Letter to the Editor
A Note on the Paper “Multipoint BVPs for Second-Order Differential Equations with Impulses” by Xuxin Yang, Zhimin He, and Jianhua Shen

Yiliang Liu and Jitai Liang

School of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning Guangxi 530006, China

Correspondence should be addressed to Yiliang Liu, yiliangliu100@126.com

Received 11 August 2010; Accepted 14 September 2010

Copyright © 2010 Y. Liu and J. Liang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We give a counter example to the comparison principle for the multipoint BVPs (by Xuxin Yang, Zhimin He, and Jianhua Shen, in Mathematical Problems in Engineering, Volume 2009, Article ID 258090, doi:10.1155/2009/258090). Then we suggest and prove a corrected version of the comparison principle.

1. Introduction and Preliminaries

Consider the following multipoint BVPs [1]:

\[-u''(t) = f(t, u(t), u(\theta(t))), \quad t \neq t_k, t \in J = [0, 1],
\]

\[\Delta u'(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \tag{1.1}\]

\[u(0) - au'(0) = cu(\eta), \quad u(1) + bu'(1) = du(\xi),\]

where \(0 \leq \theta(t) \leq t, \theta \in C(J), 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1, f\) is continuous everywhere except at \(\{t_k\} \times \mathbb{R}^2; f(t_k, \cdot, \cdot)\) and \(f(t_k, \cdot, \cdot)\) exist with \(f(t_k, \cdot, \cdot) = f(t_k, \cdot, \cdot); I_k \in C(\mathbb{R}, \mathbb{R}), \) and \(\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-), a \geq 0, b \geq 0, 0 \leq c \leq 1, 0 \leq d \leq 1, a + c > 0, b + d > 0, 0 < \eta, \xi < 1.\)

Let \(PC(J) = \{x : J \to \mathbb{R}; x(t)\) be continuous everywhere except for some \(t_k\) at which \(x(t_k^+)\) and \(x(t_k^-)\) exist and \(x(t_k) = x(t_k^+), k = 1, 2, \ldots, m\}; PC^1(J) = \{x \in PC(J) : x'(t)\) is continuous everywhere expect for some \(t_k\) at which \(x'(t_k^+), x'(t_k^-)\) exist and \(x'(t_k) =\)
\[ x'(t_k), k = 1, 2, \ldots, m \]. Let \( J^- = J \setminus \{ t_k, k = 1, 2, \ldots, m \} \), and \( E = PC^1(J, R) \cap C^2(J^-, R) \). A function \( x \in E \) is called a solution of BVPS (1.1) if it satisfies (1.1).

The purpose of this note is to point out that the results basing on the comparison principle [1, Theorem 2.1] are not true. Then we give a new comparison principle.

\section{Problem and Statement}

The authors [1] proved some existence results for multipoint BVPs (1.1) by use of the following comparison principle [1, Theorem 2.1].

Assume that \( u \in E \) satisfies

\[
-u''(t) + Mu(t) + Nu(\theta(t)) \leq 0, \quad t \neq t_k, \quad t \in J = [0, 1],
\]

\[
\Delta u'(t_k) \geq L_k u(t_k), \quad k = 1, 2, \ldots, m, \tag{2.1}
\]

\[
u(0) - au'(0) \leq cu(\eta), \quad u(1) + bu'(1) \leq du(\xi), \tag{2.2}
\]

where \( a \geq 0, b \geq 0, 0 \leq c \leq 1, 0 \leq d \leq 1, a + c > 0, b + d > 0, 0 < \eta, \xi < 1, L_k \geq 0 \), and constants \( M, N \) satisfy

\[
M > 0, N \geq 0, \quad \frac{M + N}{2} + \sum_{k=1}^{m} L_k \leq 1. \tag{2.2}
\]

Then \( u(t) \leq 0 \) for \( t \in J \).

However, the comparison principle above is not true.

\section*{A Counter Example}

Let

\[
u(t) = \begin{cases} 
\frac{3}{2} t^2 + 20, & t \in \left[0, \frac{1}{2}\right], \\
\frac{5}{2} t^2 + 3, & t \in \left(\frac{1}{2}, 1\right].
\end{cases} \tag{2.3}
\]
Then

\[ u'(t) = \begin{cases} 
3t, & t \in [0, \frac{1}{2}], \\
5t, & t \in \left(\frac{1}{2}, 1\right], 
\end{cases} \]

\[ u''(t) = \begin{cases} 
3, & t \in [0, \frac{1}{2}], \\
5, & t \in \left(\frac{1}{2}, 1\right]. 
\end{cases} \]

(2.4)

And let \( M = N = 1/1000, \ a = b = c = d = 1, m = 1, t_1 = 1/2, L_1 = 1/1000, \theta(t) = (1/2)t, \ \eta = 1/3, \ \text{and} \ \xi = 1/6. \) When \( t \in [0, 1/2], \) then

\[ \frac{1}{1000} \left( \frac{3}{2} t^2 + 20 \right) + \frac{1}{1000} \left( \frac{3}{2} \times \frac{t^2}{4} + 20 \right) \leq 3. \]  

(2.5)

When \( t \in (1/2, 1], \) then

\[ \frac{1}{1000} \left( \frac{5}{2} t^2 + 3 \right) + \frac{1}{1000} \left( \frac{5}{2} \times \frac{t^2}{4} + 3 \right) \leq 5. \]  

(2.6)

Hence \(-u''(t) + Mu(t) + Nu(\theta(t)) \leq 0. \)

\[ \Delta u'\left(\frac{1}{2}\right) = u'\left(\frac{1}{2}^+\right) - u'\left(\frac{1}{2}\right) = 5 \times \frac{1}{2} - \left(3 \times \frac{1}{2}\right) = 1, \]

(2.7)

\[ \frac{1}{1000} u\left(\frac{1}{2}\right) = \frac{1}{1000} \left( \frac{3}{2} \times \frac{1}{4} + 20 \right) = \frac{1}{1000} \times 163. \]  

(2.8)

Hence \( \Delta u'(t_1) \geq L_1 u(t_1). \)

\[ u(0) - u'(0) = 20, \quad u\left(\frac{1}{3}\right) = \frac{3}{2} \times \frac{1}{9} + 20. \]  

(2.9)

Hence \( u(0) - au'(0) \leq cu(1/3). \)

\[ u(1) + u'(1) = \frac{5}{2} + 3 + 5 = \frac{21}{2}, \quad u\left(\frac{1}{6}\right) = \frac{3}{2} \times \frac{1}{36} + 20. \]  

(2.10)
Hence \( u(1) + bu'(1) \leq du(1/6) \).

\[
\frac{M + N}{2} + \sum_{k=1}^{m} L_k = \frac{2}{1000} < 1.
\] (2.11)

But we easily show that \( u(t) > 0 \), for all \( t \in [0, 1] \), which is a contradiction with (Theorem 2.1) in [1]. In fact, we can correct Theorem 2.1 in [1] as follows.

**Theorem 2.1.** Suppose \( u \in E \cap C(J) \) such that

\[
-u''(t) + Mu(t) + Nu(\theta(t)) \leq 0 \quad t \neq t_k, \quad t \in J = [0, 1],
\]

\[
\Delta u'(t_k) \geq L_k u(t_k), \quad k = 1, 2, \ldots, m,
\]

\[
u(0) - au'(0) \leq cu(\eta), \quad u(1) + bu'(1) \leq du(\xi),
\] (2.12)

where \( a \geq 0, b > 0, 0 \leq c \leq 1, 0 \leq d \leq 1, 0 < \eta, \xi < 1, a + c > 0, b + d > 0, L_k > 0, \) and constants \( M, N \) satisfy

\[
M > 0, N > 0, \quad \frac{M + N}{2} + \sum_{k=1}^{m} L_k \leq 1.
\] (2.13)

Then \( u(t) \leq 0 \) for \( t \in J \).

**Remark 2.2.** In this Theorem, we have to add \( u \in C(J) \).

**Proof.** Suppose to contrary that there exist some \( t \in J \), such that \( u(t) > 0 \).

If \( u(1) = \max_{t \in J} u(t) > 0 \), we have \( u'(1) \geq 0, u(1) \geq u(\xi) \), and

\[
du(\xi) \leq u(1) \leq u(1) + bu'(1) \leq du(\xi).
\] (2.14)

Therefore, \( d = 1 \) and \( u(\xi) \) is maximum value.

If \( u(0) = \max_{t \in J} u(t) > 0 \), we have \( u'(0) \leq 0, u(0) \geq u(\eta) \), and

\[
cu(\eta) \leq u(0) \leq u(0) - au'(0) \leq cu(\eta).
\] (2.15)

Therefore, \( c = 1 \) and \( u(\eta) \) is maximum value.
Mathematical Problems in Engineering

So there is a $\delta \in (0, 1)$ such that

$$u(\delta) = \max_{t \in J} u(t) > 0, \quad \text{by} \quad \Delta u = 0, \quad \text{then} \quad u'(\delta^+) \leq 0, \quad u'(\delta^-) \geq 0. \quad (2.16)$$

It is obvious to see that $\delta \notin \{t_k, k = 1, 2, \ldots, m\}$ by

$$\Delta u'(\delta) = u'(\delta^+) - u'(\delta) \geq L_k u(\delta) > 0 \quad (2.17)$$

which is a contradiction because of (2.16).

(i) Suppose that $u(t) \geq 0$ for $t \in [0, \delta]$.

By $u(\delta) = \max_{t \in J} u(t) > 0$, we get $\delta \in J^-$, $u''(\delta) \leq 0$. On the other hand, by (2.12), we have

$$0 < Mu(\delta) + Nu(\theta(t)) \leq u''(\delta) \quad (2.18)$$

which is a contradiction.

(ii) Suppose there exists $t_* \in [0, \delta]$ such that $u(t_*) = \min_{t \in [0, \delta]} u(t) < 0$. By (2.12), we get

$$u''(t) \geq (M + N)u(t_*), \quad t \in [0, \delta), t \neq t_k,$$

$$\Delta u(t_k) = 0,$$

$$\Delta u'(t_k) \geq L_k u(t_k), \quad k = 1, 2, \ldots, m. \quad (2.19)$$

Integrating from $s(t_* \leq s \leq \delta)$ to $\delta$, we get

$$u'(\delta) - u'(s) \geq \int_s^\delta (M + N)u(t_*)ds + \sum_{s < t_k < \delta} L_k u(t_k)$$

$$= (\delta - s)(M + N)u(t_*) + \sum_{s < t_k < \delta} L_k u(t_k) \quad (2.20)$$

$$\geq (\delta - s)(M + N)u(t_*) + \sum_{k=1}^m L_k u(t_*).$$

Hence

$$-u'(s) \geq (\delta - s)(M + N)u(t_*) + \sum_{k=1}^m L_k u(t_*), \quad t_* \leq s \leq \delta. \quad (2.21)$$
Then integrate from $t_*$ to $\delta$ to obtain

$$-u(t_*) < u(\delta) - u(t_*)$$

$$\leq \int_{t_*}^{\delta} (M + N)u(t_*)(s-\delta)ds - \sum_{k=1}^{m} L_k u(t_*)(\delta - t_*)$$

$$= (M + N)u(t_*) \left[ -\frac{(t_*-\delta)^2}{2} \right] - \sum_{k=1}^{m} L_k u(t_*)(\delta - t_*)$$

$$\leq -\left[ \frac{M + N}{2}(\delta - t_*)^2 + \sum_{k=1}^{m} L_k \right] u(t_*)$$

$$\leq -\left( \frac{M + N}{2} + \sum_{k=1}^{m} L_k \right) u(t_*).$$  \hspace{1cm} (2.22)

By (2.13), we get $u(t_*) > 0$ which is a contradiction. We complete the proof.

This implies that in order to get the existence results of the multipoint BVPs [1], we have to require an additional continuity hypotheses on the function space.

Acknowledgments

This project is supported by NNSF of China, Grant no. 10971019 and NSF of Guangxi, Grant no. 2010GXNSFA013114.

References