Research Article

Weingarten and Linear Weingarten Type Tubular Surfaces in $\mathbb{E}^3$

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We study tubular surfaces in Euclidean 3-space satisfying some equations in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature, and the second mean curvature. This paper is a completion of Weingarten and linear Weingarten tubular surfaces in Euclidean 3-space.

1. Introduction

Let $f$ and $g$ be smooth functions on a surface $M$ in Euclidean 3-space $\mathbb{E}^3$. The Jacobi function $\Phi(f, g)$ formed with $f, g$ is defined by

$$\Phi(f, g) = \det \begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix}, \quad (1.1)$$

where $f_s = \partial f/\partial s$ and $f_t = \partial f/\partial t$. In particular, a surface satisfying the Jacobi equation $\Phi(K, H) = 0$ with respect to the Gaussian curvature $K$ and the mean curvature $H$ on a surface $M$ is called a Weingarten surface or a $W$-surface. Also, if a surface satisfies a linear equation with respect to $K$ and $H$, that is, $aK + bH = c$, $(a, b, c) \neq (0, 0, 0)$, $a, b, c \in \mathbb{R}$, then it is said to be a linear Weingarten surface or a LW-surface [1].

When the constant $b = 0$, a linear Weingarten surface $M$ reduces to a surface with constant Gaussian curvature. When the constant $a = 0$, a linear Weingarten surface $M$ reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [1].
If the second fundamental form $II$ of a surface $M$ in $E^3$ is nondegenerate, then it is regarded as a new pseudo-Riemannian metric. Therefore, the Gaussian curvature $K_{II}$ is the second Gaussian curvature on $M$ [1].

For a pair $(X,Y)$, $X \neq Y$, of the curvatures $K, H, K_{II}$ and $H_{II}$ of $M$ in $E^3$, if $M$ satisfies $\Phi(X,Y) = 0$ by $aX + bY = c$, then it said to be a $(X,Y)$-Weingarten surface or $(X,Y)$-linear Weingarten surface, respectively [1].

Several geometers have studied $W$-surfaces and $LW$-surfaces and obtained many interesting results [1–9]. For the study of these surfaces, Kühnel and Stamou investigated ruled $(X,Y)$-Weingarten surfaces in Euclidean 3-space $E^3$ [7, 9]. Also, Baikoussis and Koufogiorgos studied helicoidal $(H, K_{II})$-Weingarten surfaces [10]. Dillen, and sodsiri, and Kühnel, gave a classification of ruled $(X,Y)$-Weingarten surfaces in Minkowski 3-space $E^3$, where $(X,Y) \in \{K, H, K_{II}\}$ [2–4]. Koufogiorgos, Hasanis, and Koutroufiotis investigated closed ovaloid $(X,Y)$-linear Weingarten surfaces in $E^3$ [11, 12]. Yoon, Blair and Koufogiorgos classified ruled $(X,Y)$-linear Weingarten surfaces in $E^3$ [8, 13, 14]. Ro and Yoon studied tubes in Euclidean 3-space which are $(K, H)$, $(K, K_{II})$, $(H, K_{II})$-Weingarten, and linear Weingarten tubes, satisfying some equations in terms of the Gaussian curvature, the mean curvature, and the second Gaussian curvature [1].

Following the Jacobi equation and the linear equation with respect to the Gaussian curvature $K$, the mean curvature $H$, the second Gaussian curvature $K_{II}$, and the second mean curvature $H_{II}$, an interesting geometric question is raised: classify all surfaces in Euclidean 3-space satisfying the conditions

$$\Phi(X,Y) = 0,$$
$$aX + bY = c,$$

where $X, Y \in \{K, H, K_{II}, H_{II}\}$, $X \neq Y$ and $(a, b, c) \neq (0, 0, 0)$.

In this paper, we would like to contribute the solution of the above question by studying this question for tubes or tubular surfaces in Euclidean 3-space $E^3$.

### 2. Preliminaries

We denote a surface $M$ in $E^3$ by

$$M(s,t) = (m_1(s,t), m_2(s,t), m_3(s,t)).$$

Let $U$ be the standard unit normal vector field on a surface $M$ defined by

$$U = \frac{M_s \wedge M_t}{\|M_s \wedge M_t\|},$$

where $M_s = \partial M(s,t)/\partial s$. Then, the first fundamental form $I$ and the second fundamental form $II$ of a surface $M$ are defined by, respectively,

$$I = Eds^2 + 2Fds \, dt + Gdt^2,$$
$$II = eds^2 + 2fds \, dt + gdt^2,$$
where

\[ E = \langle M_s, M_s \rangle, \quad F = \langle M_s, M_t \rangle, \quad G = \langle M_t, M_t \rangle, \]
\[ e = -(M_s, U_s) = \langle M_{ss}, U \rangle, \quad f = -(M_s, U_t) = \langle M_{st}, U \rangle, \quad g = -(M_t, U_t) = \langle M_{tt}, U \rangle, \]

(2.4)

On the other hand, the Gaussian curvature \( K \) and the mean curvature \( H \) are

\[ K = \frac{eg - f^2}{EG - F^2}, \]
\[ H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}, \]

respectively. From Brioschi’s formula in a Euclidean 3-space, we are able to compute \( K_{II} \) and \( H_{II} \) of a surface by replacing the components of the first fundamental form \( E, F, \) and \( G \) by the components of the second fundamental form \( e, f, \) and \( g, \) respectively [14]. Consequently, the second Gaussian curvature \( K_{II} \) of a surface is defined by

\[ K_{II} = \frac{1}{(|e| - f^2)^2} \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix}, \]

(2.6)

and the second mean curvature \( H_{II} \) of a surface is defined by

\[ H_{II} = H - \frac{1}{2\sqrt{|\text{det} II|}} \sum_{ij} \frac{\partial}{\partial u^i} \left( \sqrt{|\text{det} II|} L^{ij} \frac{\partial}{\partial u^j} (\ln \sqrt{|K|}) \right), \]

(2.7)

where \( u^i \) and \( u^j \) stand for “s” and “t”, respectively, and \( L^{ij} = (L_{ij})^{-1} \), where \( L_{ij} \) are the coefficients of the second fundamental form [3, 4].

**Remark 2.1.** It is well known that a minimal surface has a vanishing second Gaussian curvature, but that a surface with the vanishing second Gaussian curvature need not to be minimal [14].

### 3. Weingarten Tubular Surfaces

**Definition 3.1.** Let \( \alpha : [a, b] \to \mathbb{E}^3 \) be a unit-speed curve. A tubular surface of radius \( \lambda > 0 \) about \( \alpha \) is the surface with parametrization

\[ M(s, \theta) = \alpha(s) + \lambda [N(s) \cos \theta + B(s) \sin \theta], \]

(3.1)
$a \leq s \leq b$, where $N(s), B(s)$ are the principal normal and the binormal vectors of $\alpha$, respectively [1].

The curvature and the torsion of the curve $\alpha$ are denoted by $\kappa, \tau$. Then, Frenet formula of $\alpha(s)$ is defined by

\[
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix},
\]

(3.2)

Furthermore, we have the natural frame $\{M_S, M_\theta\}$ given by

\[
M_s = (1 - \lambda \kappa \cos \theta)T - \lambda \tau \sin \theta N + \lambda \tau \cos \theta B,
\]

\[
M_\theta = -\lambda \sin \theta N + \lambda \cos \theta B.
\]

(3.3)

The components of the first fundamental form are

\[
E = \lambda^2 \tau^2 + \sigma^2, \quad F = \lambda^2 \tau, \quad G = \lambda^2,
\]

(3.4)

where $\sigma = 1 - \lambda \kappa \cos \theta$.

On the other hand, the unit normal vector field $U$ is obtained by

\[
U = \frac{M_s \wedge M_\theta}{||M_s \wedge M_\theta||} = -\varepsilon \cos \theta N - \varepsilon \sin \theta B.
\]

(3.5)

As $\lambda > 0$, $\varepsilon$ is the sign of $\sigma$ such that if $\sigma < 0$, then $\varepsilon = -1$ and if $\sigma > 0$, then $\varepsilon = 1$. From this, the components of the second fundamental form of $M$ are given by

\[
e = \varepsilon \lambda \tau^2 - \varepsilon \kappa \cos \theta \sigma, \quad f = \varepsilon \lambda \tau, \quad g = \varepsilon \lambda.
\]

(3.6)

If the second fundamental form is nondegenerate, $eg - f^2 \neq 0$, that is, $\kappa, \sigma$ and $\cos \theta$ are nowhere vanishing. In this case, we can define formally the second Gaussian curvature $K_{II}$ and the second mean curvature $H_{II}$ on $M$. On the other hand, the Gauss curvature $K$, the
mean curvature \( H \), the second Gaussian curvature \( K_{II} \) and the second mean curvature \( H_{II} \) are obtained by using (2.5), (2.6) and (2.7) as follows:

\[
K = -\frac{\kappa \cos \theta}{\lambda \sigma}, \\
H = \frac{\varepsilon (1 - 2\lambda \kappa \cos \theta)}{2\lambda \sigma},
\]

\[
K_{II} = -\frac{\varepsilon \kappa (\cos^2 \theta - 6\kappa \lambda \cos^3 \theta + 4\kappa^2 \lambda^2 \cos^4 \theta + 1)}{4 \cos \theta \sigma},
\]

\[
H_{II} = \frac{1}{8 \varepsilon \kappa^3 \cos^3 \theta \sigma^3} \left( \sum_{i=0}^{6} g_i \cos^i \theta \right),
\]

and where the coefficients \( g_i \) are

\[
g_0 = 3 \lambda^2 \kappa^2 \tau^2, \\
g_1 = 2 \lambda \kappa (\kappa_s \tau - \kappa \tau_s) \sin \theta - \left( 1 + 6 \lambda^2 \tau^2 \right) \kappa^3, \\
g_2 = 2 \lambda^2 \kappa^2 (\kappa \tau_s - 4 \kappa_s \tau) \sin \theta + \lambda \left( 3 (\kappa_s)^2 + 3 \kappa^4 - 2 \kappa \kappa_{ss} - \kappa^2 \tau^2 \right), \\
g_3 = 2 \lambda^2 \kappa \left( 2 \kappa^2 \tau^2 - \kappa^3 + \kappa \kappa_{ss} - 3 (\kappa_s)^2 \right) - \kappa^3, \\
g_4 = 16 \lambda \kappa^4, \\
g_5 = -20 \lambda \kappa^5, \\
g_6 = 8 \lambda^3 \kappa^6.
\]

Differentiating \( K \), \( K_{II} \), \( H \), and \( H_{II} \) with respect to \( s \) and \( \theta \), after straightforward calculations, we get,

\[
K_s = -\frac{\kappa_s \cos \theta}{\lambda \sigma^2}, \quad K_\theta = \frac{\kappa \sin \theta}{\lambda \sigma^2}, \\
H_s = -\frac{\varepsilon \kappa_s \cos \theta}{2 \sigma^2}, \quad H_\theta = \frac{\varepsilon \kappa \sin \theta}{2 \sigma^2},
\]

\[
(K_{II})_s = \frac{\varepsilon \kappa_s (8 \lambda^3 \kappa^3 \cos^5 \theta - 18 \lambda^2 \kappa^2 \cos^4 \theta + 12 \lambda \kappa \cos^3 \theta - \cos^2 \theta - 1)}{4 \cos \theta \sigma^2},
\]

\[
(K_{II})_\theta = -\frac{\varepsilon \kappa \sin \theta (8 \lambda^3 \kappa^3 \cos^5 \theta - 18 \lambda^2 \kappa^2 \cos^4 \theta + 12 \lambda \kappa \cos^3 \theta + \sin^2 \theta - 2 \lambda \kappa \cos \theta)}{4 \cos^2 \theta \sigma^2},
\]

\[
(H_{II})_s = \frac{1}{8 \varepsilon \kappa^4 \cos^3 \theta \sigma^3} \left( \sum_{i=0}^{6} f_i \cos^i \theta \right),
\]

\[
(H_{II})_\theta = \frac{1}{8 \varepsilon \kappa^4 \cos^3 \theta \sigma^3} \left( \sum_{i=0}^{6} f_i \cos^i \theta \right).
\]
and where \( f_i \) are

\[
\begin{align*}
 f_0 &= 3\kappa^2\tau(\kappa_s\tau - 2\kappa\tau_s), \\
 f_1 &= 2\kappa(2\kappa_s(\kappa_s\tau - \kappa\tau_s) - \kappa\kappa_{ss}\tau) \sin \theta + (3\kappa\tau_s - 2\kappa_s\tau) 6\lambda\kappa^3\tau, \\
 f_2 &= 2\lambda\kappa^2(9\kappa_s(\kappa\tau_s - \kappa\tau) + 2\kappa\kappa_{ss}\tau) \sin \theta + 6\lambda^2\kappa^4\tau(3\kappa_s\tau - 2\kappa\tau_s) + \kappa_s\left(9(\kappa_s)^2 - 10\kappa\kappa_{ss}\right) \\
 &+ \kappa^2\tau(2\kappa\tau_s - \kappa_s\tau), \\
 f_3 &= 2\lambda^2\kappa^3(\kappa_s(16\kappa_s\tau - 7\kappa\tau_s) - 4\kappa\kappa_{ss}) \sin \theta \\
 &+ 2\lambda\kappa\left(15\kappa\kappa_s\kappa_{ss} - \left((\kappa_s)^2 + \kappa^4\right)\kappa_s + \kappa^2\tau(2\tau\kappa_s - 5\kappa\tau_s)\right), \\
 f_4 &= 2\lambda^2\kappa^2\left(5\kappa_s\left((\kappa_s)^2 - 2\kappa\kappa_{ss}\right) + 2\kappa^2\tau(2\kappa\tau_s - 3\tau\kappa_s) + \kappa^4\kappa_s\right) - 2\kappa^4\kappa_s, \\
 f_5 &= 6\lambda\kappa^3\kappa_s, \\
 f_6 &= -4\lambda^2\kappa^6\kappa_s, \\
 (H_{\parallel})_\theta &= \frac{1}{8\varepsilon\lambda\kappa^3\cos^4\theta\sigma^4} \left(\sum_{i=0}^{6} h_i \cos^i \theta\right),
\end{align*}
\]  

(3.14) (3.15)

and where the coefficients \( h_i \) are

\[
\begin{align*}
 h_0 &= -9\lambda\kappa^2\tau^2 \sin \theta, \\
 h_1 &= 2\kappa^3\left(1 + 15\lambda^2\tau^2\right) \sin \theta + 4\lambda\kappa(\kappa\tau_s - \kappa_s\tau), \\
 h_2 &= \lambda\left(2\kappa\kappa_{ss} - 8\kappa^4 + \kappa^2\tau^2\left(1 - 30\lambda^2\tau^2\right) - 3(\kappa_s)^2\right) \sin \theta + 6\lambda^2\kappa^2(3\kappa_s\tau - 2\kappa\tau_s), \\
 h_3 &= 4\lambda^2\kappa\left(2\kappa^4 - \kappa^2\tau^2 - 2\kappa\kappa_{ss} + 3(\kappa_s)^2\right) \sin \theta + 2\lambda\kappa\left(\kappa_s\tau - \kappa\tau_s + 4\lambda^2\kappa^2(\kappa\tau_s - 4\kappa_s\tau)\right), \\
 h_4 &= 2\lambda\kappa^3\left(3\lambda^2(2\kappa\tau_s^2 - \kappa^3 + \kappa_{ss}) + \kappa\right) \sin \theta + 2\lambda^2\kappa^2\left(4(\kappa\tau_s - \tau\kappa_s) - 9\lambda(\kappa_s)^3\right), \\
 h_5 &= 6\lambda^2\kappa^3\left(4\kappa_s\tau_s - \kappa\tau_s\right) - \kappa^2 \sin \theta, \\
 h_6 &= 4\lambda^3\kappa^6 \sin \theta.
\end{align*}
\]  

(3.16)

Now, we consider a tubular surface \( M \) in \( E^3 \) satisfying the Jacobi equation \( \Phi(K, H_{\parallel}) = 0 \). By using (3.9), (3.13), and (3.15), we obtain \( \Phi(K, H_{\parallel}) \) in the following form:

\[
K_s(H_{\parallel})_\theta - K_\theta(H_{\parallel})_s = -\varepsilon 4\lambda^2\kappa^6\sigma^3\cos^3\theta \sum_{i=0}^{4} \mu_i \cos^i \theta,
\]  

(3.17)
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with respect to the Gaussian curvature $K$ and the second mean curvature $H_{II}$, where

\[
\begin{align*}
    u_0 &= -3\lambda \tau \kappa^2 (\kappa_s \tau + \kappa \tau_s) \sin \theta, \\
    u_1 &= \kappa^3 \left(6\lambda^2 \tau^2 + 1\right) \kappa_s + 6\lambda^2 \kappa \tau \tau_s \sin \theta - \lambda \kappa^2 \kappa_s \tau + \lambda \kappa^3 \tau_{ss}, \\
    u_2 &= \lambda \left(\kappa^2 \kappa_{ssss} - 4\kappa \kappa_s \kappa_{ss} - 3\kappa^4 \kappa_s + 3(\kappa_s)^3 + \kappa^3 \tau \tau_s\right) \sin \theta + \lambda \kappa^3 \left(3\kappa_s \tau_s + 4\kappa_s \tau - \kappa \tau_{ss}\right), \\
    u_3 &= \lambda \kappa \left(7\lambda \kappa \kappa_s \kappa_{ss} - \lambda \kappa^2 \kappa_{ssss} - 6\lambda (\kappa_s)^3 + 2\lambda \kappa^4 \kappa_s - 4\lambda \kappa^3 \tau \tau_s\right) \sin \theta \\
    &\quad + \left(\kappa \kappa_{ss} \tau + \kappa \kappa_s \tau_s - (\kappa_s)^2 \tau - \kappa^2 \tau_{ss}\right), \\
    u_4 &= -\lambda \kappa^2 \left(4\kappa \kappa_s \tau_s - 4\tau (\kappa_s)^2 - \kappa^2 \tau_{ss} + \kappa \kappa_{ss} \tau\right). \quad (3.18)
\end{align*}
\]

Then, by $\Phi(K, H_{II}) = 0$, (3.17) becomes

\[
\sum_{i=0}^{4} u_i \cos^i \theta = 0. \quad (3.19)
\]

Hence, we have the following theorem.

**Theorem 3.2.** Let $M$ be a tubular surface defined by (3.1) with nondegenerate second fundamental form. $M$ is a $(K, H_{II})$-Weingarten surface if and only if $M$ is a tubular surface around a circle or a helix.

*Proof.* Let us assume that $M$ is a $(K, H_{II})$-Weingarten surface, then the Jacobi equation (3.19) is satisfied. Since polynomial in (3.19) is equal to zero for every $\theta$, all its coefficients must be zero. Therefore, the solutions of $u_0 = u_1 = u_2 = u_3 = u_4 = 0$ are $\kappa_s = 0$, $\tau = 0$ and $\kappa_s = 0$, $\tau_s = 0$ that is, $M$ is a tubular surface around a circle or a helix, respectively.

Conversely, suppose that $M$ is a tubular surface around a circle or a helix, then it is easily to see that $\Phi(K, H_{II}) = 0$ is satisfied for the cases both $\kappa_s = 0$, $\tau = 0$ and $\kappa_s = 0$, $\tau_s = 0$. Thus $M$ is a $(K, H_{II})$-Weingarten surface.

We suppose that a tubular surface $M$ with nondegenerate second fundamental form in $E^3$ is $(H, H_{II})$-Weingarten surface. From (3.10), (3.13), and (3.15), $\Phi(H, H_{II})$ is

\[
H_s(H_{II})_o - H_0(H_{II})_s = \frac{1}{8\lambda \kappa^3 \sigma^3 \cos^3 \theta} \sum_{i=0}^{4} v_i \cos^i \theta, \quad (3.20)
\]

with respect to the variable $\cos \theta$, where

\[
\begin{align*}
    v_0 &= 3\lambda \tau \kappa^2 (\kappa \tau_s + \kappa_s \tau) \sin \theta, \\
    v_1 &= -\kappa^3 \left(\kappa_s + 6\lambda^2 \tau (\kappa_s \tau + \kappa \tau_s)\right) \sin \theta + \lambda \kappa^2 (\kappa_{ss} \tau - \kappa \tau_{ss}),
\end{align*}
\]
\[
v_2 = \lambda \left(3\kappa_4 \kappa_s - 3(\kappa_s)^3 + 4\kappa \kappa_s \kappa_{ss} - \kappa^3 \tau_s - \kappa^2 \kappa_{sss}\right) \sin \theta \\
+ \lambda^2 \kappa^3 (\kappa \tau_{ss} - 3 \kappa_s \tau_s - 4 \kappa_{ss} \tau), \\
v_3 = \lambda^2 \kappa \left(6(\kappa_s)^3 + \kappa^2 \kappa_{sss} - 7 \kappa \kappa_s \kappa_{ss} - 2 \kappa^4 \kappa_s + 4 \kappa^3 \tau_s \right) \sin \theta \\
+ \lambda \kappa \left(2 \tau_{ss} + (\kappa_s)^2 \tau - \kappa \kappa_s \tau - \kappa \kappa_s \tau_s \right), \\
v_4 = -\lambda^2 \kappa^2 \left(\kappa^2 \tau_{ss} - 4 \kappa \kappa_s \tau - 4 \kappa \kappa_s \tau_s + 4 (\kappa_s)^2 \tau \right).
\]

(3.21)

Then, by \( \Phi (H, H_{\parallel}) = 0 \), (3.22) becomes in following form:

\[
\sum_{i=0}^{4} v_i \cos^i \theta = 0.
\]

(3.22)

Thus, we state the following theorem.

**Theorem 3.3.** Let \( M \) be a tubular surface defined by (3.1) with nondegenerate second fundamental form. \( M \) is a \((H, H_{\parallel})\)-Weingarten surface if and only if \( M \) is a tubular surface around a circle or a helix.

**Proof.** Considering \( \Phi (H, H_{\parallel}) = 0 \) and by using (3.13), one can obtain the solutions \( \kappa_s = 0, \tau = 0, \) and \( \kappa_s = 0, \tau_s = 0 \) of the equations \( v_0 = v_1 = v_2 = v_3 = v_4 = 0 \) for all \( \theta \). Thus, it is easily proved that \( M \) is a \((H, H_{\parallel})\)-Weingarten surface if and only if \( M \) is a tubular surface around a circle or a helix.

We consider a tubular surface \( M \) is \((K_{\parallel}, H_{\parallel})\)-Weingarten surface with nondegenerate second fundamental form in \( E^3 \). By using (3.11), (3.12), (3.13), and (3.15), \( \Phi (K_{\parallel}, H_{\parallel}) \) is

\[
(K_{\parallel})_{\theta} (H_{\parallel})_{\theta} - (K_{\parallel})_{\theta} (H_{\parallel})_{s} = \frac{-1}{16 \lambda \kappa^3 \sigma^2 \cos^2 \theta} \sum_{i=0}^{9} \omega_i \cos^i \theta, 
\]

(3.23)

where

\[
\omega_0 = 3 \lambda \tau \kappa^2 (\kappa \tau_s - 2 \kappa_s \tau) \sin \theta, \\
\omega_1 = \kappa^3 \left(\kappa_s + 18 \lambda \tau (\kappa_s \tau - 2 \kappa \tau_s)\right) \sin \theta + \lambda \kappa \left(4 \kappa_s (\kappa \tau_s - \kappa_s \tau) + \kappa \kappa_s \tau_s - \kappa^2 \tau_{ss}\right), \\
\omega_2 = \left\{6 \kappa \kappa_{ss} - 18 \lambda \tau \kappa^2 \tau - 3 \kappa^4 - 6(\kappa_s)^2 - 2 \kappa^2 \tau^2\right\} \lambda \kappa_s + 4 \left(3 \lambda^2 \kappa^2 - 1 \right) \lambda \kappa^3 \tau \tau_s - \lambda \kappa^2 \kappa_{sss} \sin \theta + 3 \lambda^2 \kappa^2 \left(\kappa_s (6 \kappa_s \tau - 5 \kappa \tau_s) - 2 \kappa \kappa_{ss} \tau + \kappa^2 \tau_{ss}\right),
\]
Theorem 3.4. Let $M$ be a tubular surface defined by (3.1) with nondegenerate second fundamental form. $M$ is a $(K_{II}, H_{II})$-Weingarten surface if and only if $M$ is a tubular surface around a circle or a helix.

**Proof.** It can be easily proved similar to Theorems 3.2 and 3.3.

Consequently, we can give the following main theorem for the end of this part.
Theorem 3.5. Let \((X, Y) \in \{(K, H), (H, K), (H, K_\parallel), (K, H, H_\parallel)\}\), and let \(M\) be a tubular surface defined by (3.1) with nondegenerate second fundamental form. \(M\) is a \((X, Y)\)-Weingarten surface if and only if \(M\) is a tubular surface around a circle or a helix.

Thus, the study of Weingarten tubular surfaces in 3-dimensional Euclidean space is completed with [1].

4. Linear Weingarten Tubular Surfaces

In last part of this paper, we study on \((K, H_\parallel), (H, H_\parallel), (H, K_\parallel), (K, H, H_\parallel), (K, H, K_\parallel), (H, K_\parallel, H_\parallel), (K, K_\parallel, H_\parallel)\) linear Weingarten tubular surfaces in \(E^3\). \((K, H), (K, K_\parallel),\) and \((H, K_\parallel)\) linear Weingarten tubes are studied in [1].

Let \(a_1, a_2, a_3, a_4,\) and \(b\) be constants. In general, a linear combination of \(K, H, K_\parallel\) and \(H_\parallel\) can be constructed as

\[
a_1 K + a_2 H + a_3 K_\parallel + a_4 H_\parallel = b. \tag{4.1}
\]

By the straightforward calculations, we obtained the reduced form of (4.1)

\[
8b\kappa^2 \varepsilon \sigma^3 \cos^3 \theta + \sum_{i=0}^{8} p_i \cos^i \theta = 0, \tag{4.2}
\]

where the coefficients are

\[
p_0 = 3a_4 \lambda \kappa^2 \tau^2,
\]
\[
p_1 = a_4 \kappa \left(2 \lambda (\kappa \tau - \kappa \tau_\parallel) \sin \theta - \kappa^2 \left(6 \lambda^2 \tau^2 + 1\right)\right),
\]
\[
p_2 = a_4 \lambda \left(2 \lambda \kappa^2 (\kappa \tau_\parallel - 4 \kappa_\tau_\parallel) \sin \theta + \kappa^2 \left(3 \kappa^2 - \tau^2\right) - 2 \kappa \kappa_\parallel + 3 \kappa_\parallel^2\right) + 2 a_3 \lambda \kappa^4,
\]
\[
p_3 = a_4 \kappa \left(2 \lambda^2 (\kappa \kappa_\parallel - \kappa^4 + 2 \kappa^2 \tau^2 - 3 \kappa_\parallel^2) - 5 \kappa^2\right) - 4 a_2 \kappa^3 - 4 a_3 \lambda^2 \kappa^3,
\]
\[
p_4 = 8 a_1 \epsilon \kappa^4 + 16 a_2 \lambda \kappa^4 + 2 a_3 \lambda \kappa^4 \left(1 + \lambda^2 \kappa^2\right) + 17 a_4 \lambda \kappa^4,
\]
\[
p_5 = -16 a_1 \epsilon \lambda \kappa^5 - 20 a_2 \lambda^2 \kappa^5 - 16 a_3 \lambda^2 \kappa^5 - 20 a_4 \lambda^2 \kappa^5,
\]
\[
p_6 = 8 a_1 \epsilon \lambda^2 \kappa^6 + 8 a_2 \lambda^3 \kappa^6 + 34 a_3 \lambda^3 \kappa^6,
\]
\[
p_7 = -28 a_3 \lambda^4 \kappa^7,
\]
\[
p_8 = 8 a_3 \lambda^5 \kappa^8.
\]

Then, \(p_0, p_1, p_2, p_7,\) and \(p_8\) are zero for any \(b \in IR\). If \(a_4 \neq 0\) or \(a_5 \neq 0\), from \(p_0 = p_1 = p_7 = p_8 = 0\), one has \(\kappa = 0\). Hence, we can give the following theorems.
**Theorem 4.1.** Let \((X, Y) \in \{(K, H_{II}), (H, H_{II}), (K_{II}, H_{II})\}\). Then, there are no \((X, Y)\)-linear Weingarten tubular surfaces \(M\) in Euclidean 3-space defined by (3.1) with nondegenerate second fundamental form.

**Theorem 4.2.** Let \((X, Y, Z) \in \{(H, K_{II}, H_{II}), (K, K_{II}, H_{II}), (K, H, H_{II}), (K, H, K_{II})\}\). Then, there are no \((X, Y, Z)\)-linear Weingarten tubular surfaces \(M\) in Euclidean 3-space defined by (3.1) with nondegenerate second fundamental form.

**Theorem 4.3.** Let \(M\) be a tubular surface defined by (3.1) with nondegenerate second fundamental form. Then, there are no \((K, H, K_{II}, H_{II})\)-linear Weingarten surface in Euclidean 3-space.

Consequently, the study of linear Weingarten tubular surfaces in 3-dimensional Euclidean space is completed with [1].

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**References**


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