. Introduction
Jacobi’s celebrated two-square theorem is as follows.

Theorem 1.1. ([7]). Let \( r\{ \Box + \Box \}(n) \) denote the number of representations of \( n \) as a sum of two squares and \( d_{4,3}(n) \) denote the number of positive divisors of \( n \) congruent to \( i \) modulo \( j \). Then
\[
r\{ \Box + \Box \}(n) = 4(d_{1,4}(n) - d_{3,4}(n)). \tag{1}
\]

Simple proofs of (1) can be seen in [2] and [4]. Similar representation theorems involving squares and triangular numbers were found by Dirichlet [3], Lorenz [10], Legendre [9], and Ramanujan [1]. For example, another classical result due to Lorenz [10] is stated below.

Theorem 1.2. Let \( r\{ l\Box + m\Box \}(n) \) denote the number of representations of \( n \) as a sum of \( l \) times a square and \( m \) times a square. Then
\[
r\{ \Box + 3\Box \}(n) = 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)). \tag{2}
\]
In [5], M.D. Hirschhorn obtained sixteen identities (including those obtained by Legendre and Ramanujan) simply by dissecting the \( q \)-series representations of the identities obtained by Jacobi, Dirichlet and Lorenz. Hirschhorn [6] further extended his work and obtained twenty-nine more identities involving squares, triangular numbers, pentagonal numbers and octagonal numbers. For more work on this topic one can see [8], [11] and [12]. In [12], R. S. Melham presented an informal account of analogues of Jacobi’s two-square theorem which are verified using computer algorithms.

In this paper, we find twenty-five more such identities involving squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers, by employing Ramanujan’s theta function identities.

For \( k \geq 3 \), the \( n^{th} \) \( k \)-gonal number \( F_k(n) \) is given by

\[
F_k := F_k(n) = \frac{(k - 2)n^2 - (k - 4)n}{2}.
\]

By allowing the domain for \( F_k(n) \) to be the set of all integers, we see that the generating function \( G_k(q) \) of \( F_k(n) \) is given by

\[
G_k(q) = \sum_{n=-\infty}^{\infty} q^{F_k(n)} = \sum_{n=-\infty}^{\infty} q^{\frac{(k - 2)n^2 - (k - 4)n}{2}}.
\]

We note an exception for the case \( k = 3 \). We observe that \( G_3(q) \) generates each triangular number twice while \( G_6(q) \) generates each only once. As such, we take \( G_6(q) \) as the generating function for triangular numbers instead of \( G_3(q) \). We further observe that

\[
G_k(q) = f(q, q^{k-3}),
\]

where \( f(a, b) \) is Ramanujan’s general theta function defined by [1, p. 34, Eq. (18.1)]:

\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.
\]

Two important special cases of \( f(a, b) \) are

\[
\varphi(q) := f(q, q),
\]

\[
\psi(q) := f(q, q^3).
\]

In view of (3), the respective generating functions of squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers,
hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers are

\[
\begin{align*}
G_4(q) &= f(q, q) = \varphi(q), \\
G_6(q) &= f(q, q^3) = \psi(q), \\
G_5(q) &= f(q, q^2), \\
G_7(q) &= f(q, q^4), \\
G_8(q) &= f(q, q^5), \\
G_{10}(q) &= f(q, q^7), \\
G_{11}(q) &= f(q, q^8), \\
G_{12}(q) &= f(q, q^9),
\end{align*}
\]

and

\[
G_{18}(q) = f(q, q^{15}).
\]

In Section 2, we give dissections of \(\varphi(q), \psi(q), G_5(q),\) and \(G_{12}(q)\) and recall some identities established in [5] and [6]. In the remaining five sections, we successively present sets of identities involving decagonal numbers, hendecagonal numbers, dodecagonal numbers, heptagonal numbers, and octadecagonal numbers.

2. Preliminary Results

Let \(U_n = a^{n(n+1)/2}b^{n(n-1)/2}\) and \(V_n = a^{n(n-1)/2}b^{n(n+1)/2}\) for each integer \(n\). Then we have [1, p. 48, Entry 31]

\[
f(a, b) = f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f \left( \frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r} \right).
\]

Replacing \(a\) by \(q^a\) and \(b\) by \(q^b\), we find that

\[
f(q^a, q^b) = \sum_{r=0}^{n-1} \left( \frac{a+b}{2} \right) r^2 + \left( \frac{a-b}{2} \right) r
\]

\[
\times f \left( q^{\left( \frac{a+b}{2} \right) n^2 + (a+b)nr + \left( \frac{a-b}{2} \right) n}, q^{\left( \frac{a+b}{2} \right) n^2 - (a+b)nr - \left( \frac{a-b}{2} \right) n} \right).
\]

(4)

Setting \(a = b = 1\) and then letting \(n = 3, 5\) and \(8\) in (4), we obtain

\[
\begin{align*}
\varphi(q) &= \varphi(q^3) + 2qG_8(q^3), \\
\varphi(q) &= \varphi(q^{25}) + 2qA(q^5) + 2q^4G_{12}(q^5),
\end{align*}
\]

(5)  (6)
and

$$\varphi(q) = \varphi(q^{64}) + 2qB(q^{16}) + 2q^4\psi(q^{32}) + 2q^8G_{10}(q^{16}) + 2q^{16}\psi(q^{128}),$$  \hspace{1cm} (7)

respectively, where \( A(q) = f(q^3, q^7) \) and \( B(q) = f(q^3, q^5) \).

Setting \( a = 1, b = 3 \) and then putting \( n = 2, 4 \) and \( 6 \) in (4), we deduce that

$$\psi(q) = B(q^2) + qG_{10}(q^2),$$  \hspace{1cm} (8)

$$\psi(q) = f(q^{28}, q^{36}) + qf(q^{20}, q^{44}) + f(q^{12}, q^{52}) + q^6G_{18}(q^4),$$  \hspace{1cm} (9)

and

$$\psi(q) = f(q^{66}, q^{78}) + qB(q^{18}) + q^3f(q^{42}, q^{102}) + q^6f(q^{30}, q^{114}) + q^{10}G_{10}(q^{18}) + q^{15}G_{20}(q^6),$$  \hspace{1cm} (10)

respectively.

Setting \( a = 1, b = 0 \) and then choosing \( n = 3, 5 \) in (4) and noting that \( \psi(q) = \frac{1}{2}f(1, q) \), we obtain

$$\psi(q) = G_5(q^3) + q\psi(q^9)$$  \hspace{1cm} (11)

and

$$\psi(q) = C(q^5) + qG_7(q^5) + q^3\psi(q^{25}),$$  \hspace{1cm} (12)

respectively, where \( C(q) = f(q^2, q^3) \).

Next, setting \( a = 1, b = 2 \) and \( n = 3 \) in (4), we find that

$$G_5(q) = f(q^{12}, q^{15}) + qf(q^6, q^{21}) + q^2G_{11}(q^3).$$  \hspace{1cm} (13)

Again, setting \( a = 1, b = 9 \) and \( n = 2 \) in (4), we obtain

$$G_{12}(q) = A(q^4) + qG_7(q^8).$$  \hspace{1cm} (14)

We also require a few identities deduced in [5] and [6]. Throughout the sequel, \( r\{lF_i + mF_j\}(n) \) denotes the number of representations of \( n \) as a sum of \( l \) times a polygonal number \( F_i \) and \( m \) times a polygonal number \( F_j \). Note that \( r\{2\Box + \triangle\}(n) \) that appears in (16) is \( r\{2F_4 + F_6\}(n) \). However, we have kept the former notation in those cases which involve squares and/or triangular numbers. The first seven of the following identities appeared in [5] as equations (1.1), (1.3), (1.4), (1.5), (1.11), (1.12), and (1.14), respectively, while the last six identities appeared in [6] as equations (1.2), (1.3), (1.4), (1.6), (1.13), and (1.14), respectively.
\[ r\{\triangle + \triangle\}(n) = d_{1,4}(4n + 1) - d_{3,4}(4n + 1), \]
\[ r\{\square + \triangle\}(n) = d_{1,4}(8n + 1) - d_{3,4}(8n + 1), \]
\[ r\{\triangle + 4\triangle\}(n) = \frac{1}{2}(d_{1,4}(8n + 5) - d_{3,4}(8n + 5)), \]
\[ r\{\triangle + 2\triangle\}(n) = \frac{1}{2}(d_{1,4}(8n + 3) + d_{3,8}(8n + 3) - d_{5,8}(8n + 3) - d_{7,8}(8n + 3)), \]
\[ r\{6\square + \triangle\}(n) = d_{3,8}(8n + 1) - d_{2,3}(8n + 1), \]
\[ r\{\triangle + 12\triangle\}(n) = \frac{1}{2}(d_{1,3}(8n + 13) - d_{2,3}(8n + 13)), \]
\[ r\{3\triangle + 4\triangle\}(n) = \frac{1}{2}(d_{1,3}(8n + 7) - d_{2,3}(8n + 7)), \]
\[ r\{\triangle + 4F_5\}(n) = d_{1,24}(24n + 7) + d_{19,24}(24n + 7) - d_{5,24}(24n + 7) \]
\[ \quad - d_{23,24}(24n + 7), \]
\[ r\{3\triangle + F_5\}(n) = d_{1,12}(12n + 5) - d_{11,12}(12n + 5), \]
\[ r\{3\triangle + 2F_5\}(n) = d_{1,8}(24n + 11) - d_{7,8}(24n + 11), \]
\[ r\{6\triangle + F_5\}(n) = d_{1,8}(24n + 19) - d_{7,8}(24n + 19), \]
\[ r\{3\square + F_5\}(n) = d_{1,8}(24n + 1) + d_{3,8}(24n + 1) - d_{5,8}(24n + 1) \]
\[ \quad - d_{7,8}(24n + 1), \]
\[ r\{3\square + 4F_5\}(n) = d_{1,8}(6n + 1) + d_{3,8}(6n + 1) - d_{5,8}(6n + 1) - d_{7,8}(6n + 1). \]

### 3. Identities Involving Decagonal Numbers

**Theorem 3.1.** We have

\[ r\{\square + 3F_{10}\}(n) = d_{1,3}(16n + 27) - d_{2,3}(16n + 27), \]
\[ r\{2\triangle + 3F_{10}\}(n) = \frac{1}{2}(d_{1,3}(16n + 31) - d_{2,3}(16n + 31)), \]
\[ r\{2\triangle + F_{10}\}(n) = \frac{1}{2}(d_{1,4}(16n + 13) - d_{3,4}(16n + 13)), \]
\[ r\{\square + F_{10}\}(n) = d_{1,4}(16n + 9) - d_{3,4}(16n + 9), \]
\[ r\{6\triangle + F_{10}\}(n) = \frac{1}{2}(d_{1,3}(16n + 21) - d_{2,3}(16n + 21)). \]
\[ r\{3□ + F_{10}\}(n) = d_{1,3}(16n + 9) - d_{2,3}(16n + 9), \]
\[ r\{F_8 + F_{10}\}(n) = \frac{1}{2}(d_{1,3}(48n + 43) - d_{2,3}(48n + 43)), \]
\[ r\{F_5 + 3F_{10}\}(n) = d_{1,8}(48n + 83) - d_{7,8}(48n + 83), \]
\[ r\{2F_5 + F_{10}\}(n) = d_{1,24}(48n + 31) + d_{19,24}(48n + 31) - d_{5,24}(48n + 31) - d_{23,24}(48n + 31), \]
\[ r\{\Delta + F_{10}\}(n) = \frac{1}{2}(d_{1,8}(16n + 11) + d_{3,8}(16n + 11) - d_{5,8}(16n + 11) - d_{7,8}(16n + 11)). \]

**Proof.** Identity (19) is equivalent to
\[ \varphi(q^6)\psi(q) = \sum_{n \geq 0} (d_{1,3}(8n + 1) - d_{2,3}(8n + 1))q^n. \] (38)

Employing (10) in (38), we have
\[ \varphi(q^6)(f(q^{66}, q^{78}) + qB(q^{18}) + q^3f(q^{42}, q^{102}) + q^6f(q^{30}, q^{114}) + q^{10}G_{10}(q^{18}) + q^{15}G_{26}(q^6)) = \sum_{n \geq 0} (d_{1,3}(8n + 1) - d_{2,3}(8n + 1))q^n. \] (39)

Extracting the terms involving \(q^{6n+4}\) in (39) and then dividing the resulting identity by \(q^4\) and replacing \(q^6\) by \(q\), we find that
\[ q\varphi(q)G_{10}(q^3) = \sum_{n \geq 0} (d_{1,3}(48n + 33) - d_{2,3}(48n + 33))q^n. \] (40)

Equating the coefficients of \(q^{n+1}\) on both sides of (40) and noting that \(d_{1,3}(48n + 33) = d_{1,3}(16n + 11)\) and \(d_{2,3}(48n + 33) = d_{2,3}(16n + 11)\), we arrive at (28).

Next, (20) is equivalent to
\[ \psi(q)\psi(q^{12}) = \frac{1}{2} \sum_{n \geq 0} (d_{1,3}(8n + 13) - d_{2,3}(8n + 13))q^n, \]
which, with the aid of (10), can be rewritten as
\[ \psi(q^{12})(f(q^{66}, q^{78}) + qB(q^{18}) + q^3f(q^{42}, q^{102}) + q^6f(q^{30}, q^{114}) + q^{10}G_{10}(q^{18}) + q^{15}G_{26}(q^6)) = \frac{1}{2} \sum_{n \geq 0} (d_{1,3}(8n + 13) - d_{2,3}(8n + 13))q^n. \] (41)

Collecting the terms in (41) in which the power of \(q\) is congruent to 4 modulo 6, we find that
\[ q\psi(q^3)G_{10}(q^3) = \frac{1}{2} \sum_{n \geq 0} (d_{1,3}(48n + 45) - d_{2,3}(48n + 45))q^n. \] (42)
Equating the coefficients of $q^{n+1}$ on both sides of (42) and noting that $d_{1,13}(48n + 45) = d_{1,13}(16n + 15)$ and $d_{2,13}(48n + 45) = d_{2,13}(16n + 15)$, we arrive at (29).

Identity (1) is equivalent to

$$\varphi^2(q) = 1 + 4 \sum_{n \geq 1} (d_{1,4}(n) - d_{3,4}(n))q^n, \quad (43)$$

which can be rewritten, with the aid of (7), as

$$(\varphi(q^{64}) + 2qB(q^{16}) + 2q^4\psi(q^{32}) + 2q^9G_{10}(q^{16}) + 2q^{16}\psi(q^{128}))^2$$

$$= 1 + 4 \sum_{n \geq 1} (d_{1,4}(n) - d_{3,4}(n))q^n. \quad (44)$$

Now, we extract those terms in (44) where the power of $q$ is congruent to 13 modulo 16, divide the resulting identity by $q^{13}$ and replace $q^{16}$ by $q$, to obtain

$$\psi(q^2)G_{10}(q) = \frac{1}{2} \sum_{n \geq 0} (d_{1,4}(16n + 13) - d_{3,4}(16n + 13))q^n,$$

which readily yields (30).

Next, extracting those terms in (44) where the power of $q$ is congruent to 9 modulo 16, then dividing the resulting identity by $q^9$ and replacing $q^{16}$ by $q$, we have

$$G_{10}(q)(\varphi(q^4) + 2q\psi(q^8)) = \sum_{n \geq 0} (d_{1,4}(16n + 9) - d_{3,4}(16n + 9))q^n. \quad (45)$$

But, setting $a = b = 1$ and $n = 2$ in (4), or from [1, p. 40, Entries 25(i) and 25(ii)], we have

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \quad (46)$$

Employing (46) in (45), we find that

$$\varphi(q)G_{10}(q) = \sum_{n \geq 0} (d_{1,4}(16n + 9) - d_{3,4}(16n + 9))q^n,$$

which implies (31).

Now, (2) is equivalent to

$$\varphi(q)\varphi(q^3) = 1 + 2 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n \geq 1} (d_{4,12}(n) - d_{8,12}(n))q^n$$

$$= 1 + 2 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^{4n}. \quad (47)$$
Employing (7) in (47), we have
\[
(\varphi(q^{64}) + 2qB(q^{16}) + 2q^4\psi(q^{32}) + 2q^9G_{10}(q^{16}) + 2q^{16}\psi(q^{128}))
\times (\varphi(q^{102}) + 2q^3B(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^{27}G_{10}(q^{48}) + 2q^{38}\psi(q^{384}))
\]
\[= 1 + 2 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n \geq 1}(d_{1,3}(n) - d_{2,3}(n))q^{4n}. \quad (48)
\]
Extracting the terms in (48) involving \(q^{16n+5}\), then dividing the resulting identity by \(q^5\) and replacing \(q^{16}\) by \(q\), we find that
\[q\psi(q^5)G_{10}(q) = \frac{1}{2} \sum_{n \geq 0} (d_{1,3}(16n + 5) - d_{2,3}(16n + 5))q^n,
\]
from which (32) can be easily deduced.
Again, using (5) in (40), we have
\[q(\varphi(q^{3}) + 2qG_{8}(q^{3}))G_{10}(q^{3}) = \sum_{n \geq 0}(d_{1,3}(16n + 11) - d_{2,3}(16n + 11))q^n. \quad (49)
\]
Separating the terms involving \(q^{3n+1}\) and \(q^{3n+2}\) in (49), we obtain
\[\varphi(q^{3})G_{10}(q) = \sum_{n \geq 0}(d_{1,3}(48n + 27) - d_{2,3}(48n + 27))q^n \quad (50)
\]
and
\[2G_{8}(q)G_{10}(q) = \sum_{n \geq 0}(d_{1,3}(48n + 43) - d_{2,3}(48n + 43))q^n, \quad (51)
\]
respectively. Now the identities (33) and (34) follow easily from (50) and (51), respectively.
Next, (24) is equivalent to
\[\psi(q^{3})G_{5}(q^{2}) = \sum_{n \geq 0}(d_{1,8}(24n + 11) - d_{7,8}(24n + 11))q^n. \quad (52)
\]
Invoking (8) in (52), we have
\[(B(q^6) + q^3G_{10}(q^6))G_{5}(q^2) = \sum_{n \geq 0}(d_{1,8}(24n + 11) - d_{7,8}(24n + 11))q^n. \quad (53)
\]
Extracting the terms involving \(q^{2n+1}\) in (53), we obtain
\[qG_{10}(q^3)G_{5}(q) = \sum_{n \geq 0}(d_{1,8}(48n + 35) - d_{7,8}(48n + 35))q^n. \quad (54)
\]
Comparing the coefficients of $q^{n+1}$ on both sides of (54), we arrive at (35).

Identity (22) is equivalent to

$$
\psi(q)G_5(q^4)
= \sum_{n \geq 0} (d_{1,24}(24n+7) + d_{19,24}(24n+7) - d_{5,24}(24n+7) - d_{23,24}(24n+7))q^n.
$$

Using (8) in (55), we have

$$
(B(q^2) + qG_{10}(q^2))G_5(q^4) = \sum_{n \geq 0} (d_{1,24}(24n+7) + d_{19,24}(24n+7)
- d_{5,24}(24n+7) - d_{23,24}(24n+7))q^n.
$$

Extracting the terms involving odd powers of $q$ in (56), we obtain

$$
G_{10}(q)G_5(q^2)
= \sum_{n \geq 0} (d_{1,24}(48n+31) + d_{19,24}(48n+31) - d_{5,24}(48n+31) - d_{23,24}(48n+31))q^n,
$$

which readily yields (36).

Identity (18) is equivalent to

$$
\psi(q)\psi(q^2) = \frac{1}{2} \sum_{n \geq 0} (d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3))q^n,
$$

which, with the aid of (8), can be written as

$$
(B(q^2) + qG_{10}(q^2))\psi(q^2)
= \frac{1}{2} \sum_{n \geq 0} (d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3))q^n.
$$

Extracting the terms involving $q^{2n+1}$ in (57), we obtain

$$
G_{10}(q)\psi(q)
= \frac{1}{2} \sum_{n \geq 0} (d_{1,8}(16n+11) + d_{3,8}(16n+11) - d_{5,8}(16n+11) - d_{7,8}(16n+11))q^n.
$$

Equating the coefficients of $q^n$ on both sides of (58), we arrive at (37).
4. Identities Involving Hendecagonal Numbers

**Theorem 4.1.** We have

\[
\begin{align*}
    r\{\triangle + F_{11}\}(n) &= d_{1,12}(36n + 29) - d_{11,12}(36n + 29), \\
    r\{\triangle + 2F_{11}\}(n) &= d_{1,8}(72n + 107) - d_{7,8}(72n + 107), \\
    r\{2\triangle + F_{11}\}(n) &= d_{1,8}(72n + 67) - d_{7,8}(72n + 67), \\
    r\{\square + F_{11}\}(n) &= d_{1,8}(72n + 49) + d_{3,8}(72n + 49) \\
    &\quad - d_{5,8}(72n + 49) - d_{7,8}(72n + 49), \\
    r\{\square + 4F_{11}\}(n) &= d_{1,8}(18n + 49) + d_{3,8}(18n + 49) \\
    &\quad - d_{5,8}(18n + 49) - d_{7,8}(18n + 49), \\
    r\{F_{10} + F_{11}\}(n) &= d_{1,8}(144n + 179) - d_{7,8}(144n + 179).
\end{align*}
\]

**Proof.** Identity (23) is equivalent to

\[
\psi(q^3)G_5(q) = \sum_{n \geq 0} (d_{1,12}(12n + 5) - d_{11,12}(12n + 5))q^n,
\]

which we rewrite, by (13), as

\[
\begin{align*}
    \psi(q^3)(f(q^{12}, q^{15}) + qf(q^6, q^{21}) + q^2G_{11}(q^3)) \\
    &= \sum_{n \geq 0} (d_{1,12}(12n + 5) - d_{11,12}(12n + 5))q^n.
\end{align*}
\]

Extracting the terms involving \(q^{3n+2}\) in (65), we obtain

\[
\psi(q)G_{11}(q) = \sum_{n \geq 0} (d_{1,12}(36n + 29) - d_{11,12}(36n + 29))q^n,
\]

which readily yields (59).

Next, (24) is equivalent to

\[
\psi(q^3)G_5(q^2) = \sum_{n \geq 0} (d_{1,8}(24n + 11) - d_{7,8}(24n + 11))q^n.
\]

Invoking (13) in (66), we find that

\[
\begin{align*}
    \psi(q^3)(f(q^{24}, q^{30}) + q^2f(q^{12}, q^{42}) + q^4G_{11}(q^6)) \\
    &= \sum_{n \geq 0} (d_{1,8}(24n + 11) - d_{7,8}(24n + 11))q^n.
\end{align*}
\]
Extracting the terms involving $q^{3n+1}$ in (67), we obtain
\[ q\psi(q)G_{11}(q^2) = \sum_{n \geq 0} (d_{1,8}(72n + 35) - d_{7,8}(72n + 35))q^n, \]  
(68)
from which (60) follows.
Again, (25) is equivalent to
\[ \psi(q^6)G_5(q) = \sum_{n \geq 0} (d_{1,8}(24n + 19) - d_{7,8}(24n + 19))q^n. \]  
(69)
Using (13) in (69), we have
\[ \psi(q^6)(f(q^{12}, q^{15}) + qf(q^6, q^{21}) + q^2G_{11}(q^3)) \]
\[ = \sum_{n \geq 0} (d_{1,8}(24n + 19) - d_{7,8}(24n + 19))q^n. \]  
(70)
Extracting the terms involving $q^{3n+2}$ in (70), we obtain
\[ \psi(q^2)G_{11}(q) = \sum_{n \geq 0} (d_{1,8}(72n + 67) - d_{7,8}(72n + 67))q^n, \]
which gives (61).
Identity (26) is equivalent to
\[ \varphi(q^3)G_5(q) = \sum_{n \geq 0} (d_{1,8}(24n + 1) + d_{3,8}(24n + 1) - d_{5,8}(24n + 1) - d_{7,8}(24n + 1))q^n, \]
and by (13), we have
\[ \varphi(q^3)(f(q^{12}, q^{15}) + qf(q^6, q^{21}) + q^2G_{11}(q^3)) \]
\[ = \sum_{n \geq 0} (d_{1,8}(24n + 1) + d_{3,8}(24n + 1) - d_{5,8}(24n + 1) - d_{7,8}(24n + 1))q^n. \]  
(71)
Extracting the terms involving $q^{3n+2}$ in (71), we obtain
\[ \varphi(q)G_{11}(q) \]
\[ = \sum_{n \geq 0} (d_{1,8}(72n + 49) + d_{3,8}(72n + 49) - d_{5,8}(72n + 49) - d_{7,8}(72n + 49))q^n, \]
which readily yields (62).
Identity (27) is equivalent to
\[ \varphi(q^3)G_5(q^4) = \sum_{n \geq 0} (d_{1,8}(6n + 1) + d_{3,8}(6n + 1) - d_{5,8}(6n + 1) - d_{7,8}(6n + 1))q^n. \]  
(72)
Using (13) in (72), we have
\[
\varphi(q^3)(f(q^{48}, q^{60}) + q^4f(q^{24}, q^{84}) + q^8G_{11}(q^{12})) \\
= \sum_{n \geq 0} (d_{1,8}(6n + 1) + d_{3,8}(6n + 1) - d_{5,8}(6n + 1) - d_{7,8}(6n + 1))q^n. \tag{73}
\]
Extracting the terms involving \(q^{3n+2}\) in (73), we find that
\[
q^2\varphi(q)G_{11}(q^4) = \sum_{n \geq 0} (d_{1,8}(18n + 13) + d_{3,8}(18n + 13) \\
- d_{5,8}(18n + 13) - d_{7,8}(18n + 13))q^n,
\]
which readily yields (63).

Again, employing (8) in (68), we obtain
\[
q(B(q^2) + qG_{10}(q^2))G_{11}(q^2) = \sum_{n \geq 0} (d_{1,8}(72n + 35) - d_{7,8}(72n + 35))q^n. \tag{74}
\]
Comparing the terms in (74) where the powers of \(q\) are even, we find that
\[
qG_{10}(q)G_{11}(q) = \sum_{n \geq 0} (d_{1,8}(144n + 35) - d_{7,8}(144n + 35))q^n. \tag{75}
\]
Equating the coefficients of \(q^{n+1}\) in (75), we arrive at (64).

\[\square\]

5. Identities Involving Dodecagonal Numbers

**Theorem 5.1.** We have
\[
r\{5\Diamond + F_{12}\}(n) = d_{1,4}(5n + 4) - d_{3,4}(5n + 4), \tag{76}
\]
\[
r\{F_{12} + F_{12}\}(n) = d_{1,4}(5n + 8) - d_{3,4}(5n + 8), \tag{77}
\]
\[
r\{5\Delta + F_{12}\}(n) = \frac{1}{2}(d_{1,4}(20n + 17) - d_{3,4}(20n + 17)). \tag{78}
\]

**Proof.** Employing (6) in (43), we find that
\[
(\varphi(q^{25}) + 2qA(q^5) + 2q^4G_{12}(q^5))^2 = 1 + 4 \sum_{n \geq 1} (d_{1,4}(n) - d_{3,4}(n))q^n. \tag{79}
\]
Extracting those terms in (79) in which the power of \(q\) is congruent to 4 modulo 5, we obtain
\[
\varphi(q^5)G_{12}(q) = \sum_{n \geq 0} (d_{1,4}(5n + 4) - d_{3,4}(5n + 4))q^n,
\]
from which (76) follows.

Again, extracting the terms involving $q^{5n+3}$ in (79), we have

$$qG_{12}^2(q) = \sum_{n \geq 0} (d_{1,4}(5n + 3) - d_{3,4}(5n + 3))q^n,$$  \hspace{1cm} (80)

which immediately gives (77).

Furthermore, extracting the terms involving $q^{5n+2}$ in (79), we find that

$$A^2(q) = \sum_{n \geq 0} (d_{1,4}(5n + 2) - d_{3,4}(5n + 2))q^n.$$  \hspace{1cm} (81)

But, from [1, p. 46, Entries 30(v) and 30(vi)], we have

$$A^2(q) = f^2(q^3, q^7) = A(q^2)\varphi(q^{10}) + 2q^3G_{12}(q^4)\psi(q^{20}).$$  \hspace{1cm} (82)

From (81) and (82), we obtain

$$A(q^2)\varphi(q^{10}) + 2q^3G_{12}(q^4)\psi(q^{20}) = \sum_{n \geq 0} (d_{1,4}(5n + 2) - d_{3,4}(5n + 2))q^n.$$  \hspace{1cm} (83)

Collecting the terms involving $q^{4n+3}$ in (83), we find that

$$2G_{12}(q)\psi(q^5) = \sum_{n \geq 0} (d_{1,4}(20n + 17) - d_{3,4}(20n + 17))q^n,$$

which readily yields (78).

\[ \square \]

6. Identities Involving Heptagonal Numbers

**Theorem 6.1.** We have

$$r\{F_7 + F_7\}(n) = d_{1,4}(20n + 9) - d_{3,4}(20n + 9),$$  \hspace{1cm} (84)

$$r\{5\Delta + F_7\}(n) = \frac{1}{2} (d_{1,4}(20n + 17) - d_{3,4}(20n + 17)), $$  \hspace{1cm} (85)

$$r\{2F_{12} + F_7\}(n) = \frac{1}{2} (d_{1,4}(40n + 73) - d_{3,4}(40n + 73)).$$  \hspace{1cm} (86)

**Proof.** With the aid of (14), we rewrite (80) as

$$q(A(q^4) + qG_7(q^8))^2 = \sum_{n \geq 0} (d_{1,4}(5n + 3) - d_{3,4}(5n + 3))q^n.$$  \hspace{1cm} (87)

Extracting the terms involving $q^{8n+3}$ in (87), we find that

$$G_{12}^2(q) = \sum_{n \geq 0} (d_{1,4}(40n + 18) - d_{3,4}(40n + 18))q^n.$$  \hspace{1cm} (88)
Equating the coefficients of $q^n$ in (88) and noting the fact that $d_{1.4}(40n + 18) = d_{1.4}(20n + 9)$ and $d_{3.4}(40n + 18) = d_{3.4}(20n + 9)$, we arrive at (84).

Next, (15) is equivalent to

$$
\psi^2(q) = \sum_{n \geq 0} (d_{1.4}(4n + 1) - d_{3.4}(4n + 1))q^n. \tag{89}
$$

Invoking (12) in (89), we obtain

$$(C(q^5) + qG_7(q^5) + q^3\psi(q^{25}))^2 = \sum_{n \geq 0} (d_{1.4}(4n + 1) - d_{3.4}(4n + 1))q^n. \tag{90}$$

Extracting the terms involving $q^{5n+4}$ in (90), we get

$$2G_7(q)\psi(q^5) = \sum_{n \geq 0} (d_{1.4}(20n + 17) - d_{3.4}(20n + 17))q^n. \tag{91}$$

Equating the coefficients of $q^n$ in (91), we easily arrive at (85).

Next, (16) is equivalent to

$$\varphi(q^2)\psi(q) = \sum_{n \geq 0} (d_{1.4}(8n + 1) - d_{3.4}(8n + 1))q^n. \tag{92}$$

Using (6) and (12) in (92), we find that

$$(\varphi(q^{50}) + 2q^2A(q^{10}) + 2q^8G_{12}(q^{10}))(C(q^5) + qG_7(q^5) + q^3\psi(q^{25}))
= \sum_{n \geq 0} (d_{1.4}(8n + 1) - d_{3.4}(8n + 1))q^n. \tag{93}$$

Extracting the terms involving $q^{5n+4}$ in (93), we obtain

$$2qG_{12}(q^2)G_7(q) = \sum_{n \geq 0} (d_{1.4}(40n + 33) - d_{3.4}(40n + 33))q^n,$$

from which (86) can be deduced by equating the coefficients of $q^{n+1}$. \qed

7. Identities Involving Octadecagonal Numbers

**Theorem 7.1.** We have

$$r\{F_5 + F_{18}\}(n) = d_{1.24}(96n + 151) + d_{19.24}(96n + 151)
- d_{5.24}(96n + 151) - d_{23.24}(96n + 151), \tag{94}$$

$$r\{\triangle + F_{18}\}(n) = \frac{1}{2}(d_{1.4}(32n + 53) - d_{3.4}(32n + 53)), \tag{95}$$

$$r\{3\triangle + F_{18}\}(n) = \frac{1}{2}(d_{1.3}(32n + 61) - d_{2.3}(32n + 61)). \tag{96}$$
Proof. Identity (22) is equivalent to
\[ \psi(q)G_5(q^4) \]
\[ = \sum_{n \geq 0} (d_{1,24}(24n + 7) + d_{19,24}(24n + 7) - d_{5,24}(24n + 7) - d_{23,24}(24n + 7))q^n. \] (97)

Employing (9) in (97), we have
\[ (f(q^{28}, q^{36}) + qf(q^{36}, q^{44}) + q^3f(q^{12}, q^{50}) + q^6G_{18}(q^4))G_5(q^4) \]
\[ = \sum_{n \geq 0} (d_{1,24}(24n + 7) + d_{19,24}(24n + 7) - d_{5,24}(24n + 7) - d_{23,24}(24n + 7))q^n. \] (98)

Extracting those terms in (98) in which the power of \( q \) is congruent to 2 modulo 4, we obtain
\[ qG_{18}(q))G_5(q) \]
\[ = \sum_{n \geq 0} (d_{1,24}(96n + 55) + d_{19,24}(96n + 55) - d_{5,24}(96n + 55) - d_{23,24}(96n + 55))q^n, \]
which readily implies (94).

Again, (17) is equivalent to
\[ \psi(q)\psi(q^4) = \frac{1}{2} \sum_{n \geq 0} (d_{1,4}(8n + 5) - d_{3,4}(8n + 5))q^n. \] (99)

Using (9) in (99), we have
\[ (f(q^{28}, q^{36}) + qf(q^{36}, q^{44}) + q^3f(q^{12}, q^{50}) + q^6G_{18}(q^4))\psi(q^4) \]
\[ = \frac{1}{2} \sum_{n \geq 0} (d_{1,4}(8n + 5) - d_{3,4}(8n + 5))q^n. \] (100)

Extracting the terms involving \( q^{4n+2} \) from both sides of the above, we obtain
\[ qG_{18}(q)\psi(q) = \frac{1}{2} \sum_{n \geq 0} (d_{1,4}(32n + 21) - d_{3,4}(32n + 21))q^n, \]
which readily implies (95).

Next, (21) is equivalent to
\[ \psi(q^3)\psi(q^4) = \frac{1}{2} \sum_{n \geq 0} (d_{1,3}(8n + 7) - d_{2,3}(8n + 7))q^n. \] (101)
With the help of (9) and (11), we rewrite (101) as
\[
(f(q^{104}, q^{108}) + q^3 f(q^{60}, q^{132}) + q^9 f(q^{36}, q^{156}) + q^{18} G_{18}(q^{12})(G_{5}(q^{12}) + q^4 \psi(q^{36}))
\]
\[
= \frac{1}{2} \sum_{n \geq 0} (d_{1,3}(8n + 7) - d_{2,3}(8n + 7))q^n.
\]
(102)

Extracting the terms involving \(q^{12n+10}\) in (102), we obtain
\[
q G_{18}(q) \psi(q^2) = \frac{1}{2} \sum_{n \geq 0} (d_{1,3}(96n + 87) - d_{2,3}(96n + 87))q^n.
\]

Equating the coefficients of \(q^{n+1}\) and noting that \(d_{1,3}(96n + 87) = d_{1,3}(32n + 29)\)
and \(d_{2,3}(96n + 87) = d_{2,3}(32n + 29)\), we deduce (96) to finish the proof. \(\square\)

Acknowledgment. The authors would like to thank the referee for his/her helpful comments.

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