Research Article

Existence Results for a Nonlinear Semipositone Telegraph System with Repulsive Weak Singular Forces

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Using the fixed point theorem of cone expansion/compression, we consider the existence results of positive solutions for a nonlinear semipositone telegraph system with repulsive weak singular forces.

1. Introduction

In this paper, we are concerned with the existence of positive solutions for the nonlinear telegraph system:

\begin{align}
  u_{tt} - u_{xx} + c_1 u_t + a_1(t, x) u &= f(t, x, v), \\
  v_{tt} - v_{xx} + c_2 v_t + a_2(t, x) v &= g(t, x, u),
\end{align}

with doubly periodic boundary conditions

\begin{align}
  u(t + 2\pi, x) &= u(t, x + 2\pi) = u(t, x), \quad (t, x) \in \mathbb{R}^2, \\
  v(t + 2\pi, x) &= v(t, x + 2\pi) = v(t, x), \quad (t, x) \in \mathbb{R}^2.
\end{align}

In particular, the function \( f(t, x, v) \) may be singular at \( v = 0 \) or superlinear at \( v = +\infty \), and \( g(t, x, u) \) may be singular at \( u = 0 \) or superlinear at \( u = +\infty \).
In the latter years, the periodic problem for the semilinear singular equation

\[ x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t), \] (1.3)

with \( a, b, c \in L^1[0,T] \) and \( \lambda > 0 \), has received the attention of many specialists in differential equations. The main methods to study (1.3) are the following three common techniques:

(i) the obtainment of a priori bounds for the possible solutions and then the applications of topological degree arguments;

(ii) the theory of upper and lower solutions;

(iii) some fixed point theorems in a cone.

We refer the readers to see [1–7] and the references therein.

Equation (1.3) is related to the stationary version of the telegraph equation

\[ u_{tt} - u_{xx} + cu + \lambda u = f(t,x,u), \] (1.4)

where \( c > 0 \) is a constant and \( \lambda \in \mathbb{R} \). Because of its important physical background, the existence of periodic solutions for a single telegraph equation or telegraph system has been studied by many authors; see [8–16]. Recently, Wang utilize a weak force condition to enable the achievement of new existence criteria for positive doubly periodic solutions of nonlinear telegraph system through a basic application of Schauder’s fixed point theorem in [17]. Inspired by these papers, here our interest is in studying the existence of positive doubly periodic solutions for a semipositone nonlinear telegraph system with repulsive weak singular forces by using the fixed point theorem of cone expansion/compression.

**Lemma 1.1** (see [18]). Let \( E \) be a Banach space, and let \( K \subset E \) be a cone in \( E \). Assume that \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2 \), and let \( T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K \) be a completely continuous operator such that either

(i) \( \|Tu\| \leq \|u\|, \ u \in K \cap \partial \Omega_1 \) and \( \|Tu\| \geq \|u\|, \ u \in K \cap \partial \Omega_2 \); or

(ii) \( \|Tu\| \geq \|u\|, \ u \in K \cap \partial \Omega_1 \) and \( \|Tu\| \leq \|u\|, \ u \in K \cap \partial \Omega_2 \).

Then, \( T \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

This paper is organized as follows: in Section 2, some preliminaries are given; in Section 3, we give the main results.

**2. Preliminaries**

Let \( T^2 \) be the torus defined as

\[ T^2 = \left( \frac{\mathbb{R}}{2\pi \mathbb{Z}} \right) \times \left( \frac{\mathbb{R}}{2\pi \mathbb{Z}} \right). \] (2.1)

Doubly \( 2\pi \)-periodic functions will be identified to be functions defined on \( T^2 \). We use
the notations
\[
L^p(\mathbb{T}^2), C(\mathbb{T}^2), C^\alpha(\mathbb{T}^2), D(\mathbb{T}^2) = C^\infty(\mathbb{T}^2), \ldots
\] (2.2)
to denote the spaces of doubly periodic functions with the indicated degree of regularity. The space \(D'(\mathbb{T}^2)\) denotes the space of distributions on \(\mathbb{T}^2\).

By a doubly periodic solution of (1.1)-(1.2) we mean that a \((u,v) \in L^1(\mathbb{T}^2) \times L^1(\mathbb{T}^2)\) satisfies (1.1)-(1.2) in the distribution sense; that is,
\[
\begin{align*}
\int_{\mathbb{T}^2} u(q_{tt} - q_{xx} - c_1q_t + a_1(t,x)q) \, dt \, dx &= \int_{\mathbb{T}^2} f(t,x,v) \, dt \, dx, \\
\int_{\mathbb{T}^2} v(q_{tt} - q_{xx} - c_2q_t + a_2(t,x)q) \, dt \, dx &= \int_{\mathbb{T}^2} g(t,x,u) \, dt \, dx,
\end{align*}
\] (2.3)

First, we consider the linear equation
\[
u_{tt} - u_{xx} + c_iu_t - \lambda_iu = h_i(t,x), \quad \text{in } D'(\mathbb{T}^2),
\] (2.4)

where \(c_i > 0, \lambda_i \in \mathbb{R}\), and \(h_i(t,x) \in L^1(\mathbb{T}^2), \quad (i = 1, 2)\).

Let \(\mathcal{E}_{\lambda_i}\) be the differential operator
\[
\mathcal{E}_{\lambda_i} = u_{tt} - u_{xx} + c_iu_t - \lambda_iu,
\] (2.5)
acting on functions on \(\mathbb{T}^2\). Following the discussion in [14], we know that if \(\lambda_i < 0\), then \(\mathcal{E}_{\lambda_i}\) has the resolvent \(R_{\lambda_i}^*\):
\[
R_{\lambda_i} : L^1(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2), \quad h_i \mapsto u_i,
\] (2.6)

where \(u_i\) is the unique solution of (2.4), and the restriction of \(R_{\lambda_i}\) on \(L^p(\mathbb{T}^2)\) \((1 < p < \infty)\) or \(C(\mathbb{T}^2)\) is compact. In particular, \(R_{\lambda_i} : C(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)\) is a completely continuous operator.

For \(\lambda_i = -c_i^2/4\), the Green function \(G_i(t,x)\) of the differential operator \(\mathcal{E}_{\lambda_i}\) is explicitly expressed; see lemma 5.2 in [14]. From the definition of \(G_i(t,x)\), we have
\[
\begin{align*}
G_i := \text{ess inf } G_i(t,x) &= \frac{e^{-c_i \|/2}}{(1 - e^{-c_i \|})^2}, \\
\overline{G}_i := \text{ess sup } G_i(t,x) &= \frac{(1 + e^{-c_i \|})}{2(1 - e^{-c_i \|})^2}.
\end{align*}
\] (2.7)

Let \(E\) denote the Banach space \(C(\mathbb{T}^2)\) with the norm \(\|u\| = \max_{(t,x) \in \mathbb{T}^2}|u(t,x)|\), then \(E\) is an ordered Banach space with cone
\[
K_0 = \big\{ u \in E \mid u(t,x) \geq 0, \ \forall (t,x) \in \mathbb{T}^2 \big\}.
\] (2.8)

For convenience, we assume that the following condition holds throughout this paper:
\begin{enumerate}[(H1)]
\item \(a_i(t,x) \in C(\mathbb{T}^2, \mathbb{R}^+), \ 0 < a_i(t,x) \leq c_i^2/4\text{ for } (t,x) \in \mathbb{T}^2, \text{ and } \int_{\mathbb{T}^2} a_i(t,x) \, dt \, dx > 0.\)
\end{enumerate}
Next, we consider (2.4) when \( -\lambda \) is replaced by \( a_i(t,x) \). In [10], Li has proved the following unique existence and positive estimate result.

**Lemma 2.1.** Let \( h_i(t,x) \in L^1(\mathbb{T}^2); \) is the Banach space \( C(\mathbb{T}^2) \). Then; (2.4) has a unique solution \( u_i = P_i h_i; \) \( P_i : L^1(\mathbb{T}^2) \to C(\mathbb{T}^2) \) is a linear bounded operator with the following properties;

(i) \( P_i : C(\mathbb{T}^2) \to C(\mathbb{T}^2) \) is a completely continuous operator;

(ii) if \( h_i(t,x) > 0, \) then a.e. \( (t,x) \in \mathbb{T}^2, \ P_i[h_i(t,x)] \) has the positive estimate

\[
G_i\|h_i\|_{L^1} \leq P_i[h_i(t,x)] \leq \frac{G_i}{G_i\|a_i\|_{L^1}}\|h_i\|_{L^1}. \tag{2.9}
\]

**3. Main Result**

In this section, we establish the existence of positive solutions for the telegraph system

\[
\begin{align*}
v_{tt} - v_{xx} + c_1v_t + a_1(t,x)v &= f(t,x,u), \\
v_{tt} - v_{xx} + c_2v_t + a_2(t,x)v &= g(t,x,u).
\end{align*}
\tag{3.1}
\]

where \( a_i \in C(\mathbb{R}^2, \mathbb{R}^+) \) and \( f(t,x,v) \) may be singular at \( v = 0 \). In particular, \( f(t,x,v) \) may be negative or superlinear at \( v = +\infty \). \( g(t,x,u) \) has the similar assumptions. Our interest is in working out what weak force conditions of \( f(t,x,v) \) at \( v = 0, g(t,x,u) \) at \( u = 0 \) and what superlinear growth conditions of \( f(t,x,v) \) at \( v = +\infty, g(t,x,u) \) at \( u = +\infty \) are needed to obtain the existence of positive solutions for problem (1.1)-(1.2).

We assume the following conditions throughout.

(H2) \( f, g : \mathbb{T}^2 \times (0, \infty) \to \mathbb{R} \) is continuous, and there exists a constant \( M > 0 \) such that

\[
f_1(t,x,u) + M \geq 0, \quad f_2(t,x,u) + M \geq 0, \quad \forall (t,x) \in \mathbb{T}^2 \text{ and } u,v \in (0, \infty). \tag{3.2}
\]

(H3) \( F(t,x,v) = f(t,x,v) + M \leq j_1(v) + h_1(v) \) for \( (t,x,v) \in \mathbb{T}^2 \times (0, \infty) \) with \( j_1 > 0 \) continuous and nonincreasing on \( (0, \infty) \), \( h_1 \geq 0 \) continuous on \( (0, \infty) \) and \( h_1/j_1 \) nondecreasing on \( (0, \infty) \).

\[
G(t,x,u) = g(t,x,u) + M \leq j_2(u) + h_2(u) \text{ for } (t,x,u) \in \mathbb{T}^2 \times (0, \infty) \text{ with } j_2 > 0 \text{ continuous and nonincreasing on } (0, \infty), \ h_2 \geq 0 \text{ continuous on } (0, \infty) \text{ and } h_2/j_2 \text{ nondecreasing on } (0, \infty).
\]

(H4) \( F(t,x,v) = f(t,x,v) + M \geq j_3(v) + h_3(v) \) for all \( (t,x,v) \in T^2 \times (0, \infty) \) with \( j_3 > 0 \) continuous and nonincreasing on \( (0, \infty) \), \( h_3 \geq 0 \) continuous on \( (0, \infty) \) with \( h_3/j_3 \) nondecreasing on \( (0, \infty) \);

\[
G(t,x,u) = g(t,x,u) + M \geq j_4(u) + h_4(u) \text{ for all } (t,x,u) \in \mathbb{T}^2 \times (0, \infty) \text{ with } j_4 > 0 \text{ continuous and nonincreasing on } (0, \infty), \ h_4 \geq 0 \text{ continuous on } (0, \infty) \text{ with } h_4/j_4 \text{ nondecreasing on } (0, \infty).
\]

(H5) There exists

\[
r > \frac{M\|\omega_1\|}{\delta_1},
\tag{3.3}
\]
such that

\[ r \geq \frac{4\pi^2 G_1}{G_1 \| a_1 \|_{L^1}} I_1 \cdot I_2, \]  

(3.4)

here

\[ I_1 = j_1 \left( \frac{G_2 j_4 (r) \left( 1 + \frac{h_4 (\delta_1 r - M \| \omega_1 \|)}{j_4 (\delta_1 r - M \| \omega_1 \|)} \right) 4\pi^2 - M \| \omega_2 \|}{G_2 \| a_2 \|_{L^1}} \right), \]

\[ I_2 = 1 + \frac{h_1 \left( \left( \frac{4\pi^2 G_1}{G_2 \| a_2 \|_{L^1}} j_2 (\delta_1 r - M \| \omega_1 \|) \{ 1 + \frac{h_2 (r)}{j_2 (r)} \} \right) 4\pi^2 - M \| \omega_2 \|}{j_1 \left( \left( \frac{4\pi^2 G_1}{G_2 \| a_2 \|_{L^1}} j_2 (\delta_1 r - M \| \omega_1 \|) \{ 1 + \frac{h_2 (r)}{j_2 (r)} \} \right) 4\pi^2 - M \| \omega_2 \| \right)}, \]  

(3.5)

where \( \delta_1 = \frac{G_1^2 \| a_1 \|_{L^1}}{G_2} \in (0, 1) \), and \( \omega_1(t, x) \) is the unique solution to problem:

\[ u_{tt} - u_{xx} + c_1 u_t + a_1(t, x) u = 1, \]

\[ u(t + 2\pi, x) = u(t, x + 2\pi) = u(t, x), \]  

(3.6)

(H6) There exists \( R > r \), such that

\[ 4\pi^2 G_1 I_3 \cdot I_4 \geq R, \]

\[ \delta_2 j_4 (R) \left( 1 + \frac{h_4 (\delta_1 R - M \| \omega_1 \|)}{j_4 (\delta_1 R - M \| \omega_1 \|)} \right) > M, \]  

(3.7)

where

\[ I_3 = G_1 j_3 \left( \frac{4\pi^2 G_2}{G_2 \| a_2 \|_{L^1}} j_2 (\delta_1 R - M \| \omega_1 \|) \left( 1 + \frac{h_2 (R)}{j_2 (R)} \right) \right), \]

\[ I_4 = 1 + \frac{h_3 \left( \frac{G_2 j_4 (R) \left( 1 + \frac{h_4 (\delta_1 R - M \| \omega_1 \|)}{j_4 (\delta_1 R - M \| \omega_1 \|)} \right) 4\pi^2 - M \| \omega_2 \|}{j_3 \left( \frac{G_2 j_4 (R) \left( 1 + \frac{h_4 (\delta_1 R - M \| \omega_1 \|)}{j_4 (\delta_1 R - M \| \omega_1 \|)} \right) 4\pi^2 - M \| \omega_2 \| \right)} \right), \]  

(3.8)

**Theorem 3.1.** Assume that (H1)–(H6) hold. Then, the problem (1.1)–(1.2) has a positive doubly periodic solution \((u, v)\).

**Proof.** To show that (1.1)–(1.2) has a positive solution, we will prove that

\[ u_{tt} - u_{xx} + c_1 u_t + a_1(t, x) u = F(t, x, v - M \omega_2), \]

\[ v_{tt} - v_{xx} + c_2 v_t + a_2(t, x) v = G(t, x, u - M \omega_1) \]  

(3.9)
has a solution $(\tilde{u}, \tilde{v}) = (u + M\omega_1, v + M\omega_2)$ with $\tilde{u} > M\omega_1, \tilde{v} > M\omega_2$ for $(t, x) \in \mathbb{T}^2$. In addition, by Lemma 2.1, it is clear to see that $(u, v) \in C^2(\mathbb{T}^2) \times C^2(\mathbb{T}^2)$ is a solution of (3.9) if and only if $(u, v) \in C(\mathbb{T}^2) \times C(\mathbb{T}^2)$ is a solution of the following system:

\begin{equation}
\begin{aligned}
u &= P_1(F(t, x, v - M\omega_2)), \\
v &= P_2(G(t, x, u - M\omega_1)).
\end{aligned}
\end{equation}

(3.10)

Evidently, (3.10) can be rewritten as the following equation:

\begin{equation}
u = P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)).
\end{equation}

(3.11)

Define a cone $K \subset E$ as

\begin{equation}
K = \{u \in E : u \geq 0, u \geq \delta_1\|u\|\}.
\end{equation}

(3.12)

We define an operator $T : E \to K$ by

\begin{equation}(Tu)(t, x) = P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2))
\end{equation}

(3.13)

for $u \in E$ and $(t, x) \in \mathbb{T}^2$. We have the conclusion that $T : E \to E$ is completely continuous and $T(K) \subseteq K$. The complete continuity is obvious by Lemma 2.1. Now, we show that $T(K) \subseteq K$. For any $u \in K$, we have

\begin{equation}
Tu = P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)).
\end{equation}

(3.14)

From (H1)--(H3) and Lemma 2.1, we have

\begin{equation}
\begin{aligned}
Tu &= P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)) \\
&\geq G_1\|F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)\|_{L^1}, \\
\|Tu\| &= P(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)) \\
&\leq \frac{G_1}{G_1\|a_1\|_{L^1}}\|F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)\|_{L^1}.
\end{aligned}
\end{equation}

(3.15)

So, we get

\begin{equation}
Tu \geq \frac{G_1^2\|a_1\|_{L^1}}{G_1}\|Tu\| \geq \delta_1\|Tu\|,
\end{equation}

(3.16)

namely, $T(K) \subseteq K$.

Let

\begin{equation}
\Omega_r = \{u \in E : \|u\| < r\}, \quad \Omega_R = \{u \in E : \|u\| < R\}.
\end{equation}

(3.17)

Since $r \leq \|u\| \leq R$ for any $u \in K \cap (\overline{\Omega_R} \setminus \Omega_r)$, we have $0 < \delta_1 r - M\|\omega\| \leq u - M\omega_1 \leq R$. 


First, we show

$$
\|Tu\| \leq \|u\|, \quad \text{for } u \in K \cap \partial \Omega_r.
$$  \hfill (3.18)

In fact, if $u \in K \cap \partial \Omega_r$, then $\|u\| = r$ and $u \geq \delta_1 r > M\|\omega_1\|$ for $(t, x) \in \Omega$. By (H3) and (H4), we have

$$
P_2(G(t, x, u - M\omega_1)) \leq \frac{G_2}{G_2\|a_2\|_{L_1}} \|G(t, x, u - M\omega_1)\|_{L_1},
$$

$$
\leq \frac{G_2}{G_2\|a_2\|_{L_1}} \left\| \frac{h_2(u - M\omega_1)}{j_2(u - M\omega_1)} \right\|_{L_1},
$$

$$
\leq \frac{G_2}{G_2\|a_2\|_{L_1}} j_2(\delta_1 r - M\|\omega_1\|) \left(1 + \frac{h_2(r)}{j_2(r)}\right) 4\pi^2,
$$

$$
P_2(G(t, x, u - M\omega_1)) \geq \frac{G_2}{G_2\|a_2\|_{L_1}} \|G(t, x, u - M\omega_1)\|_{L_1},
$$

$$
\geq \frac{G_2}{G_2\|a_2\|_{L_1}} j_4(u - M\omega_1) \left(1 + \frac{h_4(u - M\omega_1)}{j_4(u - M\omega_1)}\right) \|\omega_1\|_{L_1},
$$

$$
\geq \frac{G_2}{G_2\|a_2\|_{L_1}} j_4(r) \left(1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right) 4\pi^2.
$$  \hfill (3.20)

In addition, we also have

$$
P_2(G(t, x, u - M\omega_1)) \geq \frac{G_2 j_4(r)}{G_2\|a_2\|_{L_1}} \left(1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right) 4\pi^2
$$

$$
\geq \frac{G_2 j_4(R)}{G_2\|a_2\|_{L_1}} \left(1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right) 4\pi^2
$$

$$
> \frac{G_2}{G_2\|a_2\|_{L_1}} \cdot M4\pi^2
$$

$$
> M\omega_2,
$$  \hfill (3.21)

by (H5), (H6), and (3.20).

So, we have

$$
Tu = P_1(F(t, x, v - M\omega_2))
$$

$$
\leq \frac{G_1}{G_1\|a_1\|_{L_1}} \|F(t, x, v - M\omega_2)\|_{L_1},
$$

$$
\leq \frac{G_1}{G_1\|a_1\|_{L_1}} \left\| j_1(v - M\omega_2) \left(1 + \frac{h_1(v - M\omega_2)}{j_1(v - M\omega_2)}\right) \right\|_{L_1}.
for \((t, x) \in T^2\), since \(\delta_1 r - M\|\omega_1\| \leq u - M\omega_1 \leq r\).

This implies that \(\|Tu\| \leq \|u\|\); that is, (3.18) holds.

Next, we show

\[ \|Tu\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_R. \]  

(3.23)

If \(u \in K \cap \partial\Omega_R\), then \(\|u\| = R\) and \(u \geq \delta R > M\|\omega_1\|\) for \((t, x) \in T^2\). From (H4) and (H6), we have

\[
Tu = P_1(F(t, x, v - M\omega_1))
\]

\[
\geq G_1 \left\| j_3(v - M\omega_2) \left\{ 1 + \frac{h_3(v - M\omega_2)}{j_5(v - M\omega_2)} \right\} \right\|_{L^1}
\]

\[
\geq G_1 \left\| j_3(P_2(G(t, x, u - M\omega_1)) - M\omega_2) \times \left\{ 1 + \frac{h_3(P_2(G(t, x, u - M\omega_1)) - M\omega_2)}{j_5(P_2(G(t, x, u - M\omega_1)) - M\omega_2)} \right\} \right\|_{L^1}
\]

\[
\geq G_1 \left\| j_3 \left( \frac{G_2}{G_2\|a_2\|_{L^1}} j_2(\delta_1 R - M\|\omega_1\|) \left\{ 1 + \frac{h_2(R)}{j_2(R)} \right\} 4\pi^2 \right) \right\|_{L^1}
\]

\[
\times \left\{ 1 + \frac{h_3(G_2 j_4(R) \{ 1 + h_4(\delta_1 R - M\|\omega_1\|) / j_4(\delta_1 R - M\|\omega_1\|) \} 4\pi^2 - M\|\omega_2\|) \} \right\} \right\|_{L^1}
\]

\[
\geq R = \|u\|
\]  

(3.24)

for \((t, x) \in T^2\), since \(\delta_1 R - M\|\omega_1\| \leq u - M\omega_1 \leq R\).

This implies that \(Tu \geq \|u\|\); that is, (3.23) holds.

Finally, (3.18), (3.23), and Lemma 1.1 guarantee that \(T\) has a fixed point \(u \in K \cap \partial\Omega_R \setminus \Omega\) with \(r \leq \|u\| \leq R\). Clearly, \(u > M\omega_1\).
Since
\[
P_2(G(t, x, u - M\omega_1)) \geq G_2\|G(t, x, M\omega_1)\|_{L^1},
\]
\[
\geq G_2\left\| f_1(u - M\omega_1) \left( 1 + \frac{h_4(u - M\omega_1)}{j_4(u - M\omega_1)} \right) \right\|_{L^1},
\]
\[
\geq G_2f_4(R) \left\{ 1 + \frac{h_4(\delta_1r - M\|\omega_1\|)}{j_4(\delta_1r - M\|\omega_1\|)} \right\} 4\pi^2
\]
\[
> \frac{G_2}{G_2\|a_2\|_{L^1}} M4\pi^2
\]
\[
\geq M\omega_2,
\]
then we have a doubly periodic solution \((u, v)\) of (3.9) with \(u > M\omega_1, v > M\omega_2\), namely, \((u - M\omega_1, v - M\omega_2) > (0, 0)\) is a positive solution of (1.1) with (1.2).

Similarly, we also obtain the following result.

**Theorem 3.2.** Assume that (H1)–(H4) hold. In addition, we assume the following.

(H7) There exists
\[
r > \frac{M\|\omega_2\|}{\delta_2},
\]

such that
\[
r \geq 4\pi^2 \frac{G_2}{G_2\|a_2\|_{L^1}} I_5 \cdot I_6,
\]

here
\[
I_5 = j_2 \left( 4\pi^2 \frac{G_1}{G_1\|a_1\|_{L^1}} \right) \left\{ 1 + \frac{h_3(\delta_2r - M\|\omega_2\|)}{j_3(\delta_2r - M\|\omega_2\|)} \right\} - M\|\omega_1\|,
\]
\[
I_6 = 1 + \frac{h_2 \left( 4\pi^2 \frac{G_1}{G_1\|a_1\|_{L^1}} \right) j_1(\delta_2r - M\|\omega_2\|) \left\{ 1 + h_1(r)/j_1(r) \right\}}{j_2 \left( 4\pi^2 \frac{G_1}{G_1\|a_1\|_{L^1}} \right) j_1(\delta_2r - M\|\omega_2\|) \left\{ 1 + h_1(r)/j_1(r) \right\}}.
\]

(H8) There exists \(R > r\), such that
\[
4\pi^2 G_2 I_7 \cdot I_8 \geq R,
\]
\[
\delta_1j_3(R) \left\{ 1 + \frac{h_3(\delta_2r - M\|\omega_2\|)}{j_3(\delta_2r - M\|\omega_2\|)} \right\} > M,
\]

\[
\geq M\omega_2,
\]
then we have a doubly periodic solution \((u, v)\) of (3.9) with \(u > M\omega_1, v > M\omega_2\), namely, \((u - M\omega_1, v - M\omega_2) > (0, 0)\) is a positive solution of (1.1) with (1.2).
where

\[ I_7 = j_4 \left( \frac{4\pi^2 G_1}{G_2 \|a_1\|_L^1} j_1 (\delta_2 R - M \|\omega_2\|) \left\{ 1 + \frac{h_1(R)}{j_1(R)} \right\} \right), \]

\[ I_8 = 1 + \frac{h_4 \left( 4\pi^2 G_1 j_3(R) \left\{ 1 + h_3(\delta_2 R - M \|\omega_2\|) / j_3(\delta_2 R - M \|\omega_2\|) \right\} - M \|\omega_1\| \right)}{j_4 \left( 4\pi^2 G_1 j_3(R) \left\{ 1 + h_3(\delta_2 R - M \|\omega_2\|) / j_3(\delta_2 R - M \|\omega_2\|) \right\} - M \|\omega_1\| \right)}. \]

Then, problem (1.1)-(1.2) has a positive periodic solution.

4. An Example

Consider the following system:

\[
\begin{align*}
    u_{tt} - u_{xx} + 2u_t + \sin^2(t + x)u &= \mu \left( v^{-\alpha} + v^\beta + k_1(t, x) \right), \\
    v_{tt} - v_{xx} + 2v_t + \cos^2(t + x)v &= \lambda \left( u^{-\alpha} + u^\beta + k_2(t, x) \right), \\
    u(t + 2\pi, x) &= u(t, x + 2\pi) = u(t, x), \quad (t, x) \in \mathbb{R}^2, \\
    v(t + 2\pi, x) &= v(t, x + 2\pi) = v(t, x), \quad (t, x) \in \mathbb{R}^2,
\end{align*}
\]

where \( c_1 = c_2 = 2, \mu, \lambda > 0, \alpha, \tau > 0, \beta, \sigma > 1, a_1(t, x) = \sin^2(t + x), a_2(t, x) = \cos^2(t + x) \in C(\mathbb{T}^2, \mathbb{R}^+), k_i : \mathbb{T}^2 \to \mathbb{R} \) is continuous. When \( \mu \) is chosen such that

\[ \mu < \sup_{u \in ((M\|\omega_1\|)/\delta_1, \infty)} \frac{G\|a_1\|_L^1}{G^4\pi^2} \frac{I^1}{I^2}, \]

here we denote

\[
\begin{align*}
    I^1 &= u \left( \frac{G\lambda u^{-\tau}}{G\|a_1\|_L^1} \left\{ 1 + (\delta_1 u - M \|\omega_1\|)^{\sigma+\tau} \right\} 4\pi^2 - M \|\omega_2\| \right)^\sigma, \\
    I^2 &= 1 + \left( \frac{G}{G\|a_2\|_L^1} \lambda (\delta_1 u - M \|\omega_1\|)^{-\tau} (1 + u^\sigma + 2Hu^\tau) 4\pi^2 \right)^{\beta+\sigma} + 2H \left( \frac{G}{G\|a_2\|_L^1} \lambda (\delta_1 u - M \|\omega_1\|)^{-\tau} (1 + u^\sigma + 2Hu^\tau) 4\pi^2 \right),
\end{align*}
\]

where \( H = \max\{\|k_1\|, \|k_2\|\} \) and the Green function \( G_1 = G_2 = G \). Then, problem (4.1) has a positive solution.
Mathematical Problems in Engineering

To verify the result, we will apply Theorem 3.1 with $M = \max \{\mu H, \lambda H\}$ and

\[
\begin{align*}
j_1(v) &= j_3(v) = \mu v^\alpha, \quad h_1(v) = \mu (v^\beta + 2H), \quad h_3(v) = \mu v^\beta, \\
j_2(u) &= j_4(u) = \lambda u^\tau, \quad h_2(u) = \mu (u^\sigma + 2H), \quad h_4(u) = \mu u^\sigma.
\end{align*}
\]

(4.4)

Clearly, (H1)–(H4) are satisfied.

Set

\[
T(u) = \frac{G\|a_1\|_{L^1}}{G^4\pi^2} \frac{I_1^1}{I_2^2}, \quad u \in \left( \frac{(M\|\omega_1\|)}{\delta_1}, +\infty \right).
\]

(4.5)

Obviously, $T((M\|\omega_1\|)/\delta_1) = 0$, $T(\infty) = 0$, then there exists $r \in ((M\|\omega_1\|)/\delta_1, +\infty)$ such that

\[
T(r) = \sup_{u \in ((M\|\omega_1\|)/\delta_1, \infty)} \frac{G\|a_1\|_{L^1}}{G^4\pi^2} \frac{I_1^1}{I_2^2}.
\]

(4.6)

This implies that there exists

\[
r \in \left( \frac{(M\|\omega_1\|)}{\delta_1}, +\infty \right),
\]

(4.7)

such that

\[
\mu < \sup_{u \in ((M\|\omega_1\|)/\delta_1, \infty)} \frac{G\|a_1\|_{L^1}}{G^4\pi^2} \frac{I_1^1}{I_2^2}.
\]

(4.8)

So, (H5) is satisfied.

Finally, since

\[
\frac{R \left( \frac{G\|a_2\|_{L^1}}{G^4\pi^2} \lambda (\delta_1 R - M\|\omega_1\|)^\tau (1 + R^\sigma + 2HR^\tau)4\pi^2 \right)^\alpha}{\mu G \left[ 1 + \left( \frac{G\lambda R^\tau}{G^4\pi^2} \left( 1 + (\delta_1 R - M\|\omega_1\|)^\sigma + 2HR^\tau \right) \right] 4\pi^2 - M\|\omega_2\| \right) \alpha} \to 0 \quad \text{as} \quad R \to \infty,
\]

(4.9)

this implies that there exists $R$. In addition, for fixed $r, R$, choosing $\lambda$ sufficiently large, we have

\[
\delta_2 \lambda R^\tau \left( 1 + (\delta_1 r - M\|\omega_1\|)^\sigma + \tau \right) > M.
\]

(4.10)

Thus, (H6) is satisfied. So, all the conditions of Theorem 3.1 are satisfied.

**References**


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