Research Article

Computing Exact Solutions to a Generalized Lax-Sawada-Kotera-Ito Seventh-Order KdV Equation

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The Cole-Hopf transform is used to construct exact solutions to a generalization of both the seventh-order Lax KdV equation (Lax KdV7) and the seventh-order Sawada-Kotera-Ito KdV equation (Sawada-Kotera-Ito KdV7).

1. Introduction

Many direct and computational methods have been used to handle nonlinear partial differential equations (NLPDE’s). Some methods used in a satisfactory way to obtain exact solutions to NLPDE’s are inverse scattering method [1], Hirota bilinear method [2, 3], Backlund transformations [4], Painlevé analysis [5], Lie groups [6], the tanh method [7], the generalized tanh method [8, 9], the extended tanh method [10–12], the improved tanh-coth method [13, 14], the Exp-function method [15–17], the projective Riccati equation method [18], the generalized projective Riccati equations method [19–24], the extended hyperbolic function method [25], variational iteration method [26, 27], He’s polynomials [28], homotopy perturbation method [29], and many other methods [30]. However, there is not a unified method that could be used to handle all NLPDE’s; in this sense, the implementation of new
methods or variants of the some well-known methods is relevant. The principal objective of this paper consists in obtaining exact traveling wave solutions which include periodic and soliton solutions to a particular case of the general seventh-order KdV (KdV7), which is a generalization of the seventh-order Sawada-Kotera-Ito (SKI-KdV7) equation, by using a variant of the exp-function method. The general seventh-order KdV (KdV7) equation [31] reads

\[ u_t + au^3 u_x + bu^3 x + cu_x u_{xx} + du^2 u_{xxx} + eu_{2x} u_{3x} + fu_x u_{4x} + gu u_{5x} + u_{7x} = 0. \]  

(1.1)

The (KdV7) was introduced initially by Pomeau et al. [32] for discussing the structural stability of KdV equation under a singular perturbation. Some particular cases of (1.1) are

(i) seventh-order Lax KdV equation [1, 6] \((a = 140, b = 70, c = 280, d = 70, e = 70, f = 42, g = 14)\):

\[ u_t + 140u^3 u_x + 70u^3 x + 280 u_x u_{xx} + 70u^2 u_{xxx} + 70 u_{2x} u_{3x} + 42 u_x u_{4x} + 14 u u_{5x} + u_{7x} = 0; \]  

(1.2)

(ii) seventh-order Sawada-Kotera-Ito equation [1, 8–10] \((a = 252, b = 63, c = 378, d = 126, e = 63, f = 42, g = 21)\):

\[ u_t + 252u^3 u_x + 63u^3 x + 378 u_x u_{xx} + 126u^2 u_{xxx} + 63 u_{2x} u_{3x} + 42 u_x u_{4x} + 21 uu_{5x} + u_{7x} = 0; \]  

(1.3)

(iii) seventh-order Kaup-Kupershmidt equation [1, 7] \((a = 2016, b = 630, c = 2268, d = 504, e = 252, f = 147, g = 42)\):

\[ u_t + 2016u^3 u_x + 630u^3 x + 2268 u_x u_{xx} + 504u^2 u_{xxx} + 252 u_{2x} u_{3x} + 147 u_x u_{4x} + 42 uu_{5x} + u_{7x} = 0. \]

(1.4)

2. Generalization of the Lax KdV7 and the Sawada-Kotera-Ito KdV7

Observe that (1.2) and (1.3) satisfy the relation

\[ a = \frac{d}{63} (e + f + g). \]  

(2.1)

For this reason we will study equation

\[ u_t + \frac{d}{63} (e + f + g) u^3 u_x + bu^3 x + cu_x u_{xx} + du^2 u_{xxx} + eu_{2x} u_{3x} + fu_x u_{4x} + gu u_{5x} + u_{7x} = 0. \]  

(2.2)

We seek solutions to (2.2) in the Cole-Hopf form

\[ u(t, x) = A \partial_x \tanh(\xi), \]  

(2.3)
where $A$ is some constant to be determined later and

$$\zeta = \zeta(t, x) = \mu(x + \lambda t + \delta), \quad \mu, \delta, \lambda = \text{const.} \quad (2.4)$$

Substituting (2.3) into (2.2), we obtain a polynomial equation in the variable $\zeta = \exp(\xi)$. Equating the coefficients of the different powers of $\xi$ to zero, we obtain following algebraic system:

$$\lambda + 64\mu^6 = 0,$$

$$64\mu^5(A(e + f + g) - 247\mu) + 5\lambda = 0,$$

$$64\mu^4 \left(A^2(b + c + d) - 3A\mu(5e + 9f + 19g) + 4293\mu^2 \right) + 9\lambda = 0,$$

$$64\mu^3 \left(A^3 d(e + f + g) - 63A^2\mu(3b + 5c + 11d) + 126A\mu^2(28e + 46f + 151g) - 983997\mu^3 \right) + 315\lambda = 0. \quad (2.5)$$

Eliminating $A, \lambda, \text{and } \mu$ from system (2.5) gives

$$b = d + \frac{1}{126}(e + f + g)(e - 5f + 10g),$$

$$c = \frac{5}{21}g(e + f + g) - 2d. \quad (2.6)$$

It is easy to verify that (1.2) and (1.3) are particular cases of general KdV7 equation (1.1) subject to (2.1) and (2.6). This motivates us to define the generalized Lax-Sawada-Kotera-Ito seventh-order equation (LSKI KdV7) as follows:

$$u_t + \frac{1}{63}d(e + f + g)u^3u_x + \left(d + \frac{1}{126}(e + f + g)(e - 5f + 10g) \right)u_x^3$$

$$+ \left(\frac{5}{21}g(e + f + g) - 2d \right)uu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + fu_{x}u_{4x} + guu_{5x} + u_{7x} = 0. \quad (2.7)$$

### 3. Solutions to Generalized LSKI KdV7

In order to look for solutions to (2.7), we will use the exp ansatz

$$u(\xi) = p + \frac{q}{1 + r \exp(-\xi) + s \exp(\xi)}, \quad (3.1)$$

where $p, q, r, \text{and } s$ are some constants. Substituting (3.1) into (2.7) gives an algebraic system. Solving it, we obtain

$$\lambda = -\frac{1}{63}d(e + f + g)p^3 - \mu^2 \left(dp^2 + g\mu^2 + \mu^4 \right), \quad q = \frac{126\mu^2}{e + f + g}, \quad s = \frac{1}{4r}, \quad r = r, \quad \mu = \mu. \quad (3.2)$$
From (2.4), (3.1), and (3.2), we obtain following solution to (2.7) subject:

\[
    u(x, t) = p + \frac{126\mu^2}{(e + f + g)\left(1 + r \exp(\xi) + (1/4r) \exp(-\xi)\right)},
\]

\[
    \xi = \mu(x + \lambda t + \delta),
\]

\[
    \lambda = -\frac{1}{63} d(e + f + g)p^3 - \mu^2 \left(dp^2 + gp\mu^2 + \mu^4\right).
\]

In particular, if \( r = 1/2 \), equation (3.3) gives

\[
    u(x, t) = p + \frac{63\mu^2}{e + f + g} \sech^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right),
\]

\[
    \lambda = -\frac{1}{63} d(e + f + g)p^3 - \left(dp^2 + gp\mu^2 + \mu^4\right)\mu^2.
\]

Replacing \( \mu \) with \( \mu\sqrt{-1} \) gives the following periodic solutions:

\[
    u(x, t) = p - \frac{63\mu^2}{e + f + g} \sec^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right),
\]

\[
    \lambda = -\frac{1}{63} d(e + f + g)p^3 + \left(dp^2 - gp\mu^2 + \mu^4\right)\mu^2.
\]

On the other hand, if \( r = -1/2 \), equation (3.3) gives

\[
    u(x, t) = p - \frac{63\mu^2}{e + f + g} \csc^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right),
\]

\[
    \lambda = -\frac{1}{63} d(e + f + g)p^3 - \left(dp^2 - gp\mu^2 + \mu^4\right)\mu^2.
\]

Replacing \( \mu \) with \( \mu\sqrt{-1} \) gives the following periodic solutions:

\[
    u(x, t) = p - \frac{63\mu^2}{e + f + g} \csc^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right),
\]

\[
    \lambda = -\frac{1}{63} d(e + f + g)p^3 + \left(dp^2 - gp\mu^2 + \mu^4\right)\mu^2.
\]
4. Solutions to Sawada-Kotera-Ito KdV7 Equation

From (3.3)–(3.7) with \(d = 126, e = 63, f = 42\), and \(g = 21\), we obtain the following analytic solutions to equation (1.3):

\[
\begin{align*}
\frac{d u}{d x} + u \frac{d u}{d t} + p_1 u &= 0, \\
\frac{d u}{d x} + u \frac{d u}{d t} + p_2 u &= 0, \\
\frac{d u}{d x} + u \frac{d u}{d t} + p_3 u &= 0.
\end{align*}
\]

\(\frac{d u}{d x} + u \frac{d u}{d t} + p_4 u = 0, \quad \frac{d u}{d x} + u \frac{d u}{d t} + p_5 u = 0.

5. Conclusions

We exhibited an equation that generalizes both seventh-order Lax equation and seventh-order Sawada-Kotera-Ito equation. At the same time, we obtained exact solutions to these equations with the aid of a Cole-Hopf ansatz. These same ideas are suitable for the seventh-order Kaup-Kupershmidt equation. We think that some of the solutions in this work are new in the open literature. We may apply other methods to find exact solutions to a variety of nonlinear PDE’s. See [3, 12–52].

References


