ON A VARIATION OF THE COIN EXCHANGE PROBLEM FOR ARITHMETIC PROGRESSIONS

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Abstract

Let $a_1, a_2, \ldots, a_k$ be relatively prime, positive integers arranged in increasing order. Let $\Gamma^*$ denote the positive integers in the set $\{a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_j \geq 0\}$. Let

$$S^*(a_1, a_2, \ldots, a_k) = \{ n \notin \Gamma^* : n + \Gamma^* \subseteq \Gamma^* \}.$$ 

We determine $S^*(a_1, a_2, \ldots, a_k)$ in the case where the $a_j$’s are in arithmetic progression. In particular, this determines $g(a_1, a_2, \ldots, a_k)$ in this particular case.

1. Introduction

Let $a_1, a_2, \ldots, a_k$ be relatively prime, positive integers arranged in increasing order. Let $\Gamma$ denote $\{a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_j \geq 0\}$, and let $\Gamma^* = \Gamma \setminus \{0\}$. It is well known and easy to show that $\Gamma^* \simeq \mathbb{N} \setminus \Gamma$ is a finite set. We use the classical notation $g(a_1, a_2, \ldots, a_k)$ to denote the largest number in $\Gamma^*$. J.J. Sylvester [15] showed that $g(a_1, a_2) = a_1a_2 - a_1 - a_2$. In later years, the number of elements in $\Gamma^*$, denoted by $n(a_1, a_2, \ldots, a_k)$, was also studied, and it was shown that $n(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. Another function related to this is the function $s(a_1, a_2, \ldots, a_k)$ that denotes the sum of elements in $\Gamma^*$. Introduced in [4], it was shown that $s(a_1, a_2) = (a_1 - 1)(a_2 - 1)(2a_1a_2 - a_1 - a_2 - 1)/12$.

There is a neat formula for each of the functions $g$ and $n$ when the $a_j$’s are in arithmetic progression ([1],[5],[9],[16]), but other results obtained are mostly partial results ([2],[3],[6],[7],[10],[11],[12],[13],[14]) and often not as neat. Due to an obvious connection with making change given money of different denominations, this problem is also known as the Coin Exchange Problem.
2. Main Result

We study a variation of the Coin Exchange Problem in this note. We denote by $S^*(a_1, a_2, \ldots, a_k)$ the set of all $n \in \Gamma^c$ such that

$$n + \Gamma^* \subseteq \Gamma^*,$$

and let $g^*(a_1, a_2, \ldots, a_k)$ (respectively, $n^*(a_1, a_2, \ldots, a_k)$ and $s^*(a_1, a_2, \ldots, a_k)$) denote the least (respectively, the number and sum of) elements in $S^*$. Since $g(a_1, a_2, \ldots, a_k)$ is the largest element in $S^*$, we have

$$g^*(a_1, a_2, \ldots, a_k) \leq g(a_1, a_2, \ldots, a_k),$$

and $n^*(a_1, a_2, \ldots, a_k) \geq 1$, with equality if and only if $g^* = g$. This problem arises from looking at the generators for the Derivation modules of certain curves [8], and has been extensively studied.

For each $j$, $1 \leq j \leq a_1 - 1$, let $m_j$ denote the least number in $\Gamma$ congruent to $j$ (mod $a_1$). Then $m_j - a_1$ is the largest number in $\Gamma^c$ congruent to $j$ (mod $a_1$), and no number less than this in this residue class can be in $S^*$, for they would differ by a multiple of $a_1$, an element in $\Gamma^*$. Therefore,

$$S^*(a_1, a_2, \ldots, a_k) \subseteq \{ m_j - a_1 : 1 \leq j \leq a_1 - 1 \},$$

$$g^*(a_1, a_2, \ldots, a_k) \leq \left( \max_{1 \leq j \leq a_1 - 1} m_j \right) - a_1 = g(a_1, a_2, \ldots, a_k),$$

$$n^*(a_1, a_2, \ldots, a_k) \leq a_1 - 1,$$

and

$$s^*(a_1, a_2, \ldots, a_k) \leq \sum_{j=1}^{a_1-1} m_j - a_1(a_1 - 1).$$

More precisely,

$$m_j - a_1 \in S^*(a_1, a_2, \ldots, a_k) \iff (m_j - a_1) + m_i \geq m_{j+i} \text{ for } 1 \leq i \leq a_1 - 1. \tag{5}$$

We shall explicitly evaluate the set $S^*$, and as a consequence, the functions $g$, $g^*$, $n^*$ and $s^*$, when the $a_j$’s are in arithmetic progression. We write $a_j = a + (j - 1)d$ for $1 \leq j \leq k$, and assume $\gcd(a, d) = 1$. In this case, we denote the functions $g$, $g^*$, $n^*$ and $s^*$ by $g(a, d; k)$, $g^*(a, d; k)$, $n^*(a, d; k)$ and $s^*(a, d; k)$, respectively. To determine $S^*(a, d; k)$, we recall Lemma 2 from [16].

**Lemma:** For each $t$, $1 \leq t \leq a - 1$, the least integer in $\Gamma^*$ congruent to $dt$ (mod $a$)
is given by \( a(1 + \left\lceil \frac{k-1}{k-1} \right\rceil) + dt \).

**Theorem:** Let \( a, d \) be relatively prime, positive integers, and let \( k \geq 2 \). If \( a - 1 = q(k - 1) + r \), with \( 1 \leq r \leq k - 1 \), then

\[
S^*(a, d; k) = \left\{ a \left\lceil \frac{x-1}{k-1} \right\rceil + dx : a - r \leq x \leq a - 1 \right\}.
\]

**Proof:** Fix \( k \geq 2 \). Throughout this proof, and elsewhere, by \( x \mod m \) we mean \( x - x[m] \). By (1) and Lemma,

\[
S^*(a, d; k) \subseteq \left\{ a \left\lceil \frac{x-1}{k-1} \right\rceil + dx : 1 \leq x \leq a - 1 \right\}.
\]

From (5), \( n = a\left\lceil \frac{x-1}{k-1} \right\rceil + dx \in S^* \) if and only if for each \( y \) with \( 1 \leq y \leq a - 1 \),

\[
a \left( 1 + \left\lceil \frac{(x+y) \mod a}{k-1} \right\rceil \right) + d((x+y) \mod a) \leq \left\{ a \left\lceil \frac{x-1}{k-1} \right\rceil + dx \right\} + \left\{ a \left( 1 + \left\lceil \frac{y-1}{k-1} \right\rceil \right) + dy \right\},
\]

or,

\[
a \left\lceil \frac{(x+y) \mod a}{k-1} \right\rceil + d((x+y) \mod a) \leq a \left\{ \left\lceil \frac{x-1}{k-1} \right\rceil + \left\lceil \frac{y-1}{k-1} \right\rceil \right\} + d(x+y). \quad (6)
\]

Suppose \( 2 \leq k \leq a - 1 \). Let \( a - 1 = q(k - 1) + r \), with \( 1 \leq r \leq k - 1 \). Unless \( x = a - 1 \), \( x + y \leq a - 1 \) for at least one \( y \), for such a \( y \), (6) reduces to proving the inequality

\[
\left\lceil \frac{x+y-1}{k-1} \right\rceil \leq \left\lceil \frac{x-1}{k-1} \right\rceil + \left\lceil \frac{y-1}{k-1} \right\rceil.
\]

If we now write \( x = q_1(k - 1) + r_1 \), \( y = q_2(k - 1) + r_2 \), with \( 1 \leq r_1, r_2 \leq k - 1 \), the reduced inequality above fails to hold precisely when \( r_1 + r_2 \geq k \). Given \( x \), and hence \( r_1 \), the choice \( y = r_2 = k - r_1 \) will thus ensure that (6) fails to hold provided \( x + y \leq a - 1 \). However, such a choice for \( y \) is not possible precisely when \( x \geq q(k - 1) + 1 = a - r \), so that (6) always holds in only these cases. Finally, it is easy to verify that (6) holds if \( x = a - 1 \). This shows \( S^* = \{ a\left\lceil \frac{x-1}{k-1} \right\rceil + dx : a - r \leq x \leq a - 1 \} \) if \( 2 \leq k \leq a - 1 \).

If \( k \geq a \), (6) reduces to \( d((x+y) \mod a) \leq d(x+y) \). Thus, \( S^* = \{ dx : 1 \leq x \leq a - 1 \} \), as claimed, since \( r = a - 1 \) and \( \left\lceil \frac{x-1}{k-1} \right\rceil = 0 \) in this case. This completes the proof. \( \square \)

**Corollary:** If \( a, d \) be relatively prime, positive integers, \( k \geq 2 \), and \( a - 1 = q(k - 1) + r \), with \( 1 \leq r \leq k - 1 \), then

\[
g(a, d; k) = aq + d(a - 1),
\]
\[ g^*(a, d; k) = aq + d(a - r), \]
\[ n^*(a, d; k) = r, \]

and

\[ s^*(a, d; k) = aqr + \frac{1}{2}dr(2a - r - 1). \]

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**References**


