Research Article

Coupled Fixed-Point Theorems for Contractions in Partially Ordered Metric Spaces and Applications

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Bhaskar and Lakshmikantham (2006) showed the existence of coupled coincidence points of a mapping \( F \) from \( X \times X \) into \( X \) and a mapping \( g \) from \( X \) into \( X \) with some applications. The aim of this paper is to extend the results of Bhaskar and Lakshmikantham and improve the recent fixed-point theorems due to Bessem Samet (2010). Indeed, we introduce the definition of generalized \( g \)-Meir-Keeler type contractions and prove some coupled fixed point theorems under a generalized \( g \)-Meir-Keeler-contractive condition. Also, some applications of the main results in this paper are given.

1. Introduction

The Banach contraction principle [1] is a classical and powerful tool in nonlinear analysis and has been generalized by many authors (see [2–15] and others).

Recently, Bhaskar and Lakshmikantham [16] introduced the notion of a coupled fixed-point of the given two variables mapping. More precisely, let \( X \) be a nonempty set and \( F : X \times X \to X \) be a given mapping. An element \( (x, y) \in X \times X \) is called a coupled fixed-point of the mapping \( F \) if

\[
F(x, y) = x, \quad F(y, x) = y.
\]  

(1.1)
They also showed the uniqueness of a coupled fixed-point of the mapping $F$ and applied their theorems to the problems of the existence and uniqueness of a solution for a periodic boundary value problem.

**Theorem 1.1** (see Zeidler [15]). Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on $X$. Assume that there exists $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)]$$

(1.2)

for all $x \geq u$ and $y \leq v$. Moreover, if there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \quad y_0 \geq F(y_0, x_0),$$

(1.3)

then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Later, in [17], Lakshmikantham and Ćirić investigated some more coupled fixed-point theorems in partially ordered sets, and some others obtained many results on coupled fixed-point theorems in cone metric spaces, intuitionistic fuzzy normed spaces, ordered cone metric spaces and topological spaces (see, e.g., [18–25]).

In [9], Meir and Keeler generalized the well-known Banach fixed-point theorem [1] as follows.

**Theorem 1.2** (Meir and Keeler [9]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a given mapping. Suppose that, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$e \leq d(x, y) < e + \delta(\epsilon) \implies d(T(x), T(y)) < \epsilon$$

(1.4)

for all $x, y \in X$. Then $T$ admits a unique fixed-point $x_0 \in X$ and, for all $x \in X$, the sequence $\{T^n(x)\}$ converges to $x_0$.

**Proposition 1.3** (see [17]). Let $(X, d)$ be a partially ordered metric space and $F : X \times X \to X$ be a given mapping. If the contraction (1.2) is satisfied, then $F$ is a generalized Meir-Keeler type contraction.

Motivated by the results of Bhaskar and Lakshmikantham [16], Lakshmikantham and Ćirić [17], and Samet [26], in this paper, we introduce the definition of $g$-Meir-Keeler-contractive mappings and prove some coupled fixed-point theorems under a generalized $g$-Meir-Keeler contractive condition.

**2. Main Results**

Let $X$ be a nonempty set. We note that an element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$ for all $x, y \in X$. Also, we say that $F$ and $g$ are commutative (or commuting) if $g(F(x, y)) = F(g(x), g(y))$ for all $x, y \in X$.

We introduce the following two definitions.
**Definition 2.1.** Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \to X\) and \(g : X \to X\). We say that \(F\) has the mixed strict \(g\)-monotone property if, for any \(x, y \in X\),

\[
x_1, x_2 \in X, \quad g(x_1) < g(x_2) \implies F(x_1, y) < F(x_2, y),
\]

\[
y_1, y_2 \in X, \quad g(y_1) < g(y_2) \implies F(x, y_1) > F(x, y_2).
\]

\[(2.1)\]

**Definition 2.2.** Let \((X, \leq)\) be a partially ordered set and \(d\) be a metric on \(X\). Let \(F : X \times X \to X\) and \(g : X \to X\) be two given mappings. We say that \(F\) is a generalized \(g\)-Meir-Keeler type contraction if, for all \(\epsilon > 0\), there exists \(\delta(\epsilon) > 0\) such that, for all \(x, y, u, v \in X\) with \(g(x) \leq g(u)\) and \(g(y) \geq g(v)\),

\[
e \leq \frac{1}{2} [d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta(\epsilon) \implies d(F(x, y), F(u, v)) < \epsilon.
\]

\[(2.2)\]

**Lemma 2.3.** Let \((X, \leq)\) be a partially ordered set and \(d\) be a metric on \(X\). Let \(F : X \times X \to X\) and \(g : X \to X\) be two given mappings. If \(F\) is a generalized \(g\)-Meir-Keeler type contraction, then we have

\[
d(F(x, y), F(u, v)) < \frac{1}{2} [d(g(x), g(u)) + d(g(y), g(v))]
\]

\[(2.3)\]

for all \(x, y, u, v\) with \(g(x) < g(u), g(y) \geq g(v)\) or \(g(x) \leq g(u), g(y) > g(v)\).

**Proof.** Let \(x, y, u, v \in X\) such that \(g(x) < g(u)\) and \(g(y) \geq g(v)\) or \(g(x) \leq g(u)\) and \(g(y) > g(v)\). Then \(d(g(x), g(u)) + d(g(y), g(v)) > 0\). Since \(F\) is a generalized \(g\)-Meir-Keeler type contraction, for \(\epsilon = (1/2)[d(g(x), g(u)) + d(g(y), g(v))]\), there exists \(\delta(\epsilon) > 0\) such that, for all \(x_0, y_0, u_0, v_0 \in X\) with \(g(x_0) \leq g(u_0)\) and \(g(y_0) \geq g(v_0)\),

\[
e \leq \frac{1}{2} [d(g(x_0), g(u_0)) + d(g(y_0), g(v_0))] < \epsilon + \delta(\epsilon) \implies d(F(x_0, y_0), F(u_0, v_0)) < \epsilon.
\]

\[(2.4)\]

Therefore, putting \(x_0 = x, y_0 = y, u_0 = u\) and \(v_0 = v\), we have

\[
d(F(x, y), F(u, v)) < \frac{1}{2} [d(g(x), g(u)) + d(g(y), g(v))].
\]

\[(2.5)\]

This completes the proof. \(\square\)

From now on, we suppose that \((X, \leq)\) is a partially ordered set, and there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space.
Theorem 2.4. Let $F : X \times X \to X$ and $g : X \to X$ be such that $F(X \times X) \subseteq g(X)$, $g$ is continuous and commutative with $F$. Also, suppose that

(a) $F$ has the mixed strict $g$-monotone property;
(b) $F$ is a generalized $g$-Meir-keeler type contraction;
(c) there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$.

Then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$; that is, $F$ and $g$ have a coupled coincidence in $X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again, from $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$.

Continuing this process, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$g(x_{n+1}) = F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n)$$

for all $n \geq 0$.

Now, we show that

$$g(x_n) < g(x_{n+1}), \quad g(y_n) > g(y_{n+1})$$

for all $n \geq 0$. For $n = 0$, we have

$$g(x_0) < F(x_0, y_0) = g(x_1), \quad g(y_0) > F(y_0, x_0) = g(y_1).$$

Since $F$ has the mixed strict $g$-monotone property, then we have

$$g(x_0) < g(x_1) \Rightarrow F(x_0, y_1) < F(x_1, y_1),$$

$$g(y_0) > g(y_1) \Rightarrow F(x_0, y_0) < F(x_0, y_1).$$

It follows that $F(x_0, y_0) < F(x_1, y_1)$, that is, $g(x_1) < g(x_2)$.

Similarly, we have

$$g(y_1) < g(y_0) \Rightarrow F(y_1, x_0) < F(y_0, x_0),$$

$$g(x_1) > g(x_0) \Rightarrow F(y_1, x_1) < F(y_1, x_0).$$

Thus it follows that $F(y_1, x_1) < F(y_0, x_0)$, that is, $g(y_2) < g(y_1)$.

Again, we have

$$g(x_1) < g(x_2) \Rightarrow F(x_1, y_2) < F(x_2, y_2),$$

$$g(y_1) > g(y_2) \Rightarrow F(x_1, y_1) < F(x_1, y_2).$$

Thus it follows that $F(x_1, y_1) < F(x_2, y_2)$, that is, $g(x_2) < g(x_3)$. 
Similarly, we have
\[ g(y_2) < g(y_1) \implies F(y_2, x_1) < F(y_1, x_1), \]
\[ g(x_2) > g(x_1) \implies F(y_2, x_2) < F(y_1, x_1). \]  
(2.12)

Thus it follows that \( F(y_2, x_2) < F(y_1, x_1) \), that is, \( g(y_3) < g(y_2) \).

Continuing this process for each \( n \geq 1 \), we get the following:
\[ g(x_0) < g(x_1) < g(x_2) < \cdots < g(x_n) < g(x_{n+1}) < \cdots, \]
\[ g(y_0) > g(y_1) > g(y_2) > \cdots > g(y_n) > g(y_{n+1}) > \cdots. \]  
(2.13)

Denote that
\[ \delta_n := d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})). \]  
(2.14)

Since \( g(x_{n-1}) < g(x_n) \) and \( g(y_{n-1}) > g(y_n) \), it follows from (2.6) and Lemma 2.3 that
\[ d(g(x_n), g(x_{n+1})) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \]
\[ < \frac{1}{2} [d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))]. \]  
(2.15)

Thus it follows from (2.14)–(2.16) that \( \delta_n < \delta_{n-1} \). This means that the sequence \( \{\delta_n/2\} \) is monotone decreasing. Therefore, there exists \( \delta^* \geq 0 \) such that \( \lim_{n \to \infty} \delta_n/2 = \delta^* \), that is,
\[ \lim_{n \to \infty} \frac{1}{2} [d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))] = \delta^*. \]  
(2.17)

Now, we show that \( \delta^* = 0 \). Suppose that \( \delta^* > 0 \) hold. Let \( \delta^* = \epsilon \). Then there exists a positive integer \( m \) such that
\[ \epsilon \leq \frac{1}{2} [d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] < \epsilon + \delta(\epsilon). \]  
(2.18)

Then, by using (2.7) and the condition (b), we have
\[ d(F(x_m, y_m), F(x_{m+1}, y_{m+1})) < \epsilon, \]  
(2.19)
and so, by (2.6), we have
\[ d(g(x_{m+1}), g(x_{m+2})) < \epsilon. \] (2.20)

On the other hand, by (2.15), we have
\[ \frac{1}{2} \left[ d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1})) \right] < \epsilon, \] (2.21)
which is a contradiction with (2.18). Thus we have \( \epsilon = \delta^* = 0 \), that is,
\[ \lim_{n \to \infty} \frac{1}{2} \left[ d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \right] = 0, \] (2.22)
that is,
\[ \lim_{n \to \infty} \delta_n = 0. \] (2.23)

Now, we prove that \( \{g(x_n)\} \) and \( \{g(y_n)\} \) are Cauchy sequences in \( X \). Suppose that at least one of \( \{g(x_n)\} \) or \( \{g(y_n)\} \) is not a Cauchy sequence. Then there exist \( \epsilon > 0 \) and two subsequences \( \{l_k\}, \{m_k\} \) of integers such that \( m_k > l_k \geq k \) and
\[ d(g(x_{l_k}), g(x_{m_k})) \geq \frac{\epsilon}{2}, \quad d(g(y_{l_k}), g(y_{m_k})) \geq \frac{\epsilon}{2} \] (2.24)
for all \( k \geq 1 \). Thus we have
\[ r_k = d(g(x_{l_k}), g(x_{m_k})) + d(g(y_{l_k}), g(y_{m_k})) \geq \epsilon \] (2.25)
for all \( k \geq 1 \). Let \( m_k \) be the smallest number exceeding \( l_k \) such that (2.25) holds. Then we have
\[ d(g(x_{l_k}), g(x_{m_{k-1}})) + d(g(y_{l_k}), g(y_{m_{k-1}})) < \epsilon. \] (2.26)
Thus, from (2.14), (2.25), (2.26) and the triangle inequality, it follows that
\[ \epsilon \leq r_k \]
\[ \leq d(g(x_{l_k}), g(x_{m_{k-1}})) + d(g(x_{m_{k-1}}), g(x_{m_k})) \]
\[ + d(g(y_{l_k}), g(y_{m_{k-1}})) + d(g(y_{m_{k-1}}), g(y_{m_k})) \]
\[ < \epsilon + \delta_{m_{k-1}} \]
and so
\[ \epsilon \leq \lim_{k \to \infty} r_k \leq \lim_{k \to \infty} (\epsilon + \delta_{m_{k-1}}). \] (2.27)
Hence, by (2.23), we have
\[
\lim_{k \to \infty} r_k = e^+.
\] (2.29)

It follows from (2.6), (2.14), and the triangle inequality that
\[
\begin{align*}
\quad r_k &= d(g(x_k), g(x_{m_k})) + d(g(y_k), g(y_{m_k})) \\
&\leq d(g(x_k), g(x_{k+1})) + d(g(x_{k+1}), g(x_{m_k})) + d(g(x_{m_k}), g(x_{m_k+1})) \\
&\quad + d(g(y_k), g(y_{k+1})) + d(g(y_{k+1}), g(y_{m_k})) + d(g(y_{m_k}), g(y_{m_k+1})) \\
&= \delta_{m_k} + \delta_{m_k} + d(g(x_{k+1}), g(x_{m_k})) + d(g(y_{k+1}), g(y_{m_k})) \\
&\quad + d(F(x_k, y_k), F(x_{m_k}, y_{m_k})) + d(F(y_k, x_k), F(y_{m_k}, x_{m_k})).
\end{align*}
\] (2.30)

Form (2.13) we have \(g(x_k) < g(x_{m_k})\) and \(g(y_k) > g(y_{m_k})\). Now, it follows from Lemma 2.3 and (2.30) that
\[
\begin{align*}
\quad r_k < \delta_{l_k} + \delta_{m_k} + d(g(x_k), g(x_{m_k})) + d(g(y_k), g(y_{m_k})),
\end{align*}
\] (2.31)

that is,
\[
\begin{align*}
\quad r_k < \delta_{l_k} + \delta_{m_k} + r_k.
\end{align*}
\] (2.32)

This is a contradiction. Therefore, \(\{g(x_n)\}\) and \(\{g(y_n)\}\) are Cauchy sequences. Since \(X\) is complete, there exist \(x, y \in X\) such that
\[
\begin{align*}
\quad \lim_{n \to \infty} g(x_n) = x, \quad \lim_{n \to \infty} g(y_n) = y.
\end{align*}
\] (2.33)

Since \(\{g(x_n)\}\) is monotone increasing and \(\{g(y_n)\}\) is monotone decreasing, we have
\[
\begin{align*}
\quad g(x_n) < x, \quad g(y_n) > y
\end{align*}
\] (2.34)
for all \(n \geq 1\). Thus it follows from (2.33) and the continuity of \(g\) that
\[
\begin{align*}
\quad \lim_{n \to \infty} g(x_n) = g(x), \quad \lim_{n \to \infty} g(y_n) = g(y).
\end{align*}
\] (2.35)

Thus, for all \(m \geq 1\), there exists a positive integer \(n_0\) such that, for all \(n \geq n_0\),
\[
\begin{align*}
\quad d(g(x_n), g(x)) < \frac{1}{4m}, \quad d(g(y_n), g(y)) < \frac{1}{4m}.
\end{align*}
\] (2.36)
Theorem 2.6. Suppose that all the hypotheses of Theorem 2.4 hold and, further, for all $(x, y), (x^*, y^*) \in X \times X$, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then $F$ and $g$ have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that

$$x = g(x) = F(x, y), \quad y = g(y) = F(y, x).$$

Corollary 2.5. Let $F : X \times X \rightarrow X$ be a mapping satisfying the following conditions:

(a) $F$ has the mixed strict monotone property;

(b) $F$ is a generalized Meir-Keeler type contraction;

(c) there exists $x_0, y_0 \in X$ such that $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$.

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. The conclusion follows from Theorem 2.4 by putting $g = I$ (: the identity mapping) on $X$. 

Now, we introduce the product space $X \times X$ with the following partial order: for all $(x, y), (u, v) \in X \times X$,

$$(u, v) \leq (x, y) \iff u < x, \quad v \geq y.$$
Proof. By Theorem 2.4, the set of coupled coincidences of the mapping $F$ and $g$ is nonempty. First, we show that, if $(x, y)$ and $(x^*, y^*)$ are coupled coincidence points of $F$ and $g$, that is, if

$$
g(x) = F(x, y), \quad g(y) = F(y, x), \quad g(x^*) = F(x^*, y^*), \quad g(y^*) = F(y^*, x^*),$$

(2.41)

then we have

$$
g(x) = g(x^*), \quad g(y) = g(y^*).$$

(2.42)

Put $u_0 = u, v_0 = v$ and choose $u_1, v_1 \in X$ such that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then, similarly as in the proof of Theorem 2.4, we can inductively define the sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$
g(u_{n+1}) = F(u_n, v_n), \quad g(v_{n+1}) = F(v_n, u_n)$$

(2.43)

for all $n \geq 0$. Also, if we set $x_0 = x, y_0 = y, x_0^* = x^*, \text{ and } y_0^* = y^*$, then we can define the sequences $\{g(x_n)\}, \{g(y_n)\}, \{g(x_n^*)\}, \text{ and } \{g(y_n^*)\}$ as follows:

$$
g(x_{n+1}) = F(x_n, y_n), \quad g(y_{n+1}) = F(y_n, x_n),$$

$$
g(x_{n+1}^*) = F(x_n^*, y_n^*), \quad g(y_{n+1}^*) = F(y_n^*, x_n^*)$$

(2.44)

for all $n \geq 0$. Since

$$
(F(x, y), F(y, x)) = (g(x_1), g(y_1)) = (g(x), g(y)),
$$

$$
(F(u, v), F(v, u)) = (g(u_1), g(v_1))
$$

(2.45)

are comparable each other, then $g(x) < g(u_1)$ and $g(y) \geq g(v_1)$. It is easy to show that $g(x), g(y)$, and $(g(u_n), g(v_n))$ are comparable each other, that is, $g(x) < g(u_n)$ and $g(y) \geq g(v_n)$ for all $n \geq 1$. Thus it follows from Lemma 2.3 that

$$
d(g(x), g(u_{n+1})) + d(g(y), g(v_{n+1}))
$$

$$
= d(F(x, y), F(u_n, v_n)) + d(F(y, x), F(v_n, u_n))
$$

$$
< \frac{1}{2} [d(g(x), g(u_n)) + d(g(y), g(v_n))] + \frac{1}{2} [d(g(y), g(v_n)) + d(g(x), g(u_n))]
$$

(2.46)

$$
= d(g(x), g(u_n)) + d(g(y), g(v_n))
$$

and so

$$
\frac{1}{2} [d(g(x), g(u_{n+1})) + d(g(y), g(v_{n+1}))] < \frac{1}{2^n} [d(g(x), g(u_1)) + d(g(y), g(v_1))] \rightarrow 0
$$

(2.47)
as \( n \to \infty \). Therefore, we have

\[
\lim_{n \to \infty} d\left(g(x), g(u_{n+1})\right) = 0, \quad \lim_{n \to \infty} d\left(g(y), g(v_{n+1})\right) = 0. \tag{2.48}
\]

Similarly, we can prove that

\[
\lim_{n \to \infty} d\left(g(x^*), g(u_{n+1})\right) = 0, \quad \lim_{n \to \infty} d\left(g(y^*), g(v_{n+1})\right) = 0. \tag{2.49}
\]

Thus, by the triangle inequality, (2.48) and (2.49), we have

\[
d\left(g(x), g(x^*)\right) \leq d\left(g(x), g(u_{n+1})\right) + d\left(g(x^*), g(u_{n+1})\right) \to 0,
\]

\[
d\left(g(y), g(y^*)\right) \leq d\left(g(y), g(v_{n+1})\right) + d\left(g(y^*), g(v_{n+1})\right) \to 0 \tag{2.50}
\]

as \( n \to \infty \), which imply that \( g(x) = g(x^*) \) and \( g(y) = g(y^*) \).

Now, we prove that \( g(x) = x \) and \( g(y) = y \). Denote that \( g(x) = z \) and \( g(x) = w \). Since \( g(x) = F(x, y) \) and \( g(y) = F(y, x) \), by the commutativity of \( F \) and \( g \), we have

\[
g(z) = g(g(x)) = g(F(x, y)) = F(g(x), g(y)) = F(z, w), \tag{2.51}
\]

\[
g(w) = g(g(y)) = g(F(y, x)) = F(g(y), g(x)) = F(w, z). \tag{2.52}
\]

Thus, \((z, w)\) is a coupled coincidence point of \( F \) and \( g \).

Putting \( x^* = z \) and \( y^* = w \) in (2.52), it follows from (2.42) that

\[
z = g(x) = g(x^*) = g(z), \quad w = g(y) = g(y^*) = g(w) \tag{2.53}
\]

and so, from (2.51) and (2.52),

\[
z = g(z) = F(z, w), \quad w = g(w) = F(w, z). \tag{2.54}
\]

Therefore, \((z, w)\) is a coupled common fixed-point of \( F \) and \( g \).

Finally, to prove the uniqueness of the coupled common fixed-point of \( F \) and \( g \), assume that \((p, q)\) is another coupled common fixed-point of \( F \) and \( g \). Then, by (2.42), we have \( p = g(p) = g(z) = z \) and \( q = g(q) = g(w) = w \). This completes the proof. \( \square \)

**Corollary 2.7.** Suppose that all the hypotheses of Corollary 2.5 hold and, further, for all \((x, y)\) and \((x^*, y^*) \in X \times X\), there exists \((u, v) \in X \times X\) that is comparable with \((x, y)\) and \((x^*, y^*)\). Then there exists a unique \( x \in X \) such that \( x = F(x, x) \).
3. Applications

Now, we give some applications of the main results in Section 2.

**Theorem 3.1.** Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings. Assume that there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(a) $\varphi(0) = 0$ and $\varphi(t) > 0$ for any $t > 0$;

(b) $\varphi$ is nondecreasing and right continuous;

(c) for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, for all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$,

\[
\epsilon \leq \varphi\left(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]\right) < \epsilon + \delta(\epsilon) \implies \varphi[d(F(x, y), F(u, v))] < \epsilon.
\]

(3.1)

Then $F$ is a generalized $g$-Meir-Keeler type contraction.

**Proof.** For any $\epsilon > 0$, it follows from (a) that $\varphi(\epsilon) > 0$ and so there exists $\alpha > 0$ such that, for all $u, v, u^*, v^* \in X$ with $g(u) \leq g(u^*)$ and $g(v) \geq g(v^*)$,

\[
\varphi(\epsilon) \leq \varphi\left(\frac{1}{2}[d(g(u), g(u^*)) + d(g(v), g(v^*))]\right) < \varphi(\epsilon) + \alpha
\]

\[
\implies \varphi[d(F(u, v), F(u^*, v^*))] < \varphi(\epsilon).
\]

(3.2)

From the right continuity of $\varphi$, there exists $\delta > 0$ such that $\varphi(\epsilon + \delta) < \varphi(\epsilon) + \alpha$. For any $x, y, u, v \in X$ such that $g(x) \leq g(u)$, $g(y) \geq g(v)$ and

\[
\epsilon \leq \frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))] < \epsilon + \delta,
\]

(3.3)

since $\varphi$ is nondecreasing function, we get the following:

\[
\varphi(\epsilon) \leq \varphi\left(\frac{1}{2}[d(g(x), g(u)) + d(g(y), g(v))]\right) < \varphi(\epsilon + \alpha) < \varphi(\epsilon) + \alpha.
\]

(3.4)

By (3.2), we have $\varphi[d(F(x, y), F(u, v))] < \varphi(\epsilon)$ and so $d(F(x, y), F(u, v)) < \epsilon$. Therefore, it follows that $F$ is a generalized $g$-Meir-Keeler type contraction. This completes the proof. \(\Box\)

**Corollary 3.2** (see [26, Theorem 3.1]). Let $F : X \times X \to X$ be a given mapping. Assume that there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(a) $\varphi(0) = 0$ and $\varphi(t) > 0$ for any $t > 0$;

(b) $\varphi$ is nondecreasing and right continuous;
(c) for any \( e > 0 \), there exists \( \delta(e) > 0 \) such that \( x \leq u, y \geq v \) and

\[
e \leq \varphi \left( \frac{1}{2} [d(x, u) + d(y, v)] \right) < e + \delta(e) \iff \varphi \left[ d(F(x, y), F(u, v)) \right] < e.
\]

(3.5)

Then \( F \) is a generalized Meir-Keeler type contraction.

The following result is an immediate consequence of Theorems 2.4 and 3.1.

**Corollary 3.3.** Let \( F : X \times X \to X \) and \( g : X \to X \) be two given mappings such that \( F(X \times X) \subseteq g(X) \), \( g \) is continuous and commutative with \( F \). Also, suppose that

(a) \( F \) has the mixed strict \( g \)-monotone property;

(b) for any \( e > 0 \), there exists \( \delta(e) > 0 \) such that, for all \( x, y, u, v \in X \) with \( g(x) \leq g(u) \) and \( g(y) \geq g(v) \),

\[
e \leq \int_0^{(1/2)[d(g(x), g(u)) + d(g(y), g(v))]} \varphi(t)dt < e + \delta(e) \iff \int_0^{d(F(x, y), F(u, v))} \varphi(t)dt < e,
\]

where \( \varphi \) is a locally integrable function from \([0, +\infty)\) into itself satisfying the following condition:

\[
\int_0^s \varphi(t)dt > 0
\]

for all \( s > 0 \);

(c) there exist \( x_0, y_0 \in X \) such that \( g(x_0) < F(x_0, y_0) \) and \( g(y_0) > F(y_0, x_0) \).

Then there exists \( (x, y) \in X \times X \) such that \( g(x) = F(x, y) \) and \( g(y) = F(y, x) \). Moreover, if \( g(x_0) \) and \( g(y_0) \) are comparable to each other, then \( F \) and \( g \) have a unique coupled common fixed-point in \( X \times X \).

**Corollary 3.4.** Let \( F : X \times X \to X \) be a mapping satisfying the following conditions:

(a) \( F \) has the mixed strict monotone property;

(b) for any \( e > 0 \), there exists \( \delta(e) > 0 \) such that \( x \leq u, y \geq v \) and

\[
e \leq \int_0^{(1/2)[d(x, u) + d(y, v)]} \varphi(t)dt < e + \delta(e) \iff \int_0^{d(F(x, y), F(u, v))} \varphi(t)dt < e,
\]

(3.8)
where $\varphi$ is a locally integrable function from $[0, +\infty)$ into itself satisfying

$$
\int_0^s \varphi(t) \, dt > 0
$$

for all $s > 0$;

(c) there exist $x_0, y_0 \in X$ such that $x_0 < F(x_0, y_0)$ and $y_0 > F(y_0, x_0)$.

Then there exists $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$. Moreover, if $x_0$ and $y_0$ are comparable to each other, then $F$ has a unique coupled common fixed-point in $X \times X$.

**Corollary 3.5.** Let $F : X \times X \to X$ and $g : X \to X$ be two given mappings such that $F(X \times X) \subseteq g(X)$, $g$ is continuous and commutes with $F$. Also, suppose that

(a) $F$ has the mixed strict $g$-monotone property;

(b) for any $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$,

$$
\int_0^{[d(F(x,y),F(u,v))]} \varphi(t) \, dt \leq k \int_0^{(1/2)\left[r(d(\varphi(x),g(u)))+r(d(\varphi(y),g(v)))\right]} \varphi(t) \, dt,
$$

where $k \in (0, 1)$ and $\varphi$ is a locally integrable function from $[0, +\infty)$ into itself satisfying

$$
\int_0^s \varphi(t) \, dt > 0
$$

for all $s > 0$;

(c) there exist $x_0, y_0 \in X$ such that $g(x_0) < F(x_0, y_0)$ and $g(y_0) > F(y_0, x_0)$.

Then there exists $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. Moreover, if $g(x_0)$ and $g(y_0)$ are comparable to each other, then $F$ and $g$ have a unique coupled common fixed-point in $X \times X$.

**Proof.** For any $e > 0$, if we take $\delta(e) = (1/k - 1)e$ and apply Corollary 3.3, then we can get the conclusion.

**Corollary 3.6.** Let $F : X \times X \to X$ be a mapping satisfying the following conditions:

(a) $F$ has the mixed strict monotone property,

(b) for any $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$,

$$
\int_0^{d(F(x,y),F(u,v))} \varphi(t) \, dt \leq k \int_0^{(1/2)\left[r(d(x,u)+r(y,v))\right]} \varphi(t) \, dt,
$$

for all $\varphi$ a locally integrable function from $[0, +\infty)$ into itself satisfying

$$
\int_0^s \varphi(t) \, dt > 0
$$

for all $s > 0$. 

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Theorem 3.8. Let \( k \in (0, 1) \) and \( \varphi \) is a locally integrable function from \([0, +\infty)\) into itself satisfying

\[
\int_0^s \varphi(t) \, dt > 0
\]

for all \( s > 0; \)

(c) there exist \( x_0, y_0 \in X \) such that \( x_0 < F(x_0, y_0) \) and \( y_0 > F(y_0, x_0) \).

Then there exist \( x, y \in X \) such that \( x = F(x, y) \) and \( y = F(y, x) \). Moreover, if \( x_0 \) and \( y_0 \) are comparable to each other, then \( F \) has a unique coupled common fixed-point in \( X \times X \).

Finally, by using the above results, we show the existence of solutions for the following integral equation:

\[
(x(t), y(t)) = \left( \int_0^T G(t, s) \left[ (f(s, x(s)) + \lambda x(s)) - (f(s, y(s)) + \lambda y(s)) \right] \, ds, \right. \\
\left. \int_0^T G(t, s) \left[ (f(s, y(s)) + \lambda y(s)) - (f(s, x(s)) + \lambda x(s)) \right] \, ds \right),
\]

where \( x, y \in C(I, \mathbb{R}) \) (: the set of continuous functions from \( I \) into \( \mathbb{R} \)), \( T > 0 \), \( f : I \times \mathbb{R} \to \mathbb{R} \) is a continuous function and

\[
G(t, s) = \begin{cases} 
  e^{(1+s-1)}, & \text{if } 0 \leq s < t \leq T; \\
  e^{1T - 1}, & \text{if } 0 \leq t < s \leq T.
\end{cases}
\]

Definition 3.7. A lower solution for the integral equation (3.14) is an element \((\alpha, \beta) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})\) such that

\[
\alpha'(t) + \lambda \beta(t) \leq f(t, \alpha(t)) - f(t, \beta(t)), \quad \alpha(0) < \alpha(T), \\
\beta'(t) + \lambda \alpha(t) \geq f(t, \beta(t)) - f(t, \alpha(t)), \quad \beta(0) \geq \beta(T),
\]

where \( C^1(I, \mathbb{R}) \) denotes the set of differentiable functions from \( I \) into \( \mathbb{R} \).

Now, we prove the existence of solutions for the integral equation (3.14) by using the existence of a lower solution for the integral equation (3.14).

Theorem 3.8. Let \( \mathcal{A} \) be the class of the functions \( \varphi : [0, \infty) \to [0, \infty) \) satisfying the following conditions:

(a) \( \varphi \) is increasing;
(b) for any \( x \geq 0 \), there exists \( k \in [0, 1) \) such that \( \varphi(x) < (k/2)x \).
In the integral equation (3.14), suppose that there exists $\lambda > 0$ such that, for all $x, y \in \mathbb{R}$ with $y > x$,

$$0 < f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \varphi(y - x),$$  \hspace{1cm} (3.17)

where $\varphi \in A$. If a lower solution of the integral equation (3.14) exists, then a solution of the integral equation (3.14) exists.

Proof. Define a mapping $F : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$F(x(t), y(t)) = \int_0^T G(t, s)[(f(s, x(s)) + \lambda x(s)) - (f(s, y(s)) + \lambda y(s))]ds.$$  \hspace{1cm} (3.18)

Note that, if $(x(t), y(t)) \in C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a coupled fixed-point of $F$, then $(x(t), y(t))$ is a solution of the integral equation (3.14).

Now, we check the hypotheses in Corollary 2.5 as follows:

1. $X \times X = C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is a partially ordered set if we define the order relation in $X \times X$ as follows:

$$(u(t), v(t)) \leq (x(t), y(t)) \quad \text{iff} \quad u(t) < x(t), \quad v(t) \geq y(t)$$  \hspace{1cm} (3.19)

for all $(x(t), y(t)), (u(t), v(t)) \in X \times X$ and $t \in I$.

2. $(X, d)$ is a complete metric space if we define a metric $d$ as follows:

$$d(x(t), y(t)) = \sup_{t \in I} \{|x(t) - y(t)| : x(t), y(t) \in X\}.$$  \hspace{1cm} (3.20)

3. The mapping $F$ has the mixed strict monotone property. In fact, by hypothesis, if $x_2 > x_1$, then we have

$$f(t, x_2) + \lambda x_2 > f(t, x_1) + \lambda x_1,$$  \hspace{1cm} (3.21)

which implies that, for any $t \in I$,

$$\int_0^T \left[ f(s, x_2(s)) + \lambda x_2(s) - f(s, y(s)) - \lambda y(s) \right] G(t, s) ds$$

$$> \int_0^T \left[ f(s, x_1(s)) + \lambda x_1(s) - f(s, y(s)) - \lambda y(s) \right] G(t, s) ds,$$  \hspace{1cm} (3.22)

that is,

$$F(x_2(t), y(t)) > F(x_1(t), y(t)).$$  \hspace{1cm} (3.23)
Similarly, if \( y_1 < y_2 \), then we have

\[
f(t, y_2) + \lambda y_2 > f(t, y_1) + \lambda y_1,
\]

which implies that, for any \( t \in I \),

\[
\begin{align*}
&\int_0^T \left[ f(s, x(s)) + \lambda x(s) - f(s, y_2(s)) - \lambda y_2(s) \right] G(t, s) ds \\
&\quad < \int_0^T \left[ f(s, x(s)) + \lambda x(s) - f(s, y_1(s)) - \lambda y_1(s) \right] G(t, s) ds,
\end{align*}
\]

that is,

\[
F(x(t), y_2(t)) < F(x(t), y_1(t)).
\]

Now, we show that \( F \) satisfies (1.2). In fact, let \((x, y) \leq (u, v)\) and \( t \in I \). Then we have

\[
\begin{align*}
d(F(x(t), y(t)), F(u(t), v(t))) &= \sup \{ |F(x(t), y(t)) - F(u(t), v(t))| : t \in I \} \\
&= \sup_{t \in I} \left\{ \int_0^T G(t, s) \left[ f(s, x(s)) + \lambda x(s) - f(s, u(s)) - \lambda u(s) \right] ds \\
&\quad - \int_0^T G(t, s) \left[ f(s, v(s)) + \lambda v(s) - f(s, y(s)) - \lambda y(s) \right] ds \right\} \\
&\leq \sup_{t \in I} \int_0^T G(t, s) \left[ f(s, x(s)) + \lambda x(s) - f(s, u(s)) - \lambda u(s) \\
&\quad + f(s, v(s)) + \lambda v(s) - f(s, y(s)) - \lambda y(s) \right] ds.
\end{align*}
\]

Since the function \( \varphi(x) \) is increasing and \((x, y) \leq (u, v)\), we have

\[
\varphi(x(s) - u(s)) \leq \varphi(d(x(s), u(s))), \quad \varphi(v(s) - y(s)) \leq \varphi(d(v(s), y(s))),
\]
we obtain the following:

\[
d(F(x(t), y(t)), F(u(t), v(t)))
\]

\[
\leq \sup_{t \in I} \int_{0}^{T} G(t, s) \left| \lambda \varphi(x(s) - u(s)) + \lambda \varphi(v(s) - y(s)) \right| ds
\]

\[
\leq \lambda \sup_{t \in I} \int_{0}^{T} G(t, s) \left| \varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s))) \right| ds
\]

\[
= \lambda (\varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s)))) \cdot \sup_{t \in I} \int_{0}^{T} G(t, s) ds
\]

\[
= \lambda (\varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s)))) \cdot \frac{1}{1 - e^{\lambda T}} \left( \left[ \frac{1}{\lambda} e^{(T+s-t)} \right]_{0}^{t} + \left[ \frac{1}{\lambda} e^{(s-t)} \right]_{t}^{T} \right)
\]

\[
= \lambda (\varphi(d(x(s), u(s))) + \varphi(d(v(s), y(s)))) \cdot \frac{1}{1 - e^{\lambda T}} (e^{\lambda T} - 1)
\]

\[
< \frac{k}{2} [d(x(s), u(s)) + d(v(s), y(s))]
\]

\[
\leq \frac{k}{2} \sup \{|x(t) - u(t)| : t \in I\} + \frac{k}{2} \sup \{|v(t) - y(t)| : t \in I\}
\]

\[
= \frac{k}{2} [d(x(t), u(t)) + d(y(t), v(t))].
\]

(3.29)

Then, by Proposition 1.3, \( F \) is a generalized Meir-Keeler type contraction.

Finally, let \((\alpha(t), \beta(t)) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})\) be a lower solution for the integral equation (3.14). Then we show that

\[
\alpha < F(\alpha, \beta), \quad \beta \geq F(\beta, \alpha).
\]

(3.30)

Indeed, we have \(\alpha'(t) + \lambda \beta(t) \leq f(t, \alpha(t)) - f(t, \beta(t))\) for any \(t \in I\) and so

\[
\alpha'(t) + \lambda \alpha(t) \leq f(t, \alpha(t)) - f(t, \beta(t)) + \lambda \alpha(t) - \lambda \beta(t)
\]

(3.31)

for any \(t \in I\). Multiplying by \(e^{\lambda t}\) in (3.31), we get the following:

\[
\left( \alpha(t)e^{\lambda t} \right)' \leq \left[ (f(t, \alpha(t)) + \lambda \alpha(t)) - (f(t, \beta(t)) + \lambda \beta(t)) \right] e^{\lambda t}
\]

(3.32)

for any \(t \in I\), which implies that

\[
\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_{0}^{t} \left[ (f(s, \alpha(s)) + \lambda \alpha(s)) - f(s, \beta(s)) - \lambda \beta(s) \right] e^{\lambda s} ds
\]

(3.33)
for any \( t \in I \). This implies that

\[
\alpha(0)e^{\lambda t} < \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T [f(s, \alpha(s)) + \lambda \alpha(s) - f(s, \beta(s)) - \lambda \beta(s)]e^{\lambda s} \, ds \quad (3.34)
\]

and so

\[
\alpha(0) < \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s) - f(s, \beta(s)) - \lambda \beta(s)] \, ds. \quad (3.35)
\]

Thus it follows from (3.35) and (3.33) that

\[
\alpha(t)e^{\lambda t} < \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s) - f(s, \beta(s)) - \lambda \beta(s)] \, ds
\]

\[
+ \int_0^t \frac{e^{\lambda (T+s-t)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s) - f(s, \beta(s)) - \lambda \beta(s)] \, ds, \quad (3.36)
\]

and so

\[
\alpha(t) < \int_0^T \frac{e^{\lambda (T+s-t)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s) - f(s, \beta(s)) - \lambda \beta(s)] \, ds
\]

\[
+ \int_t^T \frac{e^{\lambda (s-t)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s) - f(s, \beta(s)) - \lambda \beta(s)] \, ds, \quad (3.37)
\]

Hence we have

\[
\alpha(t) < \int_0^T G(t,s) [f(s, \alpha(s)) + \lambda \alpha(s) - f(s, \beta(s)) - \lambda \beta(s)] \, ds = F(\alpha(t), \beta(t)) \quad (3.38)
\]

for any \( t \in I \).

Similarly, we have \( \beta(t) \geq F(\beta(t), \alpha(t)) \). Therefore, by Corollary 2.5, \( F \) has a coupled fixed-point.

\( \square \)

**Example 3.9.** In the integral equation (3.14), we put \( \lambda = 1.5, f(u, v) = u-v \) for all \((u, v) \in I \times \mathbb{R}\) and \( T = 0.5 \). Then \( f \) is a continuous function, and we have

\[
(x(t), y(t)) = \left( \int_0^{0.5} G(t,s) [0.5x(s) - 0.5y(s)] \, ds, \int_0^{0.5} G(t,s) [0.5y(s) - 0.5x(s)] \, ds \right),
\]

(3.39)
where \( x, y \in C(I, \mathbb{R}) \), and
\[
G(t, s) = \begin{cases} 
  e^{1.5(0.5s-t)} & \text{if } 0 \leq s < t \leq 0.5, \\
  e^{1.5(s-t)} & \text{if } 0 \leq t < s \leq 0.5.
\end{cases} 
\]

Also, \((\alpha(t), \beta(t)) = (-2e^{-0.5t}, 3e^{-0.5t})\) is a lower solution of (3.39). Moreover, if we define \( \varphi(x) = x/3 \) for all \( x \in [0, \infty) \), then \( \varphi \) is increasing and, for any \( x > 0 \), there exists \( k = 1/1.1 \in [0, 1) \) such that \( \varphi(x) = x/3 < (k/2)x = x/2.2 \). For all \( x, y \in \mathbb{R} \) with \( y > x \), we have
\[
0 < f(t, y) + \lambda y - [f(t, x) + \lambda x] = 0.5(y - x) \leq \lambda \varphi(y - x) = 1.5 \frac{y - x}{3} = 0.5(y - x).
\]

Therefore, all the conditions of Theorem 3.8 hold, and a solution of (3.39) exists.

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