Research Article
Nonnegativity Preservation under Singular Values Perturbation

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Received 30 December 2008; Accepted 18 May 2009

Recommended by David Chelidze

We study how singular values and singular vectors of a matrix $A$ change, under matrix perturbations of the form $A + au_jv_j^*$ and $A + au_jv_q^*$, $p \neq q$, where $a \in \mathbb{R}$. $A$ is an $m \times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$, and $u_j, v_k, j = 1, \ldots, m; k = 1, \ldots, n$, are the left and right singular vectors, respectively. In particular we give conditions under which this kind of perturbations preserve nonnegativity and certain matrix structures.

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1. Introduction

A singular value decomposition of a matrix $A \in \mathbb{C}^{m \times n}$ is a factorization $A = U\Sigma V^*$, where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \in \mathbb{R}^{m \times n}$, $r = \min\{m, n\}$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and both $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary. The diagonal entries of $\Sigma$ are called the singular values of $A$. The columns $u_j$ of $U$ are called left singular vectors of $A$ and the columns $v_j$ of $V$ are called right singular vectors of $A$. Every $A \in \mathbb{C}^{m \times n}$ has a singular value decomposition $A = U\Sigma V^*$ and the following relations hold: $Av_j = \sigma_j u_j$. $A^*u_j = \sigma_j v_j$, and $u_j^*Av_j = \sigma_j$. If $A \in \mathbb{R}^{m \times n}$, then $U$ and $V$ may be taken to be real (see [1]).

Let $A$ be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$ and left and right singular vectors $u_j, v_k, j = 1, \ldots, m; k = 1, \ldots, n$, respectively. In this paper we study how singular values and singular vectors of $A$ change, under matrix perturbations of the form $A + au_jv_j^*$ and $A + au_jv_q^*$, $p \neq q$, $a \in \mathbb{R}$. Perturbations of the form $A + au_jv_j^*$ were used in [2] to construct nonnegative matrices with prescribed extremal singular values. Both kinds of perturbations are closely related to the inverse singular value problem (ISVP), which is the problem of constructing a structured matrix from its singular values. ISVP arises in many areas of application, such as circuit theory, computed tomography, irrigation theory, mass distributions, and so forth (see [3]). The ISVP can be seen as an extension of the inverse eigenvalue problem (IEP), which look for necessary and
Theorem 1.1. Let $A$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$. Let

$$U = (u_1 | u_2 | \cdots | u_p), \quad V = (v_1 | v_2 | \cdots | v_p), \quad p \leq r,$$  \hspace{1cm} (1.1)

be matrices of order $m \times p$ and $n \times p$, whose columns are the left and right singular vectors, respectively, corresponding to $\sigma_i$, $i = 1, \ldots, p$. Let $D = \text{diag}\{d_1, d_2, \ldots, d_p\}$ with $\sigma_i + d_i \geq 0$. Then $A + UDV^*$ has singular values

$$\{\sigma_1 + d_1, \ldots, \sigma_p + d_p, \sigma_p+1, \ldots, \sigma_r\}. \hspace{1cm} (1.2)$$

Note that the singular values of $A + UDV^*$ are not necessarily in nondecreasing order. However we can reorderer them by using an appropriate permutation.

Corollary 1.2. Let $A$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$. Let $u_i$ and $v_i$, respectively, the left and right singular vectors corresponding to $\sigma_i$, $i = 1, \ldots, r$. Let $\alpha \in \mathbb{R}$ such that $\alpha + \sigma_i \geq 0$, $i = 1, \ldots, r$. Then $A + \alpha u_i v_i^*$ has singular values

$$\sigma_1, \ldots, \sigma_{i-1}, \sigma_i + \alpha, \sigma_{i+1}, \ldots, \sigma_r. \hspace{1cm} (1.3)$$

Remark 1.3. The perturbation given by Theorem 1.1 allow us to have certain control on the spectral condition number of the perturbed matrix. That is, if $\kappa_2(A) = \sigma_1 / \sigma_r$, then we may choose $0 < \alpha_1 \leq \sigma_1 - \sigma_2$ and $0 < \alpha_r \leq \sigma_{r-1} - \sigma_r$ in such a way that

$$\kappa_2\left(A - \alpha_1 u_1 v_1^* + \alpha_r u_r v_r^*\right) = \frac{\sigma_1 - \alpha_1}{\alpha_r + \alpha_r} < \kappa_2(A). \hspace{1cm} (1.4)$$

The paper is organized as follows. In Section 2 we consider perturbations of the form $A + \alpha u_i v_i^*$, which we will call simple perturbations, and give sufficient conditions under which the perturbation preserves nonnegativity. In Section 3 we discuss perturbations of the form...
A + αu_p v_q^T, p ≠ q, which, because of their different indices, we will call mixed perturbations. We also give sufficient conditions in order that mixed perturbations preserve nonnegativity. It is also shown that both, simple and mixed perturbations, preserve doubly stochastic structure. Finally, we show some examples to illustrate the results.

2. Nonnegativity Preservation under Simple Perturbations

Let A be an \( m \times n \) positive matrix with singular values \( σ_1 ≥ σ_2 ≥ \cdots ≥ σ_r > 0 \), \( r = \min\{m, n\} \). In this section we consider perturbations \( A + αu_i v_i^T \), which preserve nonnegativity. Note that if \( A \) is an \( m \times n \) nonnegative matrix, then the left and right singular vectors \( u_1 \) and \( v_1 \), corresponding to the maximal singular value \( σ_1 \), respectively, are nonnegative. Hence, in this case, the matrix \( A + αu_1 v_1^T \) is nonnegative for all \( α > 0 \).

Now, let us consider the perturbation \( A + αu_i v_i^T \), with \( i > 1 \). Let \( u_s \) and \( v_s \) be the left and right singular vectors corresponding to \( σ_s, s > 1 \), respectively. Let \( α > 0 \) and let the entry in position \((i, k)\) of \( u_s v_s^T \) be negative. That is, \((u_s v_s^T)_{ik} < 0\). Then if \( A + αu_i v_i^T \) is nonnegative, \( a_{ik} + α(u_s v_s^T)_{ik} ≥ 0 \) iff \( 0 < α ≤ \frac{a_{ik}}{|(u_s v_s^T)_{ik}|} \). (2.1)

Thus, to preserve the nonnegativity of \( A \) it is enough to choose \( α \) in the interval

\[
\left(0, \min_{i,k} \frac{a_{ik}}{|(u_s v_s^T)_{ik}|}\right),
\]

provided that \( a_{ik} > 0 \), otherwise \((u_s v_s^T)_{ik}\) must be zero. Then from (2.2) and Theorem 1.1 we have the following result.

**Lemma 2.1.** Let \( A \) be an \( m \times n \) positive matrix with singular values \( σ_1 ≥ σ_2 ≥ \cdots ≥ σ_r > 0 \), \( r = \min\{m, n\} \). Let \( α \) be in the interval

\[
\left(0, \min_s \min_{i,k} \frac{a_{ik}}{|(u_s v_s^T)_{ik}|}\right).
\]

Then \( A + αu_s v_s^T, s = 2, \ldots, r \), is nonnegative with singular values

\[
σ_1, \ldots, σ_{s-1}, σ_s + α, σ_{s+1}, \ldots, σ_r.
\] (2.4)

**Remark 2.2.** It is clear that if in Lemma 2.1, \( α \) is taken in

\[
\left(0, \min_s \min_{i,k} \frac{a_{ik}}{|(u_s v_s^T)_{ik}|}\right),
\]

then \( A + αu_s v_s^T, 2 ≤ s ≤ r \), is positive with singular values

\[
σ_1, \ldots, σ_{s-1}, σ_s + α, σ_{s+1}, \ldots, σ_r.
\] (2.6)
Moreover, for $\alpha$ in intervals in Lemma 2.1 and this remark, the nonnegativity is obtained independently of the chosen singular vectors $u_s, v_s$.

A more handle interval for $\alpha$ is given by the following lemma.

**Lemma 2.3.** Let $A$ be an $m \times n$ positive matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$. Let $\alpha$ be in the interval

$$
\left(0, \frac{\sigma_1 \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j}\right].
$$

(2.7)

Then $A + \alpha u_s v_s^T, 2 \leq s \leq r$, is nonnegative with singular values

$$
\sigma_1, \ldots, \sigma_{s-1}, \sigma_s + \alpha, \sigma_{s+1}, \ldots, \sigma_r.
$$

(2.8)

**Proof.** From (2.2) and since $|a_{ik}| \leq \sigma_1$, see [1, Corollary 3.1.3], we have

$$
\max_{i,k} \left| \left( u_i v_s^T \right)_{ik} \right| = \max_{i,k} \left| \frac{a_{ik}}{\sigma_s} - \sum_{j=1, j \neq s}^{r} \frac{\sigma_j}{\sigma_s} \left( u_j v_j^T \right)_{ik} \right|

\leq \frac{1}{\sigma_s} \max_{i,k} |a_{ik}| + \sum_{j=1, j \neq s}^{r} \max_{i,k} \frac{\sigma_j}{\sigma_s} \left( u_j v_j^T \right)_{ik}

\leq \frac{1}{\sigma_s} \max_{i,k} |a_{ik}| + \sum_{j=1, j \neq s}^{r} \max_{i,k} \frac{\sigma_j}{\sigma_s}

\leq \frac{\sigma_1}{\sigma_s} + \sum_{j=1, j \neq s}^{r} \frac{\sigma_j}{\sigma_s}

\leq \frac{1}{\sigma_s} \left( 2\sigma_1 + \sum_{j=2, j \neq s}^{r} \sigma_j \right)

(2.9)

= \frac{1}{\sigma_s} \left( 2\sigma_1 - \sigma_s + \sum_{j=2}^{r} \sigma_j \right)

= 2 \frac{\sigma_1}{\sigma_s} - 1 + \sum_{j=2}^{r} \frac{\sigma_j}{\sigma_s}

\leq 2 \frac{\sigma_1}{\sigma_r} - 1 + \sum_{j=2}^{r} \frac{\sigma_j}{\sigma_r}

= \frac{\sigma_1}{\sigma_r} + \sum_{j=1}^{r-1} \frac{\sigma_j}{\sigma_r}.
Then

\[
\frac{1}{\max_{i,k} |(u_s v_s^T)_{ik}|} \geq \frac{\sigma_r}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j},
\]

\[
\frac{\min_{i,k} a_{ik}}{\max_{i,k} |(u_s v_s^T)_{ik}|} \geq \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j}.
\] (2.10)

Hence we have

\[
\left( 0, \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j} \right] \subseteq \left( 0, \min_{i,k} \frac{a_{ik}}{|(u_s v_s^T)_{ik}|} \right],
\]

\[
\left( 0, \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j} \right] \subseteq \left( 0, \min_{s} \min_{i,k} \frac{a_{ik}}{|(u_s v_s^T)_{ik}|} \right].
\] (2.11)

Then, from Lemma 2.1 the result follows.

**Remark 2.4.** For positive \( A = (a_{ij}) \), and \( \alpha \in \mathbb{R} \), we repeat the arguments from Lemmas 2.1 and 2.3 to obtain that if \( \alpha \) is in the interval

\[
\left[ -\frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j} , \frac{\sigma_r \min_{i,k} a_{ik}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j} \right],
\] (2.12)

then \( A + \alpha u_s v_s^T, s = 2, \ldots, r \), is nonnegative with singular values

\[
\sigma_1, \ldots, \sigma_{s-1}, |\sigma_s + \alpha|, \sigma_{s+1}, \ldots, \sigma_r.
\] (2.13)

Now we consider rank-2 perturbations, \( A + UDV^* \), where \( U = (u_s, u_t), D = \text{diag}(a_1, a_2), V = (v_s, v_t) \). That is, perturbations of the form \( A + a_1 u_s v_s^T + a_2 u_t v_t^T \) as in Theorem 1.1. Let \( A \) be an \( m \times n \) positive matrix with singular values \( \sigma_1 \geq \cdots \geq \sigma_r > 0 \), \( r = \min\{m, n\} \). Then \( A + a_1 u_s v_s^T + a_2 u_t v_t^T \) will be nonnegative if

\[
a_{ik} + a_1 (u_s v_s^T)_{ik} + a_2 (u_t v_t^T)_{ik} \geq 0, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n.
\] (2.14)

From the family of straight lines

\[
a_2 = -\frac{(u_s v_s^T)_{ik}}{(u_t v_t^T)_{ik}} a_1 - \frac{a_{ik}}{(u_t v_t^T)_{ik}}, \quad i = 1, \ldots, m; \quad j = 1, \ldots, n,
\] (2.15)
it follows that they intersect the axes \( a_1 \) and \( a_2 \) at points

\[
\left( -\frac{a_{ik}}{(u_s v^T_s)_{ik}}, 0 \right), \quad \left( 0, \frac{a_{ik}}{(u_s v^T_s)_{ik}} \right),
\]

(2.16)

where \((u_s v^T_s)_{ik} \neq 0\) and \((u_i v^T_i)_{ik} \neq 0\), respectively. Let

\[
E = \max_{(u,v)^T > 0} \left( -\frac{a_{ik}}{(u_s v^T_s)_{ik}} \right),
\]

\[
F = \max_{(u,v)^T > 0} \left( -\frac{a_{ik}}{(u_i v^T_i)_{ik}} \right),
\]

\[
G = \min_{(u,v)^T < 0} \left( -\frac{a_{ik}}{(u_i v^T_i)_{ik}} \right),
\]

\[
H = \min_{(u,v)^T < 0} \left( -\frac{a_{ik}}{(u_s v^T_s)_{ik}} \right).
\]

(2.17)

\[
R_1 = \left\{ (a_1, a_2) : E \leq a_1 \leq 0 \wedge -\frac{F}{E} a_1 + F \leq a_2 \leq -\frac{H}{E} a_1 + H \right\},
\]

\[
R_2 = \left\{ (a_1, a_2) : 0 \leq a_1 \leq G \wedge -\frac{F}{G} a_1 + F \leq a_2 \leq -\frac{H}{G} a_1 + H \right\}.
\]

(2.18)

Then \( A + a_1 u_s v^T_s + a_2 u_i v^T_i \) is nonnegative for \((a_1, a_2) \in R_1 \cup R_2\).

A more handle region for \((a_1, a_2)\) is given by the following lemma.

**Lemma 2.5.** Let \( A \) be an \( m \times n \) positive matrix with singular values \( \sigma_1 \geq \cdots \geq \sigma_r > 0 \), \( r = \min\{m,n\} \). Let \( P_1 = (E,0) \), \( P_2 = (0,F) \), \( P_3 = (G,0) \), \( P_4 = (0,H) \) be the intersection points in (2.17). Let

\[
(a_1, a_2) \in S = \left\{ (x,y) : \| (x,y) \|_1 \right\} \leq \min_{k=1,2,3,4} \| P_k \|_1,
\]

(2.19)

where \( \| \cdot \|_1 \) is the \( l_1 \) norm. Then \( A + a_1 u_s v^T_s + a_2 u_i v^T_i \), \( 2 \leq s, t \leq r \), is nonnegative with singular values \( \sigma_1, \ldots, |\sigma_s + a_1|, \ldots, |\sigma_t + a_2|, \ldots, \sigma_r \).

**Example 2.6.** Let

\[
A = \begin{pmatrix}
1 & 2 & 4 \\
5 & 6 & 8 \\
4 & 9 & 4
\end{pmatrix},
\]

(2.20)
with singular values $\sigma_1 = 15.5687298$, $\sigma_2 = 3.9581084$, and $\sigma_3 = 0.9736668$. Let

$$U = (u_2 \mid u_3), \quad V = (v_2 \mid v_3), \quad D = \text{diag}\{\alpha_1, \alpha_2\},$$

(2.21)

where

$$u_2 = \begin{pmatrix} 0.4219823 \\ 0.5264618 \\ -0.7380847 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0.8658718 \\ -0.4753192 \\ 0.1560054 \end{pmatrix},$$

$$v_2 = \begin{pmatrix} 0.0257579 \\ -0.6669921 \\ 0.7446195 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -0.9106840 \\ 0.2915541 \\ 0.2926616 \end{pmatrix}.$$ (2.22)

From (2.17) we compute $E$, $F$, $G$, $H$. Then, the intersection points are

$$(-12.7300876, 0), \quad (7.1058357, 0), \quad (0, -7.9224079), \quad (0, 1.2681734),$$

(2.23)

and $S = \{(x, y) : \| (x, y) \|_1 \leq 1.2681734\}$. Thus, from Lemma 2.5, for $(\alpha_1, \alpha_2) = (-1/2, 1/2)$ we have

$$A + \alpha_1 u_2 v_2^T + \alpha_2 u_3 v_3^T = \begin{pmatrix} 0.60030 & 2.2670 & 3.9696 \\ 5.2097 & 6.1063 & 7.7344 \\ 3.9385 & 8.7766 & 4.2976 \end{pmatrix},$$

(2.24)

with singular values $\sigma_1, \sigma_2 + \alpha_1, \sigma_3 + \alpha_2$.

Remark 2.7. Let $A$ be an $m \times n$ complex matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$ and singular value decomposition $A = U\Sigma V^*$. In [11] it was defined the concept of energy of $A$ as $\mathcal{E}(A) = \sigma_1 + \sigma_2 + \cdots + \sigma_r$. If $A$ is positive, then as an application of the rank-2 perturbation result, from Lemma 2.5 and Remark 2.4, we may construct, for

$$0 < \alpha \leq \min \left\{ \frac{\alpha_{\min,j} a_{jk}}{\sigma_1 + \sum_{j=1}^{r-1} \sigma_j} \right\},$$

(2.25)

a family of nonnegative matrices $B = A + \alpha u_i v_i^T$, with $\mathcal{E}(B) = \mathcal{E}(A)$.

Now, suppose $A$ is nonnegative with $\mathcal{E}(A) \leq C$, where $C$ is an upper bound. Then from (1.2), by taking $\alpha = C - \mathcal{E}(A)$ we may construct a family of nonnegative matrices $B = A + \alpha u_i v_i^T$ with $\mathcal{E}(B) = \mathcal{E}(A) + \alpha = C$.

Now, in order to show that simple perturbations preserve doubly stochastic structure we need the following lemma. First we introduce a definition and a notation. An $n \times n$ matrix $A = (a_{ij})$ is said to be with constant row sums if $\sum_{j=1}^{n} a_{ij} = \alpha$, $i = 1, 2, \ldots, n$. We denote by $\text{CS}_\alpha$ the set of all matrices with constant row sums equal to $\alpha$. 
Lemma 2.8. Let $A$ be an $n \times n$ irreducible doubly stochastic matrix and let $Ax = \lambda x$, $x^T = (x_1, \ldots, x_n)$, with $\lambda \neq 1$. Then

$$S(x) = x_1 + x_2 + \cdots + x_n = 0. \quad (2.26)$$

Proof. Since $A$ is doubly stochastic, then

$$S(Ax) = \Sigma a_{i1}x_j + \Sigma a_{i2}x_j + \cdots + \Sigma a_{in}x_j$$
$$= x_1\Sigma a_{i1} + x_2\Sigma a_{i2} + \cdots + x_n\Sigma a_{in} \quad (2.27)$$
$$= S(x),$$

and $S(Ax) = S(\lambda x) = \lambda S(x)$. Then,

$$S(x) = \lambda S(x), \quad (2.28)$$

and since $\lambda \neq 1$, $S(x) = 0$. \hfill $\square$

The following result shows that simple perturbations preserve doubly stochastic structure.

Proposition 2.9. Let $A$ be an $n \times n$ irreducible doubly stochastic matrix. Then,

(i) \((A + \alpha_1 u_1 v_1^T) \in CS_{1+\alpha_1}, \quad (A + \alpha_1 u_1 v_1^T)^T \in CS_{1+\alpha_1}\)

(ii) \((A + \alpha_i u_i v_i^T) \in CS_{1}, \quad (A + \alpha_i u_i v_i^T)^T \in CS_{1}; \quad i = 2, \ldots, n, \quad (2.29)\)

(iii) \((A + \sum_{i=1}^{n}\alpha_i u_i v_i^T) \in CS_{1+\alpha_1}\).

Proof. Since $A, A^T \in CS_1$, then $AA^T \in CS_1$ and $A^TA \in CS_1$. Hence, the singular vectors $u_1$ and $v_1$ are

$$u_1 = v_1 = \frac{1}{\sqrt{n}} e = \frac{1}{\sqrt{n}} (1, 1, \ldots, 1)^T. \quad (2.30)$$

Then $\alpha_1 u_1 v_1^T = \alpha_1 v_1 u_1^T = (1/n)\alpha_1 ee^T$ and $\alpha_1 u_1 v_1^T \in CS_{\alpha_1}$. Thus (i) holds. From Lemma 2.8, $\alpha_i u_i v_i^T e = \alpha_i v_i^Te u_i = 0$ and (ii) holds. From (i) and (ii) we have (iii). \hfill $\square$

Example 2.10. Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \quad (2.31)$$
which is nonnegative generalized doubly stochastic, that is, $A$ is nonnegative with $A, A^T \in \text{CS}_{10}$, and $A$ has singular values $10, 2.8284, 2, 8284, 2$. Let $U\Sigma V^*$ be the singular value decomposition of $A$. Let $D = \text{diag}\{4, 3, 2, 1\}$. Then

$$B = A + UDV^* = \begin{pmatrix}
6.2596 & 4.8500 & 2.2404 & 0.6501 \\
1.0822 & 6.0081 & 4.4178 & 2.4919 \\
2.2404 & 0.6501 & 6.2596 & 4.8500 \\
4.4178 & 2.4919 & 1.0822 & 6.0081
\end{pmatrix}$$

is nonnegative generalized doubly stochastic with singular values $14, 5.8284, 4.8284, 3$.

3. Nonnegativity Preservation under Mixed Perturbations

In this section we discuss matrix perturbations of the form $A + \alpha u_k v_j^*$, with $k \neq j$, which we will call mixed perturbations, and we study how the singular values and vectors change under this kind of perturbations. We also give sufficient conditions under which mixed perturbations preserve nonnegativity and preserve doubly stochastic structure. Let us start by considering the following particular case: let $A$ be a $4 \times 3$ matrix with singular values $\sigma_1, \sigma_2, \sigma_3$. Let $A = U\Sigma V^*$ with $U = (u_1 \mid u_2 \mid u_3 \mid u_4)$ and $V = (v_1 \mid v_2 \mid v_3)$. Let $\alpha \geq 0$. That is,

$$A = (u_1 \mid u_2 \mid u_3 \mid u_4) \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
v_1^* \\
v_2^* \\
v_3^*
\end{pmatrix} = \sum_{k=1}^{3} \sigma_k u_k v_k^*.$$  \hspace{1cm} (3.1)

The matrix $\alpha u_1 v_2^*$ has the singular value decomposition:

$$\alpha u_1 v_2^* = (u_1 \mid u_2 \mid u_3 \mid u_4) \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
v_1^* \\
v_2^* \\
v_3^*
\end{pmatrix}$$

$$= (u_1 \mid u_2 \mid u_3 \mid u_4) \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
v_1^* \\
v_2^* \\
v_3^*
\end{pmatrix}$$  \hspace{1cm} (3.2)
Then,

\[ A + \alpha u_1 v_2^* = (u_1 | u_2 | u_3 | u_4) \begin{pmatrix} \sigma_1 & \alpha & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \end{pmatrix}. \] (3.3)

Now we compute the singular values of the matrix

\[ C = \begin{pmatrix} \sigma_1 & \alpha & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix}, \] (3.4)

by computing the eigenvalues of \( \tilde{C} \tilde{C}^T \), where \( \tilde{C} = \begin{pmatrix} \sigma_1 & \alpha \\ 0 & \sigma_2 \end{pmatrix} \). Since

\[ \tilde{C} \tilde{C}^T = \begin{pmatrix} \sigma_1^2 + \alpha^2 & \alpha \sigma_2 \\ \alpha \sigma_2 & \sigma_2^2 \end{pmatrix}, \] (3.5)

then

\[ \text{tr}(\tilde{C} \tilde{C}^T) = \alpha^2 + \sigma_1^2 + \sigma_2^2 = \lambda_1 + \lambda_2, \]

\[ \text{det}(\tilde{C} \tilde{C}^T) = \sigma_1^2 \sigma_2^2 = \lambda_1 \lambda_2, \] (3.6)

with \( \lambda_1, \lambda_2 \) being the eigenvalues of \( \tilde{C} \tilde{C}^* \). Thus, we obtain

\[ \lambda_1 = \frac{\alpha^2 + \sigma_1^2 + \sigma_2^2 + \sqrt{(\alpha^2 + \sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2 \sigma_2^2}}{2}, \]

\[ \lambda_2 = \frac{\alpha^2 + \sigma_1^2 + \sigma_2^2 - \sqrt{(\alpha^2 + \sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2 \sigma_2^2}}{2}. \] (3.7)

Hence, the singular values of \( A + \alpha u_1 v_2^* \) are \( \sqrt{\lambda_1}, \sqrt{\lambda_2}, \sigma_3 \).

Now we generalize these results for \( m \times n \) matrices.

**Theorem 3.1.** Let \( A \) be an \( m \times n \) matrix with singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0 \), \( r = \min\{m, n\} \), and with singular value decomposition \( A = U \Sigma V^* \), where

\[ U = (u_1 | \cdots | u_k | \cdots | u_m), \quad V = (v_1 | \cdots | v_j | \cdots | v_n), \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \quad k, j \leq r. \] (3.8)
Let $\alpha \in \mathbb{R}$. Then $A + \alpha u_k v_j^*$ has singular values,

$$\{\sigma_1, \ldots, \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_{j-1}, \tilde{\sigma}_j, \sigma_{j+1}, \ldots, \sigma_r\},$$

where

$$\tilde{\sigma}_k = \left(\frac{\alpha^2 + \sigma_k^2 + \sigma_j^2 + \sqrt{(\alpha^2 + \sigma_k^2 + \sigma_j^2)^2 - 4\sigma_k^2\sigma_j^2}}{2}\right)^{1/2},$$

$$\tilde{\sigma}_j = \left(\frac{\alpha^2 + \sigma_k^2 + \sigma_j^2 - \sqrt{(\alpha^2 + \sigma_k^2 + \sigma_j^2)^2 - 4\sigma_k^2\sigma_j^2}}{2}\right)^{1/2}.$$

Proof. Without loss of generality we may assume that $m \geq n$ and $j > k$. The matrix $\alpha u_k v_j^*$ has a singular value decomposition

$$\alpha u_k v_j^* = (\pm u_k | \cdots | u_{k-1} | u_1 | u_{k+1} | \cdots | u_m) \tilde{\Sigma} \begin{pmatrix} v_j^* \\ \vdots \\ v_{j-1}^* \\ v_1^* \\ v_{j+1}^* \\ \vdots \\ v_n^* \end{pmatrix},$$

where $\tilde{\Sigma} = \text{diag}(|\alpha|, 0, \ldots, 0)$. The decomposition in (3.12) can be written as

$$\alpha u_k v_j^* = U P_k \tilde{\Sigma} Q_j V^*,$$

where $P_k$ and $Q_j$ are $m \times m$ and $n \times n$ permutation matrices of the form

$$P_k = \begin{pmatrix} W_k & 0 \\ 0 & I_{m-k} \end{pmatrix}, \quad Q_j = \begin{pmatrix} W_j & 0 \\ 0 & I_{n-j} \end{pmatrix},$$

with

$$W_l = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

of order $l$.
Since

\[ \tilde{\Sigma} = \begin{pmatrix} \alpha \\ \vdots \\ \vdots \\ \alpha \end{pmatrix}, \quad \text{then} \quad P_k \tilde{\Sigma} Q_j = \begin{pmatrix} \alpha \\ \vdots \\ \vdots \end{pmatrix}, \quad (3.16) \]

where if \( \alpha < 0 \), we multiply the \( u_k \) vector by minus one. Then

\[ A + \alpha u_k v_j^* = U (\Sigma + P_k \tilde{\Sigma} Q_j) V^* = U \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} V^*, \quad (3.17) \]

where

\[ \Lambda = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_k \\ \alpha \\ \vdots \\ \sigma_j \\ \vdots \\ \sigma_n \end{pmatrix}. \quad (3.18) \]

By applying row and column permutations on the matrix \( \Sigma + P_k \tilde{\Sigma} Q_j \), it follows from (3.17) and (3.18) that the singular values of \( A + \alpha u_k v_j^* \) are

\[ \{ \sigma_1, \ldots, \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_{j-1}, \tilde{\sigma}_j, \sigma_{j+1}, \ldots, \sigma_n \}, \quad (3.19) \]

where \( \tilde{\sigma}_k \) and \( \tilde{\sigma}_j \) are as in (3.10) and (3.11), respectively. \( \Box \)

Observe that in Theorem 3.1, \( A + \alpha u_k v_j^* \) has singular values

\[ \{ \sigma_1, \ldots, \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_{j-1}, \tilde{\sigma}_j, \sigma_{j+1}, \ldots, \sigma_r \}, \quad (3.20) \]

if \( k, j \leq r \). If \( r < k \leq m \), then only \( \sigma_j \), corresponding to the right singular vector \( v_j \), changes and take the form \( \tilde{\sigma}_j = \sqrt{\alpha^2 + \sigma_j^2} \). A straightforward calculation shows that for \( \alpha > 0 \), \( \tilde{\sigma}_k \leq \sigma_k + \alpha \). Observe that

\[ \tilde{\sigma}_j = \sqrt{\alpha^2 + \sigma_j^2} \leq \sigma_j + \alpha. \quad (3.21) \]
Example 3.2. Let

\[
A = \begin{pmatrix}
1 & 2 & 0 \\
6 & 1 & 3 \\
0 & 4 & 1 \\
1 & 5 & 0
\end{pmatrix},
\]

(3.22)

with singular values \( \sigma_1 = 7.8207, \sigma_2 = 5.6257, \sigma_3 = 1.09 \). Let

\[
u_1 = (0.26030 \ 0.70386 \ 0.39776 \ 0.52784)^T,
\]

(3.23)

\[
u_2 = (0.63252 \ -0.71647 \ 0.29425)^T,
\]

left and right singular vectors of \( A \), respectively. Let \( \alpha = 1 \). Then, from (3.10) and (3.11), the matrix

\[
A + \alpha u_1 v_2^* = \begin{pmatrix}
1.1646 & 1.8135 & 0.076593 \\
6.4452 & 0.49571 & 3.2071 \\
0.25159 & 3.715 & 1.117 \\
1.3339 & 4.6218 & 0.15532
\end{pmatrix}
\]

(3.24)

has singular values \( \tilde{\sigma}_1 = 7.9477, \tilde{\sigma}_2 = 5.5358 \) and \( \sigma_3 \). By using Theorem 1.1 we have that \( A + \alpha u_1 v_1^* + \alpha u_2 v_2^* \) has singular values \( \sigma_1 + 1 = 8.8207, \sigma_2 + 1 = 6.6257, \) and \( \sigma_3 = 1.09 \).

Different from perturbations of the form \( A + \alpha u_i v_i^T \), the perturbation of Theorem 3.1 affects not only the singular values \( \sigma_k \) and \( \sigma_j \), but also the corresponding left and right singular vectors \( u_k \) and \( v_j \). To make this modification clear, we consider again the previous discussion to Theorem 3.1:

\[
A + \alpha u_1 v_2^* = (u_1 \mid u_2 \mid u_3 \mid u_4) \begin{pmatrix}
\sigma_1 & \alpha & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
v_1^* \\
v_2^* \\
v_3^* \\
v_4^*
\end{pmatrix}.
\]

(3.25)

Let

\[
\tilde{C} = \begin{pmatrix}
\sigma_1 & \alpha \\
0 & \sigma_2
\end{pmatrix} = \begin{pmatrix}
\omega_{11} & \omega_{12} \\
\omega_{21} & \omega_{22}
\end{pmatrix} \begin{pmatrix}
\tilde{\sigma}_1 & 0 \\
0 & \tilde{\sigma}_2
\end{pmatrix} \begin{pmatrix}
\nu_{11} & \nu_{12} \\
\nu_{21} & \nu_{22}
\end{pmatrix}^T,
\]

(3.26)
the SVD of $\tilde{C}$ with $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ obtained from (3.10) and (3.11), respectively. The left singular vectors of

$$
C = \begin{pmatrix}
\sigma_1 & a & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3 \\
0 & 0 & 0
\end{pmatrix}
$$

(3.27)

(eigenvectors of $CC^T$) are

$$
\tilde{u}_1 = (\omega_{11} \omega_{21} 0 0)^T, \\
\tilde{u}_2 = (\omega_{12} \omega_{22} 0 0)^T, \\
\tilde{u}_3 = (0 0 1 0)^T = e_3, \\
\tilde{u}_4 = (0 0 0 1)^T = e_4,
$$

and its corresponding right singular vectors (eigenvectors of $C^T C$) are

$$
\tilde{v}_1 = (v_{11} v_{21} 0)^T, \\
\tilde{v}_2 = (v_{12} v_{22} 0)^T, \\
\tilde{v}_3 = (0 0 1)^T = e_3.
$$

Thus, $A + au_1v_2^*$ can be written as $A + au_1v_2^* = \tilde{U}\tilde{\Sigma}\tilde{V}^*$, where

$$
\tilde{U} = (u_1 \mid u_2 \mid u_3 \mid u_4)(\tilde{u}_1 \mid \tilde{u}_2 \mid \tilde{u}_3 \mid \tilde{u}_4) = (\omega_{11}u_1 + \omega_{21}u_2 \mid \omega_{12}u_1 + \omega_{22}u_2 \mid u_3 \mid u_4), \\
\tilde{V} = (v_1 \mid v_2 \mid v_3)(\tilde{v}_1 \mid \tilde{v}_2 \mid \tilde{v}_3) = (v_{11}v_1 + v_{21}v_2 \mid v_{12}v_1 + v_{22}v_2 \mid v_3), \\
\tilde{\Sigma} = \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3).
$$
Now, we generalize this result. Without loss generality we assume that $k < j$. From (3.17) and (3.18), by permuting rows and columns of the matrix $\Sigma + P_k \tilde{\Sigma} Q_j$, we have

\[
A + \alpha u_k v_j^* = U_1 \begin{pmatrix} A \\ 0 \end{pmatrix} V_1^* \quad \text{with} \quad \Lambda = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_k \alpha \\ & & \sigma_j & \\ & & & \ddots \\ & & & & \sigma_n \end{pmatrix},
\]

\[
U_1 = (u_1 \mid \cdots \mid u_k \mid u_j \mid \cdots \mid u_{k+1} \mid u_{j+1} \mid \cdots \mid u_m),
\]

\[
V_1 = (v_1 \mid \cdots \mid v_k \mid v_j \mid \cdots \mid v_{k+1} \mid v_{j+1} \mid \cdots \mid v_n).
\]

The singular vectors of $\begin{pmatrix} \Lambda \\ 0 \end{pmatrix}$ are obtained from the singular vectors of the $2 \times 2$ matrix $\begin{pmatrix} \sigma_k \alpha \\ 0 \sigma_j \end{pmatrix}$. Let

\[
\begin{pmatrix} \sigma_k \alpha \\ 0 \sigma_j \end{pmatrix} = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} \tilde{\sigma}_k \ 0 \\ 0 \tilde{\sigma}_j \end{pmatrix} \begin{pmatrix} v_{11} \ v_{12} \\ v_{21} \ v_{22} \end{pmatrix}^T.
\]

Then the unitary matrices of the singular value decomposition of $\begin{pmatrix} \Lambda \\ 0 \end{pmatrix}$ are

\[
U_2 = \begin{pmatrix} I \\ \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \\ I \end{pmatrix} = (e_1 \mid \cdots \mid e_{k-1} \mid \tilde{u}_k \mid \tilde{u}_{k+1} \mid e_{k+2} \mid \cdots \mid e_m),
\]

\[
V_2 = \begin{pmatrix} I \\ \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \\ I \end{pmatrix} = (e_1 \mid \cdots \mid e_{k-1} \mid \tilde{v}_k \mid \tilde{v}_{k+1} \mid e_{k+2} \mid \cdots \mid e_n),
\]

\[
(3.31)
\]

\[
(3.32)
\]

\[
(3.33)
\]
of order $m \times m$ and $n \times n$, respectively. Then

$$A + \alpha u_k v_j^* = \tilde{U} \begin{pmatrix} \tilde{\Lambda} \\ 0 \end{pmatrix} \tilde{V}^*$$

with $\tilde{U} = U_1 U_2$, $\tilde{V} = V_1 V_2$,

$$\tilde{\Lambda} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \tilde{\sigma}_k \\ \sigma_j \\ \vdots \\ \sigma_n \end{pmatrix},$$

(3.34)

where $\tilde{\sigma}_k$ and $\tilde{\sigma}_j$ are not ordered. Since

$$\tilde{U} = (u_1 | \cdots | u_{11} u_k + \omega_{21} u_j | \omega_{12} u_k + \omega_{22} u_j | \cdots | u_m),$$

$$\tilde{V} = (v_1 | \cdots | v_{11} v_k + \omega_{21} v_j | \omega_{12} v_k + \omega_{22} v_j | \cdots | v_n),$$

(3.35)

then the singular vectors corresponding to $\sigma_k, \sigma_j$ have been modified. We have prove the following result.

**Corollary 3.3.** Let $A$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $r = \min\{m, n\}$ and with singular value decomposition $A = U \Sigma V^*$, where

$$U = (u_1 | \cdots | u_{11} u_k + \omega_{21} u_j | \omega_{12} u_k + \omega_{22} u_j | \cdots | u_m),$$

$$V = (v_1 | \cdots | v_{11} v_k + \omega_{21} v_j | \omega_{12} v_k + \omega_{22} v_j | \cdots | v_n),$$

$$\Sigma = \text{diag} \{\sigma_1, \sigma_2, \ldots, \sigma_r\}.$$  

(3.36)

Let $\alpha \in \mathbb{R}$. Then $A + \alpha u_k v_j^*$ has left singular vectors,

$$\tilde{u}_i = u_i, \quad i = 1, \ldots, m, \quad i \neq k, \quad i \neq j,$$

$$\tilde{u}_k = \omega_{11} u_k + \omega_{21} u_j,$$

$$\tilde{u}_j = \omega_{12} u_k + \omega_{22} u_j,$$

(3.37)

and right singular vectors

$$\tilde{v}_i = v_i, \quad i = 1, \ldots, n, \quad i \neq k, \quad i \neq j,$$

$$\tilde{v}_k = v_{11} v_k + v_{21} v_j,$$

$$\tilde{v}_j = v_{12} v_k + v_{22} v_j.$$  

(3.38)
Observe that if $2 \times 2$ orthogonal matrices in (3.32) are of the same type

$$
\begin{pmatrix}
  c & s \\
  -s & c
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
  c & s \\
  s & -c
\end{pmatrix},
$$

then a straight forward calculation shows that (3.37) and (3.38) become

$$
\tilde{u}_i = u_i, \quad i = 1, \ldots, m, \quad i \neq k, \quad i \neq j, \\
\tilde{u}_k = c_1 u_k - s_1 u_j, \\
\tilde{u}_j = s_1 u_k + c_1 u_j,
$$

and

$$
\tilde{v}_i = v_i, \quad i = 1, \ldots, n, \quad i \neq k, \quad i \neq j, \\
\tilde{v}_k = c_2 v_k - s_2 v_j, \\
\tilde{v}_j = s_2 v_k + c_2 v_j,
$$

respectively, while if they are of different type, then (3.38) becomes

$$
\tilde{v}_i = v_i, \quad i = 1, \ldots, m, \quad i \neq k, \quad i \neq j, \\
\tilde{v}_k = c_2 v_k + s_2 v_j, \\
\tilde{v}_j = s_2 v_k - c_2 v_j.
$$

From (3.10) and (3.11) it is clear that $\bar{\sigma}_k > \bar{\sigma}_j$ with $k < j$. The following result tells us how the new singular values $\tilde{\sigma}_k, \tilde{\sigma}_j$ relate with the previous singular values $\sigma_k, \sigma_j$.

**Corollary 3.4.** Let $A$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$ and with singular value decomposition $A = U \Sigma V^*$, where

$$
U = (u_1 | \cdots | u_k | \cdots | u_m), \quad V = (v_1 | \cdots | v_j | \cdots v_n), \quad k, j \leq r, \\
\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}.
$$

Let $\tilde{\sigma}_k, \tilde{\sigma}_j, k < j$, be respectively, as in (3.10) and (3.11), the singular values of $A + \alpha u_k v_j^*$. Then $\tilde{\sigma}_k > \sigma_k \geq \sigma_j > \tilde{\sigma}_j$. 
Proof. Since $4\sigma_j^2\alpha^2 > 0$ we have

$$
\left(\sigma_k^2 + \alpha^2\right)^2 + \sigma_j^4 + 2\sigma_j^2\left(\sigma_k^2 + \alpha^2\right) - 4\sigma_k^2\sigma_j^2 > \left(\sigma_k^2 + \alpha^2\right)^2 + \sigma_j^4 - 2\sigma_j^2\left(\sigma_k^2 + \alpha^2\right),
$$

$$
\left(\sigma_k^2 + \sigma_j^2 + \alpha^2\right)^2 - 4\sigma_k^2\sigma_j^2 > \left(\left(\sigma_k^2 + \alpha^2\right) - \sigma_j^2\right)^2,
$$

(3.44)

Thus

$$
\sigma_j > \left(1\right) \left[\frac{\sigma_k^2 + \sigma_j^2}{\sqrt{\left(\sigma_k^2 + \sigma_j^2 + \alpha^2\right)^2 - 4\sigma_k^2\sigma_j^2}}\right]^{1/2}.
$$

(3.45)

In the same way,

$$
\left(\sigma_j^2 + \alpha^2\right)^2 + \sigma_k^4 + 2\sigma_k^2\left(\sigma_j^2 + \alpha^2\right) - 4\sigma_k^2\sigma_j^2 > \left(\sigma_j^2 + \alpha^2\right)^2 + \sigma_k^4 - 2\sigma_k^2\left(\sigma_j^2 + \alpha^2\right),
$$

$$
\left(\sigma_k^2 + \sigma_j^2 + \alpha^2\right)^2 - 4\sigma_k^2\sigma_j^2 > \left(\sigma_k^2 - \left(\sigma_j^2 + \alpha^2\right)\right)^2,
$$

(3.46)

$$
\sigma_k^2 + \sigma_j^2 + \alpha^2 + \frac{\left(\sigma_k^2 + \sigma_j^2 + \alpha^2\right)^2 - 4\sigma_k^2\sigma_j^2}{\sigma_k^2 - \sigma_j^2} > 2\sigma_k^2.
$$

Then

$$
\sigma_k < \left(\frac{1}{2}\right) \left[\frac{\sigma_k^2 + \sigma_j^2 + \sqrt{\left(\sigma_k^2 + \sigma_j^2 + \alpha^2\right)^2 - 4\sigma_k^2\sigma_j^2}}{\sigma_k^2 - \sigma_j^2}\right]^{1/2}.
$$

(3.47)

Therefore,

$$
\tilde{\sigma}_k > \sigma_k \geq \sigma_j > \tilde{\sigma}_j.
$$

(3.48)

Observe that $\sigma_i \in (\tilde{\sigma}_i, \tilde{\sigma}_k)$, $i = k, k + 1, \ldots, j$. In particular for $k = 1$ and $j = n$, all singular values of $A$ are in the interval $(\tilde{\sigma}_n, \tilde{\sigma}_1)$.

Now we extend the mixed perturbation result given by Theorem 3.1 to rank-2 perturbations, that is, perturbations of the form $B = A + \alpha_1 u_k v_i^* + \alpha_2 u_k v_j^*$, with $\alpha_1$, $\alpha_2$ nonzero.
real numbers and $u_{ki}, v_{ji}, i = 1, 2, k_i < j_i, k_1 < j_1 < k_2 < j_2$, being left and right singular vectors of $A$, respectively. Then as in (3.12)

$$(\pm u_{k_1}, \pm u_{k_2}, \ldots, u_1, u_{k_1+1}, \ldots, u_{k_2-1}, u_2, u_{k_2+1}, \ldots, u_m) \tilde{\Sigma}$$

with $\tilde{\Sigma} = \text{diag}(|\alpha_1|, |\alpha_2|, 0, \ldots, 0)$. It is clear that the matrix in (3.49) can be written as

$$UP_1P_2\tilde{\Sigma}Q_1Q_2V^*,$$

where $P_1, P_2, Q_1, Q_2$ are permutation matrices of the form

$$P_1 = \begin{pmatrix} W_{k_1} & 0 \\ 0 & I_{m-k_1} \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ W_{k_2} \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} W_{j_1} & 0 \\ 0 & I_{n-j_1} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 \\ W_{j_2} \end{pmatrix}.$$

From (3.50), the matrix in (3.49) is

$$\begin{pmatrix} (\alpha_1)_{k_1,j_1} \\ \vdots \\ (\alpha_2)_{k_2,j_2} \end{pmatrix},$$

$$\begin{pmatrix} v_{j_1}^* \\ v_{j_2}^* \\ \vdots \\ v_{j_1+1}^* \\ v_{j_2+1}^* \\ \vdots \\ v_n^* \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{j_1}^* \\ v_{j_2}^* \\ \vdots \\ v_n^* \end{pmatrix}.$$
where if $\alpha_1$, $\alpha_2$ are negative, we multiply $u_{k_1}$, $u_{k_2}$ by minus one. Thus,

$$B = A + \alpha_1 u_{k_1} v^*_{j_1} + \alpha_2 u_{k_2} v^*_{j_2} = U \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} V^*, \quad (3.53)$$

where

$$\Lambda = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_k \ \cdots \ \alpha_{k_1,j_1} \\ \vdots \\ \sigma_j \ \cdots \ \alpha_{k_2,j_2} \\ \vdots \\ \sigma_n \end{pmatrix}. \quad (3.54)$$

By permuting rows and columns of $\Lambda$ it follows that singular values of $\begin{pmatrix} \Lambda \\ 0 \end{pmatrix}$ are

$$\sigma_1, \ldots, \sigma_{k_1-1}, \sigma_{k_1+1}, \ldots, \sigma_{j_1-1}, \sigma_{j_1+1}, \ldots, \sigma_{k_2-1}, \sigma_{k_2+1}, \ldots, \sigma_{j_2-1}, \sigma_{j_2+1}, \ldots, \sigma_n \quad (3.55)$$

together with the singular values of

$$C_1 = \begin{pmatrix} \sigma_{k_1} & \alpha_{k_1,j_1} \\ 0 & \sigma_{j_1} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \sigma_{k_2} & \alpha_{k_2,j_2} \\ 0 & \sigma_{j_2} \end{pmatrix}. \quad (3.56)$$
Is immediate that the singular values of these matrices are

\[
\tilde{\sigma}_k = \left( \frac{\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2 + \sqrt{\left( \frac{\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2}{2} - 4\sigma_{k_1}^2\sigma_{j_1}^2 \right)^2}}{2} \right)^{1/2},
\]

\[
\tilde{\sigma}_j = \left( \frac{\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2 - \sqrt{\left( \frac{\alpha_1^2 + \sigma_{k_1}^2 + \sigma_{j_1}^2}{2} - 4\sigma_{k_1}^2\sigma_{j_1}^2 \right)^2}}{2} \right)^{1/2},
\]

\[
\tilde{\sigma}_k = \left( \frac{\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2 + \sqrt{\left( \frac{\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2}{2} - 4\sigma_{k_2}^2\sigma_{j_2}^2 \right)^2}}{2} \right)^{1/2},
\]

\[
\tilde{\sigma}_j = \left( \frac{\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2 - \sqrt{\left( \frac{\alpha_2^2 + \sigma_{k_2}^2 + \sigma_{j_2}^2}{2} - 4\sigma_{k_2}^2\sigma_{j_2}^2 \right)^2}}{2} \right)^{1/2}.
\]

The matrix \( \mathbf{B} = \mathbf{A} + \alpha_1 \mathbf{u}_{k_1} \mathbf{v}_{j_1}^* + \alpha_2 \mathbf{u}_{k_2} \mathbf{v}_{j_2}^* \) can be written as

\[
\mathbf{B} = \bar{\mathbf{U}} \begin{pmatrix} \bar{\mathbf{\Lambda}} \\ 0 \end{pmatrix} \bar{\mathbf{V}}^* \quad \text{with} \quad \bar{\mathbf{U}} = \mathbf{U}_1 \mathbf{U}_2, \quad \bar{\mathbf{V}} = \mathbf{V}_1 \mathbf{V}_2,
\]

\[
\bar{\mathbf{\Lambda}} = \begin{pmatrix} \sigma_1 & & & & \\ & \tilde{\sigma}_k & & & \\ & & \tilde{\sigma}_j & & \\ & & & \tilde{\sigma}_k & \\ & & & & \sigma_n \end{pmatrix},
\]

(3.57)
where the $\tilde{\sigma}$'s are not ordered,

$$
U_1 = (u_{1}, u_{2} \cdots u_{k_{i}}, u_{j_{i}} \cdots u_{k_{i+1}}, u_{j_{i+1}} \cdots u_{k_{j}}, u_{j_{j}} \cdots u_{m}),
$$

$$
V_1 = (v_{1}, v_{2} \cdots v_{k_{i}}, v_{j_{i}} \cdots v_{k_{i+1}}, v_{j_{i+1}} \cdots v_{k_{j}}, v_{j_{j}} \cdots v_{n}),
$$

$$
U_2 = (e_{1} \cdots e_{k_{i}-1}, \tilde{\tilde{u}}_{k_{i}} e_{k_{i}+1}, e_{k_{i}+2} \cdots e_{k_{j}-1}, \tilde{\tilde{u}}_{k_{j}} e_{k_{j}+2} \cdots e_{m}),
$$

$$
V_2 = (e_{1} \cdots e_{k_{i}-1}, \tilde{\tilde{v}}_{k_{i}} e_{k_{i}+1}, e_{k_{i}+2} \cdots e_{k_{j}-1}, \tilde{\tilde{v}}_{k_{j}} e_{k_{j}+2} \cdots e_{n}).
$$

We have proved the following result.

**Theorem 3.5.** Let $A$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$ and SVD $A = U\Sigma V^*$, where

$$
U = (u_{1}, u_{2} \cdots u_{k_{i}} \cdots u_{k_{j}} \cdots u_{m}), \quad V = (v_{1}, v_{2} \cdots v_{j_{i}} \cdots v_{j_{j}} \cdots v_{n}),
$$

$k_{i}, j_{i} \leq r$, $k_{i} \neq j_{i}$, are $m \times m$ and $n \times n$ unitary matrices, respectively, and $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$. Let $\alpha_1, \alpha_2$ be real numbers. Then $B = A + \alpha_1 u_{k_{i}} v_{j_{i}}^* + \alpha_2 u_{k_{j}} v_{j_{j}}^*$ has singular values

$$
\bigcup_{q=1}^{r} \{\sigma_q\} \cup \{\tilde{\sigma}_{k_{i}}, \tilde{\sigma}_{j_{i}}, \tilde{\sigma}_{k_{j}}, \tilde{\sigma}_{j_{j}}\},
$$

$$
\tilde{\sigma}_{k_{i}} = \left(\frac{\alpha_1^2 + \sigma_{k_{i}}^2 + \sigma_{j_{i}}^2 + \sqrt{(\alpha_1^2 + \sigma_{k_{i}}^2 + \sigma_{j_{i}}^2)^2 - 4\sigma_{k_{i}}^2 \sigma_{j_{i}}^2}}{2}\right)^{1/2},
$$

$$
\tilde{\sigma}_{j_{i}} = \left(\frac{\alpha_2^2 + \sigma_{k_{i}}^2 + \sigma_{j_{i}}^2 - \sqrt{(\alpha_2^2 + \sigma_{k_{i}}^2 + \sigma_{j_{i}}^2)^2 - 4\sigma_{k_{i}}^2 \sigma_{j_{i}}^2}}{2}\right)^{1/2}
$$

$i = 1, 2$. The singular vectors are given by

$$
\tilde{u}_{q} = u_{q}, \quad q = 1, \ldots, m, \ q \neq k_{i}, \ q \neq j_{i}, \ i = 1, 2,
$$

$$
\tilde{u}_{k_{i}} = \omega_{11}^{(1)} u_{k_{i}} + \omega_{21}^{(1)} u_{j_{i}},
$$

$$
\tilde{u}_{j_{i}} = \omega_{12}^{(1)} u_{k_{i}} + \omega_{22}^{(1)} u_{j_{i}},
$$

$$
\tilde{u}_{k_{j}} = \omega_{11}^{(2)} u_{k_{j}} + \omega_{21}^{(2)} u_{j_{j}},
$$

$$
\tilde{u}_{j_{j}} = \omega_{12}^{(2)} u_{k_{j}} + \omega_{22}^{(2)} u_{j_{j}}.
$$
where the coefficients \( \omega \)'s are obtained as before, and

\[
\tilde{v}_q = v_q, \quad q = 1, \ldots, n, \quad q \neq k_i, \quad q \neq j_i, \quad i = 1, 2,
\]

\[
\tilde{v}_k_i = u_{11}^{(1)} v_{k_i} + u_{21}^{(1)} v_{j_i},
\]

\[
\tilde{v}_j_i = u_{12}^{(1)} v_{k_i} + u_{22}^{(1)} v_{j_i},
\]

\[
\tilde{v}_k_2 = u_{11}^{(2)} v_{k_i} + u_{21}^{(2)} v_{j_i}
\]

\[
\tilde{v}_j_2 = u_{12}^{(2)} v_{k_i} + u_{22}^{(2)} v_{j_i}.
\]  

(3.65)

Now, as in Lemma 2.1 in Section 2, we look for a condition to preserve nonnegativity when we deal with mixed perturbations. Let \( A \) be an \( m \times n \) positive matrix with singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \), \( r = \min\{m, n\} \), and with singular value decomposition \( A = U \Sigma V^* \), where

\[
U = (u_1 | \cdots | u_p | \cdots | u_m), \quad V = (v_1 | \cdots | v_q | \cdots | v_n), \quad p, q \leq r,
\]

\[
\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}.
\]  

(3.66)

Then \( A + \alpha u_p v_q^T \) is nonnegative for \( \alpha > 0 \) if \( a_{ik} + \alpha (u_p v_q^T)_{ik} \geq 0, \quad i = 1, \ldots, m, \quad k = 1, \ldots, n \). If \( (u_p v_q^T)_{ik} < 0 \), then

\[
a_{ik} + \alpha (u_p v_q^T)_{ik} \geq 0 \quad \text{iff} \quad 0 < \alpha \leq \min_{i,k} \frac{a_{ik}}{|(u_p v_q^T)_{ik}|}.
\]  

(3.67)

Then we have the following result.

**Lemma 3.6.** Let \( A \) be an \( m \times n \) positive matrix with singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \), \( r = \min\{m, n\} \). Let \( \alpha \) be in the interval

\[
\left( 0, \min \min_{p,q} \min_{i,k} \frac{a_{ik}}{|(u_p v_q^T)_{ik}|} \right).
\]  

(3.68)

Then \( A + \alpha u_p v_q^T, \quad 1 \leq p \leq m, \quad 1 \leq q \leq n, \quad p \neq q, \) is nonnegative with singular values

\[
\sigma_1, \ldots, \sigma_{p-1}, \tilde{\sigma}_p, \sigma_{p+1}, \ldots, \sigma_{q-1}, \tilde{\sigma}_q, \sigma_{q+1}, \ldots, \sigma_r
\]

(3.69)

where \( \tilde{\sigma}_p \) and \( \tilde{\sigma}_q \) are defined as in (3.62) and (3.63), respectively.
Example 3.7. Let \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \) be with singular values \( \sigma_1 = 9.5255, \sigma_2 = 0.5143 \). \( A = U \Sigma V^T \), where

\[
U = \begin{pmatrix}
0.22985 & -0.88346 & 0.40825 \\
0.52474 & -0.24078 & -0.81650 \\
0.81964 & 0.40190 & 0.40825 \\
\end{pmatrix}, \quad V = \begin{pmatrix}
0.61963 & 0.78489 \\
0.78489 & -0.61963 \\
\end{pmatrix}.
\]

Then, from (3.68) we have for \( p = 1, 2, 3; \ q = 1, 2, \)

\[
\min_{p\neq 1, 2} \min_{i \neq j} \min_{k \neq 1, 2} \left| a_{ik} \left( u_p v_q^T \right)_{ik} \right| = 1.8268
\]

and \( \alpha \in (0, 1.8268] \). For \( \alpha = 1.8268 \), we have

\[
A + \alpha u_1 v_2^T = \begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{pmatrix} + 1.8268 \begin{pmatrix}
0.18041 & -0.14242 \\
0.41186 & -0.32514 \\
0.64333 & -0.50787 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1.3296 & 1.7398 \\
3.7524 & 3.406 \\
6.1752 & 5.0722 \\
\end{pmatrix}
\]

is nonnegative with singular values \( \tilde{\sigma}_1 = 9.6996 \) and \( \tilde{\sigma}_2 = 0.5050 \).

To show that mixed perturbations preserve doubly stochastic structure, observe from Lemma 2.8 that \( u_p v_q^T e = 0 \) for \( q = 2, \ldots, n \). Then, if \( A \) is an \( n \times n \) positive doubly stochastic matrix, we have that \( A + \alpha u_p v_q^T \) is doubly stochastic.

**Acknowledgment**

This work was supported by Fondecyt 1085125, Chile.

**References**


